

GENERALIZED COHERENT STATES FOR CHARGED PARTICLE IN UNIFORM AND VARIABLE MAGNETIC FIELD

M.R. SETARE[†], A. FALLAHPOUR[‡]

Department of Science, University of Kurdistan
Pasdaran Ave., Sanandaj, Iran

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Coherent states play an important role in quantum physics. By studying the dynamical groups and recognizing their algebra relations, we can specify coherent states for these groups. In this paper, we will specify dynamical group of a charged particle in uniform and variable magnetic fields, then we obtain the coherent states for its corresponding problem.

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1. Introduction

The standard coherent state system is intimately related to a group [1–3], considered first by Weyl, the so-called Heisenberg–Weyl group. The coherent state method is particularly effective in cases where the Heisenberg–Weyl group is the dynamical symmetry group of a considered physical system. The simplest example is a quantum oscillator under the action of a variable external driving force [3]. In this case the Heisenberg equations of motion coincide with the corresponding equations for the classical variables. In the course of the time evolution, any coherent state remains coherent [4, 5], and the motion of the phase space point representing the coherent state is described by the classical equation. This fact enables one to simplify the quantum problem significantly, reducing it to the corresponding classical problem. The Heisenberg–Weyl group, of course, is not the universal dynamical symmetry group, other symmetry groups appear in many cases. For instance, the symmetry group for spin precession in a variable magnetic field is SU(2) group, and for the problem of a quantum oscillator with variable frequency, the symmetry group is SU(1, 1). Coherent states of various

[†] rezakord@ipm.ir

[‡] falahpor_a_s@yahoo.com

Lie groups, as functions of some variable z which run over the entire complex plane [6], have been successfully used in many fields of physics [1, 7]. Based on the methods of group theory there are three different approaches for defining coherent states [3]. In the first approach, the coherent states are generated by the action of group elements on a reference state of a group representation in the Hilbert space. For example, the reference state can be the ground state corresponding to a quantum mechanical Hilbert space. In the second approach, coherent states are defined as eigenstates of a lowering group generator. Again, the lowering generator can be related to the Lie algebra of the dynamical symmetry group of a quantum system, such that it acts on the Hilbert space generated by the quantum states representing the Lie algebra. The third approach for defining coherent states is related to the optimization of uncertainty relations for Hermitian generators of a Lie group. This method is the Schrödinger discovery [8] of the coherent states as the states which minimize the Heisenberg uncertainty relation. Different approaches overlap only in the special case of the Heisenberg–Weyl group, that is the dynamical symmetry group of the quantum harmonic oscillator. A question arises whether for other Lie group, systems of states exist having some properties similar to those of the standard coherent state system. The answer is positive. In [1, 2], general coherent systems related to representations of an arbitrary Lie group were constructed and investigated, elaborate methods of group theory were employed to study properties of these systems. Generalized coherent states, which were introduced in [1, 3], are relevant to an arbitrary Lie group, they are parametrized by point of homogeneous spaces where the group acts. In some cases, one can consider these spaces as generalized phase spaces for classical dynamical systems.

Coherent states play an important role in quantum physics, for instance, in non-equilibrium statistical physics that describes the evolution toward thermodynamic equilibrium for quantum systems with equidistant energy spectra set in thermostat [1], and so we can investigate the coherent states to obtain Landau diamagnetism for a free electron gas [9]. In [10], the possible occurrence of orbital magnetism for two-dimensional spinless electrons confined by a harmonic potential $\sim \frac{1}{2}m\omega_0 R^2$ in various regimes of temperature and magnetic field is studied. There are two dynamical symmetries of the $SU(2)$ and $SU(1,1)$ Lie algebra of quadratic observable. It has been shown that the coherent states can be defined as tensor products of the standard one-dimensional coherent states that correspond to tensor products of Fock harmonic oscillator eigenstates.

Coherent states can be studied for arbitrary groups, but here we confine our problem to $SU(1,1)$ dynamical group. The role of $SU(1,1)$ group in physics, especially in quantum physics, has been recognized for a long time and its coherent states have been extensively studied. There are two

well-known analytic representations of the coherent states for the $SU(1,1)$ Lie group [11]. One is the analytic representation in the unit disk based on the overcomplete set of the $SU(1,1)$ Perelomov coherent states [1] and the other is the Barut–Girardello representation based on the overcomplete basis of the Barut–Girardello states [12]. One can find interesting properties and applications of these two representations in [13–17]. In [11] it has been shown that the Barut–Girardello representation and the analytic representation in the unit disk are related through a Laplace transform. These representations are useful in many quantum mechanical problems involving dynamical systems with $SU(1,1)$ symmetry [18–21], for example, the coherent states have been used to study two-photon realization of $SU(1,1)$ in quantum optics [10]. In [22] these representations have been applied for photon states associated with the Holstein–Primakoff realization of the $SU(1,1)$ Lie algebra.

As the other application, coherent states can be considered for Pöschl–Teller potential and their revival dynamics. In [23] the properties of Barut–Girardello and Kluder–Perelomov coherent states have been compared for trigonometric Pöschl–Teller potential with $SU(1,1)$ dynamical group. In [24], the behavior of electrons in an external uniform magnetic field \vec{B} has been considered where the space coordinates perpendicular to \vec{B} have been taken as non-commuting. The authors have used the coherent states for the calculation the partition function, the magnetization and the susceptibility.

In this paper, we consider a charged particle moving in the presence of a static and uniform magnetic field in the z -direction, and we extend this problem to the magnetic field which varies as $\frac{1}{x^2}$ [25]. The dynamical groups for these systems are $SU(1,1)$ groups. Then using the definition of Kluder–Perelomov coherent states, we will write generalized coherent states for these physical systems.

2. Generalized coherent state for $SU(1,1)$ Lie algebra

We note that Lie algebra corresponding to the Lie group $SU(1,1)$ has three generators, \hat{K}_1 , \hat{K}_2 and \hat{K}_3 , or \hat{K}_+ , \hat{K}_- and \hat{K}_3 as its basis elements. The commutation relation of the $SU(1, 1)$ Lie algebra is given by [1, 3]:

$$[\hat{K}_3, \hat{K}_+] = \hat{K}_+, \quad [\hat{K}_3, \hat{K}_-] = -\hat{K}_-, \quad [\hat{K}_+, \hat{K}_-] = -2\hat{K}_3. \quad (1)$$

The Fock space on which $\{\hat{K}_+, \hat{K}_-, \hat{K}_3\}$ acts, is $\mathbf{H} \equiv \{|K, n\rangle | n \in \mathbf{N} \cup \{0\}\}$ and its actions are

$$K_+ |K, n\rangle = \sqrt{(n+1)(2K+n)} |K, n+1\rangle, \quad (2)$$

$$K_- |K, n\rangle = \sqrt{n(2K+n-1)} |K, n-1\rangle, \quad (3)$$

$$K_3 |K, n\rangle = (K+n) |K, n\rangle, \quad (4)$$

where $|K, 0\rangle$ is a normalized state:

$$K_- |K, 0\rangle = 0, \quad (5)$$

$$\langle K, 0 | K, 0 \rangle = 1. \quad (6)$$

From Eqs. (2), (3), (4), we have

$$|K, n\rangle = \frac{(K_+)^n}{\sqrt{(n!)(2K)_n}} |K, 0\rangle, \quad (7)$$

where $(a)_n$ is the Pochhammer's notation $(a)_n \equiv a(a+1)\dots(a+n-1)$. Now we would like to consider the displaced operator associated to the $SU(1,1)$ Lie algebra. This operator is given by the following relation [1]:

$$D(\xi) = e^{\xi K_+ - \bar{\xi} K_-} = \exp(\zeta K_+) \exp(\eta K_3) \exp(\zeta' K_-), \quad (8)$$

where

$$\zeta = \tanh |\xi| e^{i\varphi}, \quad \eta = 2 \ln \cos |\xi| = -\ln(1 - |\zeta|^2), \quad \zeta' = -\bar{\zeta}, \quad (9)$$

where φ is a phase and ξ is a complex number. This is also the key formula for generalized coherent operators. Applying the displacement operator $D(\xi)$ to the state vector $|\psi_0\rangle$, and using the normal form Eq. (8), we obtain another representation for the coherent states:

$$|\zeta\rangle = (1 - |\zeta|^2)^k \exp(\zeta K_+) |0\rangle, \quad (10)$$

where $|K, 0\rangle = |0\rangle$.

Expanding the exponential function and using [1], we obtain the decomposition of the coherent state over the orthonormal basis as [28]

$$|\zeta\rangle = (1 - |\zeta|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{(2K)_n}{n!}} \zeta^n |K, n\rangle. \quad (11)$$

The above equation is the coherent state of $SU(1,1)$ Lie algebra. We will use this relation in next sections in our interesting problem.

3. Charged particle in a uniform magnetic field

The Hamiltonian operator for charged particle in an external constant magnetic field is given by [25, 26]:

$$H = \frac{1}{2\mu} \left(\vec{P} + \frac{e}{c} \vec{A} \right)^2 = \frac{-\hbar^2}{2\mu} \nabla^2 - \frac{ie\hbar}{\mu c} \vec{A} \cdot \vec{\nabla} + \frac{e^2}{2\mu c^2} A^2. \quad (12)$$

We can consider the vector potential \vec{A} as $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$, also we assume $\vec{B} = B\hat{z}$, where B is a constant. Then the Schrödinger equation for this system is

$$H\Psi = \frac{-\hbar^2}{2\mu}\nabla^2\Psi + \frac{eB}{2\mu c}L_z\Psi + \frac{e^2B^2}{8\mu c^2}(x^2 + y^2)\Psi = E\Psi. \quad (13)$$

Now we solve the above equation in the cylindrical coordinates. We consider the wave function $\Psi(r) = U(\rho)e^{im\varphi}e^{ikz}$ ($m = 0, 1, 2, \dots$).

$$\frac{d^2U(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dU(\rho)}{d\rho} - \frac{m^2}{\rho^2}U(\rho) - \frac{e^2B^2}{4\hbar^2c^2}\rho^2U(\rho) + \left[\frac{2\mu E}{\hbar^2} - \frac{eBm}{\hbar c} - k^2\right]U(\rho) = 0, \quad (14)$$

Performing the change of variable, $x = \sqrt{\frac{eB}{2\hbar c}}\rho$, in the above equation, we obtain

$$\frac{d^2U(x)}{dx^2} + \frac{1}{x}\frac{dU(x)}{dx} - \frac{m^2}{x^2}U(x) + (\lambda - x^2)U(x) = 0, \quad (15)$$

where

$$\lambda = \frac{4\mu c}{eB\hbar}\left(E - \frac{\hbar^2k^2}{2\mu}\right) - 2m. \quad (16)$$

The asymptotic behavior of wave function is

$$U(x) = x^m e^{-x^2/2} F(x). \quad (17)$$

Now by solving the differential equation (15) and changing the variable to $y = x^2$, and considering that n is the principle quantum number, we have:

$$U_n(y) = N_n e^{-y/2} y^{m/2} L_n^{m-1/2}(y), \quad (18)$$

where N_n is the normalization factor:

$$N_n = \sqrt{\frac{2n!}{(n+m-\frac{1}{2})!}}. \quad (19)$$

We now address the problem of finding the creation and annihilation operators for the wave functions (18) with the factorization method [27]. The ladder operators can be constructed directly from the wave function without introducing any auxiliary variable [25]. So we can define the annihilation operator \hat{P}_- as:

$$\hat{P}_- = -y\frac{d}{dy} - \frac{y}{2} + n + \frac{m}{2}, \quad (20)$$

with the following eigenvalue,

$$p_- = \sqrt{n \left(n + m - \frac{1}{2} \right)}. \quad (21)$$

In a similar way, we get

$$\hat{P}_+ = y \frac{d}{dy} - \frac{y}{2} + n + \frac{m+1}{2}, \quad (22)$$

with the following eigenvalue,

$$p_+ = \sqrt{(n+1) \left(n + m + \frac{1}{2} \right)}. \quad (23)$$

The operators \hat{P}_- , \hat{P}_+ and \hat{P}_0 satisfy

$$[\hat{P}_0, \hat{P}_+] = \hat{P}_+, \quad [\hat{P}_0, \hat{P}_-] = -\hat{P}_-, \quad [\hat{P}_+, \hat{P}_-] = -2\hat{P}_0, \quad (24)$$

where

$$\hat{P}_0 = \hat{n} + \frac{m}{2} + \frac{1}{4}, \quad (25)$$

and

$$\hat{n}U_n(\rho) = nU_n(\rho). \quad (26)$$

Obviously Eq. (24) shows that these operators following commutation relations satisfy SU(1,1) Lie algebra. In other words our problem has the SU(1,1) dynamical group.

4. Generalized coherent state for a charged particle in the uniform magnetic field

The Klauder–Perelomov definition of coherent states consists of applying the operator $e^{\xi \hat{P}_+}$ to the ground state $|0\rangle$, such that

$$|\xi\rangle = e^{\xi \hat{P}_+ - \bar{\xi} \hat{P}_-} |0\rangle, \quad (27)$$

which leads to

$$|\zeta\rangle = (1 - |\zeta|^2)^{(m/2)+(1/4)} e^{\zeta \hat{P}_+} |0\rangle, \quad (28)$$

where $\zeta = \frac{\xi \tanh |\xi|}{|\xi|}$ is a complex number satisfying the condition $|\zeta| < 1$. The last equation can be reorganized as follows

$$|\zeta\rangle = (1 - |\zeta|^2)^{(m/2)+(1/4)} \sum_{n=0}^{\infty} \sqrt{\frac{(m + \frac{1}{2})_n}{n!}} \zeta^n |n\rangle. \quad (29)$$

By noting the generalized coherent state for SU(1,1) Lie algebra [1], and the calculations in the previous sections, we can obtain the generalized coherent state for the wave function, Eq. (18), as

$$U_\zeta(y) = \langle y | \zeta \rangle. \quad (30)$$

By substituting Eq. (28) in the above equation, we have

$$U_\zeta(y) = (1 - |\zeta|^2)^{(m/2)+(1/4)} \sum_{n=0}^{\infty} \sqrt{\frac{(m + \frac{1}{2})_n}{n!}} \sqrt{\frac{2n!}{(n + m - \frac{1}{2})!}} \\ \times e^{-y/2} y^{m/2} L_n^{m-1/2}(y) \zeta^n. \quad (31)$$

Now we define the generating function of the associated Laguerre polynomial, as

$$\sum_{n=0}^{\infty} L_n^t(x) z^n = \frac{e^{-xz/(1-z)}}{(1-z)^{t+1}}, \quad (32)$$

and substituting the above relation in Eq. (30), we obtain

$$U_\zeta(y) = \frac{(1 - |\zeta|^2)^{(m/2)+(1/4)}}{(1 - \zeta)^{(m+1/2)}} \sqrt{\frac{2}{\Gamma(m + \frac{1}{2})}} y^{m/2} e^{-y((1+\zeta)/2(1-\zeta))}. \quad (33)$$

$U_\zeta(y)$ is the generalized coherent state for the wave function of the charged particle in a constant magnetic field.

5. Charged particle in variable magnetic field

The Hamiltonian operator for a charged particle in the presence of an external variable magnetic field $B_x = 0, B_y = 0, B_z = -\frac{\beta}{x^2}$, is given by [25]

$$H = \frac{1}{2\mu} \left(\vec{P} + \frac{e}{c} \vec{A} \right)^2 = \frac{-\hbar^2}{2\mu} \nabla^2 + \frac{\hbar}{i} \frac{e}{\mu c} \beta \frac{1}{x} \frac{\partial}{\partial y} + \frac{e^2 \beta^2}{2\mu c^2} \frac{1}{x^2}, \quad (34)$$

where β is a constant. Then the Schrödinger equation becomes

$$\left(\frac{-\hbar^2}{2\mu} \nabla^2 + 2 \frac{\hbar}{i} \omega_\beta \frac{1}{x} \frac{\partial}{\partial y} + 2\mu \omega_\beta^2 \frac{1}{x^2} \right) \Phi(x, y, z) = E \Phi(x, y, z), \quad (35)$$

where

$$\omega_\beta = \frac{e\beta}{2\mu c}. \quad (36)$$

Consider the wave function

$$\Phi(x, y, z) = U(x)e^{i(k_y y + k_z z)}. \quad (37)$$

Substituting it in Eq. (34), one finds the wave function $U(x)$ satisfies the following equation

$$\frac{d^2 U(\rho)}{d\rho^2} + \left[\frac{\lambda}{\rho} - \frac{\alpha}{\rho^2} - \frac{1}{4} \right] U(\rho) = 0, \quad (38)$$

where

$$\lambda = -2k_y \sqrt{\frac{\alpha}{\tilde{E}}}, \quad \rho = \frac{-2\sqrt{\alpha}}{\lambda} k_y x, \quad (39)$$

and

$$\alpha = \frac{4\mu^2 \omega_\beta^2}{\hbar^2}, \quad \tilde{E} = -4 \left(\frac{2\mu E}{\hbar^2} - k_y^2 - k_z^2 \right). \quad (40)$$

From the behavior of the wave function at the origin and at infinity, we can consider the following ansatz for $U_n(\rho)$

$$U_n(\rho) = N_n \rho^s e^{-\frac{\rho}{2}} {}_1F_1(S + \lambda, 2s, \rho), \quad (41)$$

with

$$s = \frac{1}{2} + \sqrt{\frac{1}{4} + \alpha}, \quad (42)$$

where ${}_1F_1(S + \lambda, 2s, \rho)$, is the confluent hypergeometric function. Now, by using the relation between hypergeometric functions and Laguerre polynomials, we obtain $U_n(\rho)$ as

$$U_n(\rho) = N_n \rho^s e^{-\rho/2} L_n^{2s-1}(\rho) \quad (43)$$

where N_n is a normalized factor

$$N_n = \sqrt{\frac{n!}{(n+2s-1)!(2n+2s)}}. \quad (44)$$

By using the factorization method [27], the ladder operators can be constructed directly from the wave function without introducing any auxiliary variable [25], thus it is found that

$$\hat{L}_+ = \rho \frac{d}{d\rho} - \frac{\rho}{2} + n + s, \quad (45)$$

with the following eigenvalue

$$l_+ = \sqrt{\frac{(n+1)(n+2s)(n+s+1)}{(n+s)}}. \quad (46)$$

In a similar way, we get

$$\hat{L}_- = -\rho \frac{d}{d\rho} - \frac{\rho}{2} + n + s, \quad (47)$$

with the following eigenvalue

$$l_- = \sqrt{\frac{n(n+2s-1)(n+s-1)}{(n+s)}}. \quad (48)$$

We modify \tilde{L}_\pm as follows

$$\tilde{L}_+ = \sqrt{\frac{(n+s)}{(n+s+1)}} L_+, \quad \tilde{L}_- = \sqrt{\frac{(n+s)}{(n+s-1)}} L_-. \quad (49)$$

Now, the actions of \tilde{L}_+ and \tilde{L}_- on the eigenfunctions, respectively, are

$$\tilde{L}_+ U_n(\rho) = \sqrt{(n+1)(n+2s)} U_{n+1}(\rho), \quad (50)$$

and

$$\tilde{L}_- U_n(\rho) = \sqrt{n(2s+n-1)} U_{n-1}(\rho). \quad (51)$$

Then, we define the operators \hat{n} , as follows

$$\hat{n} U_n(\rho) = n U_n(\rho). \quad (52)$$

Using the above equations, we calculate the commutation relation $[\tilde{L}_-, \tilde{L}_+]$

$$[\tilde{L}_-, \tilde{L}_+] U_n(\rho) = 2\tilde{L}_0 U_n(\rho), \quad (53)$$

where

$$\tilde{L}_0 = \hat{n} + s, \quad (54)$$

so, we can conclude easily that operators \tilde{L}_- , \tilde{L}_+ and \tilde{L}_0 , that satisfy the following commutation relations as,

$$[\tilde{L}_0, \tilde{L}_+] = \tilde{L}_+, \quad [\tilde{L}_0, \tilde{L}_-] = -\tilde{L}_-, \quad [\tilde{L}_+, \tilde{L}_-] = -2\tilde{L}_0. \quad (55)$$

Obviously Eq. (54) shows that these operators following commutation relations satisfy SU(1,1) Lie algebra. In other words, we understand that a charged particle in variable magnetic field has the dynamical group SU(1,1).

6. Generalized coherent state for charged particle in variable magnetic field

In this section, we use the Klauder–Perelomov coherent states that consist of applying the operator $e^{\zeta \tilde{L}_+}$ to the ground state $|0\rangle$, such that

$$|\zeta\rangle = (1 - |\zeta|^2)^s e^{\zeta \tilde{L}_+} |0\rangle. \quad (56)$$

Expanding the exponential, we obtain

$$|\zeta\rangle = (1 - |\zeta|^2)^s \sum_{n=0}^{\infty} \sqrt{\frac{(2s)_n}{n!}} \zeta^n |n\rangle. \quad (57)$$

The overlapping property is

$$\langle \zeta_1 | \zeta_2 \rangle = \left[\frac{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}{(1 - \bar{\zeta}_1 \zeta_2)} \right]^s. \quad (58)$$

As mentioned above the charged particle in variable magnetic field has a dynamical group $SU(1,1)$. Now we will obtain the generalized coherent state for the wave function, as follows

$$U_\zeta(\rho) = \langle \rho | \zeta \rangle. \quad (59)$$

This formula is

$$U_\zeta(\rho) = (1 - |\zeta|^2)^s \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+2s)}{\Gamma(2s)(n+2s-1)!(2n+2s)}} \rho^s e^{-\rho/2} \zeta^n L_n^{2s-1}(\rho). \quad (60)$$

In conclusion, $U_\zeta(\rho)$ is a generalized coherent state for the wave function of charged particle in variable magnetic field.

7. Conclusion

The coherent states are mathematical tools which provide a close connection between classical and quantum formalisms. The study of coherent states is not confined to the harmonic oscillator only, but it has been generalized to various systems — for example, systems with Lie algebra $SU(1,1)$ or $SU(2)$ symmetry [1, 28, 29] or even systems with nonlinear algebraic symmetry [30–32]. In this article, using a definition of the generalized coherent states for $SU(1,1)$ Lie algebra, we presented the n th state of the wave function for a charged particle in a constant magnetic field, after that we

calculated the annihilation and creation operators by using the factorization method [27] and we showed that the dominated relation between these operators is $SU(1,1)$ Lie algebra. Then in Section 4 we obtained the generalized coherent states for this problem. In Section 5 we extend the problem to the varying magnetic field, then we obtained the corresponding generalized coherent states in Section 6.

The physical system of a charged particle under the influence of a constant magnetic field, which was considered early after the foundation of quantum mechanics, has recent ramifications in condensed matter physics. One would expect that the study of the system under a space dependent magnetic field may result in more pronounced and interesting behaviour. In this paper we have considered a simple situation where only the y -component of the vector potential exists in the form B_z proportional to $\frac{1}{x^2}$. Parallel to the [10, 24], one can use our coherent state, to obtain symbols of various involved observables. We will come back to this problem in a next work. Finally we must mention that the condition $|\zeta| < 1$ shows that the $SU(1,1)$ coherent state are defined in the interior of the unit disk. As it has been done in [11], one can show that our analytic representations of coherent state in the unit disk are related through a Laplace transform to the Barut–Girardello representation.

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