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OSCILLATOR MODEL ON LIE-ALGEBRAICALLY DEFORMED NONRELATIVISTIC SPACE-TIME

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The classical and quantum oscillator model on Lie-algebraically deformed nonrelativistic space-time is introduced and analyzed. The corresponding equations of motions are studied using mostly numerical methods. The time-dependent energy spectrum is presented as well.

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1. Introduction

Due to several theoretical arguments (see *e.g.* [1–4]) the interest in studying of space-time noncommutativity is growing rapidly. There appeared a lot of papers dealing with noncommutative classical [5–12] and quantum [13–21] mechanics, as well as with field theoretical models (see *e.g.* [22–31]), defined on quantum space-time.

At present, in accordance with Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries (see [32, 33]) one can distinguish two quite interesting kinds of quantum spaces. First of them corresponds to the well-known canonical type of noncommutativity

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}\,, \qquad \theta_{\mu\nu} = \text{const.}\,, \tag{1}$$

with antisymmetric constant tensor $\theta^{\mu\nu}$. Its relativistic and nonrelativistic Hopf-algebraic counterparts have been proposed in [34–36] and [37], respectively[1].

[1] See also [38].

The second class of mentioned deformations introduces the Lie-algebraic type of space-time noncommutativity

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta^\rho_{\mu\nu}\hat{x}_\rho\,, \tag{2}$$

with particularly chosen constant coefficients $\theta^\rho_{\mu\nu}$. The corresponding Poincaré quantum groups have been introduced in [39,40], while the suitable Galilei algebras — in [41] and [37].

Recently, there was proposed a particular type of Lie-algebraic deformation of nonrelativistic space-time with spatial directions commuting to time[2]

$$[t, \hat{x}_i] = 0 = [\hat{x}_k, \hat{x}_{\rho(\tau)}]\,, \qquad [\hat{x}_\tau, \hat{x}_\rho] = \frac{i}{\kappa}t\,, \qquad \rho, \tau, k - \text{fixed and different}\,. \tag{3}$$

The above noncommutativity has been obtained in the framework of quantum groups in [37], while its basic properties have been investigated in a context of nonrelativistic particle subjected to the external constant force [12]. In particular, there was demonstrated that such a kind of quantum space-time produces additional acceleration of moving particle coupled to the force terms generated by so-called space-like deformation [37].

In this article we extend our investigations of deformation (3) to more complicated nonrelativistic system — the classical and quantum oscillator model. It should be noted, however, that analogous studies have been already performed in a context of canonical deformation (1), *i.e.* the solution of corresponding equation of motion has been provided and analyzed in [9], while the (deformed) energy spectrum has been discussed in [18–20].

In this paper we adopt the general treatment proposed in [19] and show, that the space-time noncommutativity (3) generates a proper (explicit) time-dependence of oscillator Hamiltonian function. In such a way we discover a connection between noncommutative geometry (a quantum group) and nonrelativistic models with time-dependent mass and frequency (see *e.g.* [42–46]). Besides, we also confirm that such a deformation introduces the additional term proportional to angular momentum of considered system [18, 19]. Finally, the numerical studies of the solutions of corresponding equations of motion are presented in detail and the growing in time energy spectrum is analyzed as well[3].

In this article apart of deformation (3) we also introduced its natural generalization

$$[t, \hat{x}_i] = 0 = [\hat{x}_k, \hat{x}_{\rho(\tau)}]\,, \qquad [\hat{x}_\tau, \hat{x}_\rho] = if_\kappa(t)\,, \tag{4}$$

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[2] It will be call t-deformation of classical space.

[3] It should be noted that for fixed time parameter t the obtained energy spectrum becomes the same as one recovered in [19].

with arbitrary time-dependent function $f_\kappa(t)$ approaching zero for parameter κ running to infinity. We demonstrate that for such generalized space-time the energy spectrum of the model depends on function $f_\kappa(t)$ and becomes finite for large times. It should be noted however, that noncommutativity (4) cannot be derived in the framework of quantum groups, and for this reason, from formal point of view its geometric status remains unknown.

The paper is organized as follows. In second section we recall the Galilei Hopf structure providing quantum space (3). Further, in Section 3, we investigate the classical oscillator model on such a space-time, *i.e.* the corresponding equation of motion is provided and its solution is analyzed numerically as well. In Section 4 the energy spectrum of a proper quantum model is discovered. Section 5 deals with the classical and quantum oscillator system defined on generalized space-time (4). The results are summarized and discussed in the last section.

2. Twisted Galilei Hopf algebra and corresponding Lie-algebraically deformed space-time

In this section (following the paper [37]) we recall the Lie-algebraically deformed Galilei Hopf algebra $\mathcal{U}_\kappa(\mathcal{G})$ (κ denotes deformation parameter) providing nonrelativistic space-time (3). Its algebraic sector remains classical[4]

$$\begin{aligned}
[K_{ij}, K_{kl}] &= i\left(\delta_{il}\, K_{jk} - \delta_{jl}\, K_{ik} + \delta_{jk} K_{il} - \delta_{ik} K_{jl}\right) , \\
[K_{ij}, V_k] &= i\left(\delta_{jk}\, V_i - \delta_{ik}\, V_j\right) , \\
[K_{ij}, \Pi_k] &= i\left(\delta_{jk}\, \Pi_i - \delta_{ik}\, \Pi_j\right) , \\
[K_{ij}, \Pi_0] &= [V_i, V_j] = [\, V_i, \Pi_j] = 0 \, , \\
[V_i, \Pi_0] &= -i\Pi_i \, , \quad [\Pi_\mu, \Pi_\nu] = 0 \, ,
\end{aligned} \tag{5}$$

while the co-algebraic part takes the form (ρ, τ — fixed and different)

$$\Delta_\kappa(\Pi_0) = \Delta_0(\Pi_0) + \frac{1}{2\kappa}\Pi_\tau \wedge \Pi_\rho \, , \tag{6}$$

$$\Delta_\kappa(\Pi_i) = \Delta_0(\Pi_i) \, , \qquad \Delta_\kappa(V_i) = \Delta_0(V_i) \, , \tag{7}$$

$$\Delta_\kappa(K_{ij}) = \Delta_0(K_{ij}) + \frac{i}{2\kappa}\,[K_{ij}, V_\rho] \wedge \Pi_\tau + \frac{1}{2\kappa} V_\rho \wedge (\delta_{i\tau}\Pi_j - \delta_{j\tau}\Pi_i) \, , \tag{8}$$

the antipodes and co-units remain undeformed ($S_0(a) = -a, \epsilon(a) = 1$).

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[4] The symbols K_{ij}, V_i and Π_μ denote rotations, boosts and space-time translation generators, respectively.

The deformed co-products (6)–(8) are obtained by the twist procedure [47], *i.e.*

$$\begin{aligned}\Delta_\kappa(a) &= \mathcal{F}_\kappa \circ \Delta_0(a) \circ \mathcal{F}_\kappa^{-1}\,, \\ \Delta_0(a) &= a\otimes 1 + 1\otimes a\,, \end{aligned} \tag{9}$$

where the twist factor $\mathcal{F}_\kappa \in \mathcal{U}_\kappa(\mathcal{G}) \otimes \mathcal{U}_\kappa(\mathcal{G})$ satisfies the classical cocycle condition

$$\mathcal{F}_{\kappa 12} \cdot (\Delta_0 \otimes 1)\mathcal{F}_\kappa = \mathcal{F}_{\kappa 23} \cdot (1 \otimes \Delta_0)\mathcal{F}_\kappa \tag{10}$$

and the normalization condition

$$(\epsilon \otimes 1)\mathcal{F}_\kappa = (1\otimes \epsilon)\mathcal{F}_\kappa = 1\,, \qquad \mathcal{F}_{\kappa 12} = \mathcal{F}_\kappa \otimes 1\,, \qquad \mathcal{F}_{\kappa 23} = 1 \otimes \mathcal{F}_\kappa\,. \tag{11}$$

It looks as follows

$$\mathcal{F}_\kappa = \exp\left(\frac{i}{2\kappa} \mathit{\Pi}_\tau \wedge V_\rho\right)\,. \tag{12}$$

Obviously for deformation parameter κ approaching infinity the above Hopf structure becomes classical.

Let us now turn to the deformed space-time corresponding to the Hopf algebra $\mathcal{U}_\kappa(\mathcal{G})$. It is defined as the quantum representation spaces (Hopf module) for quantum Galilei algebra, with action of the deformed symmetry generators satisfying suitably deformed Leibnitz rules (see *e.g.* [48]). The action of Galilei group $\mathcal{U}_\kappa(\mathcal{G})$ on a Hopf module of functions depending on space-time coordinates (t, x_i) is given by

$$\begin{aligned}\mathit{\Pi}_0 \rhd f(t,\overline{x}) &= i\partial_t f(t,\overline{x})\,, \\ \mathit{\Pi}_i \rhd f(t,\overline{x}) &= i\partial_i f(t,\overline{x})\,, \end{aligned} \tag{13}$$

$$\begin{aligned}K_{ij} \rhd f(t,\overline{x}) &= i\left(x_i\partial_j - x_j\partial_i\right) f(t,\overline{x})\,, \\ V_i \rhd f(t,\overline{x}) &= it\partial_i\, f(t,\overline{x})\,. \end{aligned} \tag{14}$$

Moreover, the $\star$-multiplication of arbitrary two functions is defined as follows

$$f(t,\overline{x}) \star_\kappa g(t,\overline{x}) := \omega \circ \left(\mathcal{F}_\kappa^{-1} \rhd f(t,\overline{x}) \otimes g(t,\overline{x})\right)\,. \tag{15}$$

In the above formula $\mathcal{F}.$ denotes twist factor (12) and $\omega \circ (a\otimes b) = a\cdot b$.

Consequently, we get the following factor

$$\mathcal{F}_\kappa = \exp\left(-\frac{i}{2\kappa}\,\partial_\tau \wedge t\partial_\rho\right)\,, \tag{16}$$

and the corresponding nonrelativistic space-time (see (3))

$$[t, x_i]_{\star_\kappa} = [x_k, x_\rho]_{\star_\kappa} = [x_k, x_\tau]_{\star_\kappa} = 0\,, \qquad [x_\tau, x_\rho]_{\star_\kappa} = \frac{i}{\kappa} t\,, \tag{17}$$

with indexes ρ, τ and k different and fixed, and $[a,b]_{\star_\kappa} := a\star_\kappa b - b\star_\kappa a$. Finally, it should be mentioned that for the deformation parameter κ running to infinity the above quantum space becomes commutative.

3. Classical oscillator model

Let us start with the following Lie-algebraically deformed phase space corresponding to the quantum space-time (17) (see [12])[5]

$$\{t, \bar{x}_i\} = 0 = \{\bar{x}_k, \bar{x}_{\rho(\tau)}\}\,, \qquad \{\bar{x}_\tau, \bar{x}_\rho\} = \frac{1}{\kappa} t\,, \tag{18}$$

$$\{\bar{x}_i, \bar{p}_j\} = \delta_{ij}\,, \qquad \{\bar{p}_i, \bar{p}_j\} = 0\,, \tag{19}$$

where indices k, ρ and τ are different and fixed, $i, j = 1, 2, 3$. One can check that the relations (18), (19) satisfy the Jacobi identity and for deformation parameter κ running to infinity become classical.

We define the Hamiltonian function for isotropic harmonic oscillator with constant mass m and frequency ω as follows

$$\bar{H}(\bar{p}, \bar{x}) = \frac{1}{2m}\left(\bar{p}_\rho^2 + \bar{p}_\tau^2 + \bar{p}_k^2\right) + \frac{m\omega^2}{2}\left(\bar{x}_\rho^2 + \bar{x}_\tau^2 + \bar{x}_k^2\right)\,. \tag{20}$$

Next, in order to analyze the above system we represent the noncommutative variables $(\bar{x}_i, \bar{p}_i)$ on classical phase space (x_i, p_i) as (see *e.g.* [18, 19, 21])

$$\bar{x}_\rho = x_\rho + \frac{t}{2\kappa} p_\tau\,, \qquad \bar{x}_\tau = x_\tau - \frac{t}{2\kappa} p_\rho\,, \qquad \bar{x}_k = x_k\,, \qquad \bar{p}_i = p_i\,, \tag{21}$$

where

$$\{x_i, x_j\} = 0 = \{p_i, p_j\}\,, \qquad \{x_i, p_j\} = \delta_{ij}\,. \tag{22}$$

Then, the Hamiltonian (20) takes the form

$$\begin{aligned} H(p,x) = H(t) \;&=\; \frac{\left(p_\rho^2 + p_\tau^2\right)}{2M(t)} + \frac{M(t)\Omega^2(t)}{2}\left(x_\rho^2 + x_\tau^2\right) \\ &\quad + \frac{tm\omega^2 L_k}{2\kappa} + \frac{p_k^2}{2m} + \frac{m\omega^2 x_k^2}{2} \end{aligned} \tag{23}$$

[5] We use the correspondence relation $\{a, b\} = \frac{1}{i}[\hat{a}, \hat{b}]$ ($\hbar = 1$).

with symbol $L_k = x_\rho p_\tau - x_\tau p_\rho$ denoting angular momentum of particle in direction k. Besides, the coefficients $M(t)$ and $\Omega(t)$ present in the above formula denote the time-dependent functions given by[6, 7]

$$M(t) = \frac{m}{1+\frac{m^2\omega^2t^2}{4\kappa^2}}\,, \qquad \Omega(t) = \omega\sqrt{1+\frac{m^2\omega^2t^2}{4\kappa^2}}\,, \tag{24}$$

respectively. One can check that

$$M(t)\Omega^2(t) = m\omega^2 = \mathrm{const}\,. \tag{25}$$

As it was already mentioned in Introduction, the considered system (23) with neglected last three terms as well as with arbitrary "mass" and "frequency" functions $M(t)$, $\Omega(t)$ has been studied at classical and quantum levels in [42–46]. Obviously, in our case, due to the condition (25) the coefficient of a second term in Hamiltonian function (23) remains constant, *i.e.*

$$\frac{M(t)\Omega^2(t)}{2}\left(x_\rho^2+x_\tau^2\right) = \frac{m\omega^2}{2}\left(x_\rho^2+x_\tau^2\right)\,. \tag{26}$$

Using the formulas (22), (23) one gets the following canonical Hamiltonian equations of motions

$$\dot{x}_\rho = \frac{p_\rho}{M(t)} - \frac{tm\omega^2}{2\kappa}x_\tau\,, \qquad \dot{p}_\rho = -m\omega^2x_\rho - \frac{tm\omega^2}{2\kappa}p_\tau\,, \tag{27}$$

$$\dot{x}_\tau = \frac{p_\tau}{M(t)} + \frac{tm\omega^2}{2\kappa}x_\rho\,, \qquad \dot{p}_\tau = -m\omega^2x_\tau + \frac{tm\omega^2}{2\kappa}p_\rho\,, \tag{28}$$

$$\dot{x}_k = \frac{p_k}{m}\,, \qquad \dot{p}_k = -m\omega^2x_k\,, \tag{29}$$

which when combined yield the equations

$$\begin{cases} \ddot{x}_\rho = \frac{m\omega^2}{2\kappa}t\left(-2\dot{x}_\tau+\frac{M(t)}{\kappa}\dot{x}_\rho\right)+\frac{m\omega^2}{2\kappa}\left(\frac{m\omega^2}{2\kappa^2}M(t)t^2-1\right)x_\tau-\omega^2x_\rho\,, \\ \ddot{x}_\tau = \frac{m\omega^2}{2\kappa}t\left(2\dot{x}_\rho+\frac{M(t)}{\kappa}\dot{x}_\tau\right)+\frac{m\omega^2}{2\kappa}\left(1-\frac{m\omega^2}{2\kappa^2}M(t)t^2\right)x_\rho-\omega^2x_\tau\,, \\ \ddot{x}_k = -\omega^2x_k\,. \end{cases} \tag{30}$$

The solution of Eq. (30) has been studied numerically and the corresponding trajectories are illustrated on Fig. 1. If the parameter κ runs to infinity the equation (30) becomes undeformed and describes the periodic motion of classical harmonic oscillator [49].

[6] $\lim_{t\to\infty} M(t) = 0$, $\lim_{t\to\infty} \Omega(t) = \infty$.

[7] See Fig. 2(a).

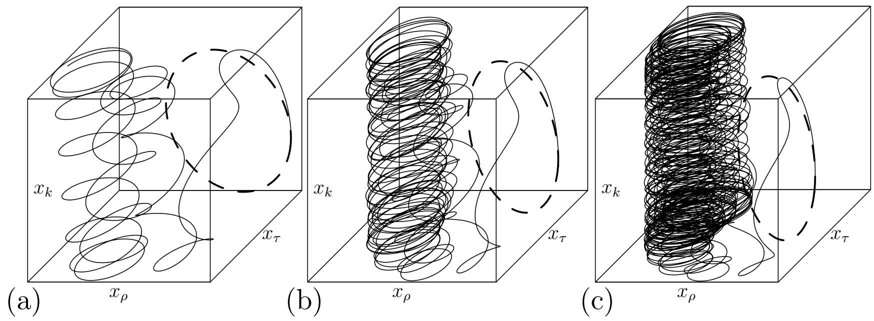

Fig. 1. The particle trajectory for parameters $m = \omega = \kappa = 1$. The dashed line corresponds to the case of classical (undeformed) oscillator and the time parameter runs from 0 to 15, 30 and 50 for figures (a), (b) and (c), respectively.

4. Quantum oscillator model

The main aim of this section is to study the spectrum of the following quantum-mechanical counterpart of Hamiltonian (23)

$$\hat{H}(t) = \frac{\left(\hat{p}_\rho^2 + \hat{p}_\tau^2\right)}{2M(t)} + \frac{m\omega^2}{2}\left(\hat{x}_\rho^2 + \hat{x}_\tau^2\right) + \frac{tm\omega^2 \hat{L}_k}{2\kappa} + \frac{\hat{p}_k^2}{2m} + \frac{m\omega^2 \hat{x}_k^2}{2}\,, \quad (31)$$

with $\hat{x}_i$ and $\hat{p}_i$ denoting the classical position and momentum operators such that

$$[\hat{x}_i, \hat{x}_j] = 0 = [\hat{p}_i, \hat{p}_j]\,, \qquad [\hat{x}_i, \hat{p}_j] = i\delta_{ij}\,. \quad (32)$$

In accordance with the scheme proposed in [19] (see also [18]) we introduce a set of time-dependent creation ($a_A^\dagger(t)$) and annihilation ($a_A(t)$) operators

$$\hat{a}_\pm(t) \;=\; \frac{1}{2}\left[\frac{(\hat{p}_\rho \pm i\hat{p}_\tau)}{\sqrt{M(t)\varOmega(t)}} - i\sqrt{M(t)\varOmega(t)}(\hat{x}_\rho \pm i\hat{x}_\tau)\right]\,, \quad (33)$$

$$\hat{a}_k \;=\; \frac{1}{\sqrt{2}}\left(\frac{\hat{p}_k}{\sqrt{m\omega}} - i\sqrt{m\omega}\hat{x}_k\right)\,, \quad (34)$$

satisfying the standard commutation relations

$$[\hat{a}_A, \hat{a}_B] = 0\,, \qquad [\hat{a}_A^\dagger, \hat{a}_B^\dagger] = 0\,, \qquad [\hat{a}_A, \hat{a}_B^\dagger] = \delta_{AB}\,, \qquad A, B = \pm, k\,. \quad (35)$$

It should be noted, that for the parameter κ running to infinity the creation/annihilation operators (33) take the classical form, *i.e.* they become time-independent. Further, one can easily check that in terms of the operators (33) and (34) the Hamiltonian function (31) looks as follows

$$\hat{H}(t) = \varOmega_+(t)\left(\hat{N}_+(t) + \frac{1}{2}\right) + \varOmega_-(t)\left(\hat{N}_-(t) + \frac{1}{2}\right) + \omega\left(\hat{N}_k + \frac{1}{2}\right)\,, \quad (36)$$

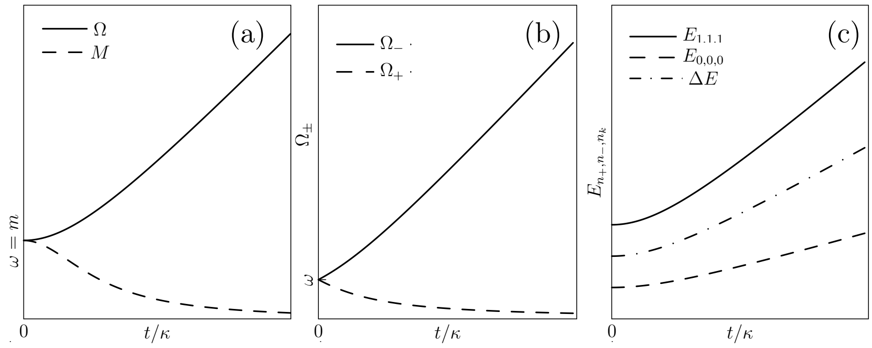

Fig. 2. The time evolution of functions $\Omega(t)$, $M(t)$, $\Omega_\pm(t)$, $E_{0,0,0}(t)$, $E_{1,1,1}(t)$ and difference $\Delta E(t) = E_{1,1,1}(t) - E_{0,0,0}(t)$ with fixed parameter κ and $m = \omega = 1$.

with coefficient $\Omega_\pm(t)$ given by[8]

$$\Omega_\pm(t) = \Omega(t) \mp \frac{tm\omega^2}{2\kappa} \,. \tag{37}$$

Besides, the following objects

$$\hat{N}_\pm(t) = \hat{a}_\pm^\dagger(t)\hat{a}_\pm(t) \,, \qquad \hat{N}_k = \hat{a}_k^\dagger \hat{a}_k \,, \tag{38}$$

play a role of particle number operators in $\pm$ and k direction, respectively.

The eigenvectors of Hamiltonian (36) are generated by creation operators $a_\pm^\dagger(t)$, $a_k^\dagger$ acting on vacuum state $|0\rangle$

$$|n_+, n_-, n_k\rangle_t = \frac{1}{\sqrt{n_+!}}\frac{1}{\sqrt{n_-!}}\frac{1}{\sqrt{n_k!}} \left(\hat{a}_+^\dagger(t)\right)^{n_+} \left(\hat{a}_-^\dagger(t)\right)^{n_-} \left(\hat{a}_k^\dagger\right)^{n_k} |0\rangle \,. \tag{39}$$

Then, the corresponding eigenvalues take the form

$$E_{n_+,n_-,n_k}(t) = \Omega_+(t)\left(n_+ + \frac{1}{2}\right) + \Omega_-(t)\left(n_- + \frac{1}{2}\right) + \omega\left(n_k + \frac{1}{2}\right) \,. \tag{40}$$

One can also see that the difference between two neighboring levels of spectrum (40) is given by the formula

$$\Delta E(t) := E_{n_++1,n_-+1,n_k+1}(t) - E_{n_+,n_-,n_k}(t) = 2\Omega(t) + \omega \,, \tag{41}$$

with function $\Delta E(t)$ growing in time. Obviously, for deformation parameter κ running to infinity the spectrum (40) as well as its difference (41) become

[8] See Fig. 2(b).

classical. As an illustration of relation (40) the time evolution of $E_{0,0,0}(t)$ and $E_{1,1,1}(t)$ eigenvalues has been presented in Fig. 2(c).

Finally, it should be noted that for fixed parameter t the above system can be identified with canonically deformed oscillator model proposed in [19], *i.e.* the spectrum (40) for time parameter $t = \kappa\theta_{\rho\tau}$ (see (1)) becomes the same as one derived in [19].

5. Beyond the quantum group

Let us consider the following generalized phase space

$$\{t, \bar{x}_i\} = 0 = \{\bar{x}_k, \bar{x}_{\rho(\tau)}\}\,, \quad \{\bar{x}_\tau, \bar{x}_\rho\} = f_\kappa(t)\,, \tag{42}$$

$$\{\bar{x}_i, \bar{p}_j\} = \delta_{ij}\,, \qquad \{\bar{p}_i, \bar{p}_j\} = 0\,, \tag{43}$$

where $f_\kappa(t)$ denotes arbitrary time-dependent function approaching to zero for parameter κ running to infinity. Obviously, the above relations satisfy the Jacobi identity and for parameter κ approaching to infinity become classical. Besides, as it was already mentioned in Introduction, the generalized noncommutativity (42) cannot be realized as a translation sector in the Hopf-algebraic framework of relativistic and nonrelativistic symmetries. Nevertheless, due to the link (for particular choices of function $f_\kappa(t)$) with oscillator models [45,46], the study of such a system appears quite interesting and shall be discussed in present section.

The relations (42) and (43) can be represented in terms of classical phase space variables as follows

$$\bar{x}_\rho = x_\rho + \frac{f_\kappa(t)}{2}p_\tau\,, \qquad \bar{x}_\tau = x_\tau - \frac{f_\kappa(t)}{2}p_\rho\,, \qquad \bar{x}_k = x_k\,, \qquad \bar{p}_i = p_i\,. \tag{44}$$

Then, the corresponding Hamiltonian function takes the form

$$H_f(t) = \frac{\left(p_\rho^2 + p_\tau^2\right)}{2M_f(t)} + \frac{m\omega^2}{2}\left(x_\rho^2 + x_\tau^2\right) + \frac{f_\kappa(t)m\omega^2 L_k}{2} + \frac{p_k^2}{2m} + \frac{m\omega^2 x_k^2}{2}\,, \tag{45}$$

with generalized coefficient $M_f(t)$

$$M_f(t) = \frac{m}{1 + \frac{m^2\omega^2}{4}f_\kappa^2(t)}\,. \tag{46}$$

By direct calculation one can also find the corresponding equation of motion

$$\left\{\begin{array}{rcl} \ddot{x}_\rho &=& \frac{m\omega^2 f_\kappa(t)}{2}\left(\dot{f}_\kappa(t)M_f(t)\dot{x}_\rho - 2\dot{x}_\tau\right) \\ && + \frac{m\omega^2 \dot{f}_\kappa(t)}{2}\left(\frac{m\omega^2 M_f(t)}{2}f_\kappa^2(t) - 1\right)x_\tau - \omega^2 x_\rho\,, \\ \ddot{x}_\tau &=& \frac{m\omega^2 f_\kappa(t)}{2}\left(\dot{f}_\kappa(t)M_f(t)\dot{x}_\tau + 2\dot{x}_\rho\right) \\ && + \frac{m\omega^2 \dot{f}_\kappa(t)}{2}\left(1 - \frac{m\omega^2 M_f(t)}{2}f_\kappa^2(t)\right)x_\rho - \omega^2 x_\tau\,, \\ \ddot{x}_k &=& -\omega^2 x_k\,, \end{array}\right. \tag{47}$$

which in the case $f_\kappa(t) = t/\kappa$ leads to the system (18), (19) related with quantum group [37]. Besides, for the choice $f_\kappa(t) = \theta^{\rho\tau}$ we get the equations of motion for canonical deformation of Galilei algebra proposed in [9].

The above equations have been investigated numerically for two particular choices of function $f_\kappa(t)$ with fixed parameter κ

$$f_\kappa(t) \;=\; \sin\left(\frac{t}{\kappa}\right) \tag{48a}$$

and

$$f_\kappa(t) \;=\; \left(e^{-\frac{t}{\kappa}} - 1\right)\,. \tag{48b}$$

First of them corresponds to the periodic time evolution of generalized coefficient (46) and can be compared with the oscillator system studied in [45,46][9]. The second choice introduces, as we shall see below, the finite for large times energy spectrum of a proper quantum oscillator model. Of course, for $\kappa \to \infty$ the both functions (48a)) and (48b)) approach to zero, and the phase space (42), (43) becomes classical. We add that the corresponding trajectories are illustrated in Figs. 3 and 4.

Let us now turn to the energy spectrum of quantum oscillator model defined on generalized phase space (42), (43). If we introduce the set of the following generalized creation/annihilation operator

$$\hat{a}_{f\pm}(t) \;=\; \frac{1}{2}\left[\frac{(\hat{p}_\rho \pm i\hat{p}_\tau)}{\sqrt{M_f(t)\Omega_f(t)}} - i\sqrt{M_f(t)\Omega_f(t)}(\hat{x}_\rho \pm i\hat{x}_\tau)\right]\,, \tag{49}$$

[9] More preciously, there was considered in [45, 46] the oscillator model described by the Hamiltonian function (45) with neglected L_k-, x_k- and p_k-terms as well as with time-periodic coefficient $M_f(t)$ ($m = M_f$) and constant frequency ω.

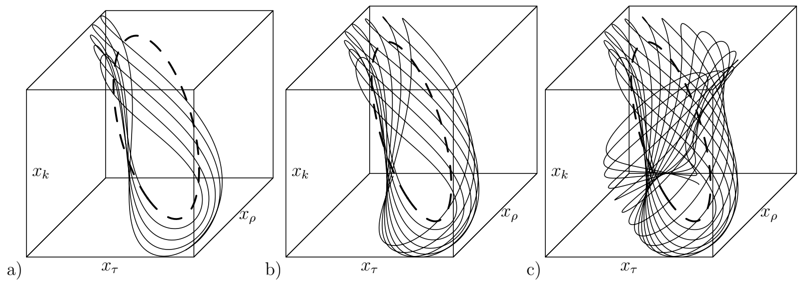

Fig. 3. The particle trajectory for function $f_\kappa(t) = \sin\left(\frac{t}{\kappa}\right)$ and parameters $m = \omega = \kappa = 1$. The dashed line corresponds to the case of classical (undeformed) oscillator and the time parameter runs from 0 to 30, 50 and 100 for figures (a), (b) and (c), respectively.

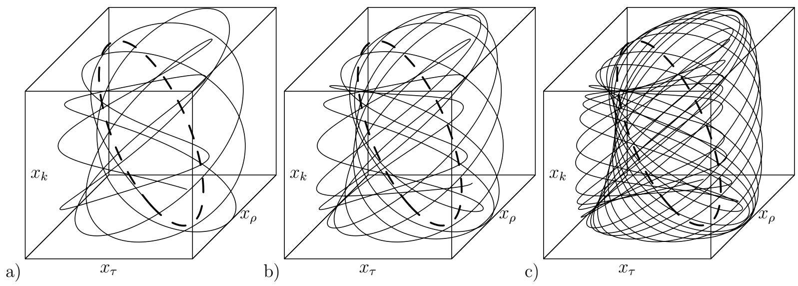

Fig. 4. The particle trajectory for function $f_\kappa(t) = \left(e^{-\frac{t}{\kappa}} - 1\right)$ and parameters $m = \omega = \kappa = 1$. The dashed line corresponds to the case of classical (undeformed) oscillator and the time parameter runs from 0 to 30, 50 and 100 for figures (a), (b) and (c), respectively.

$$\hat{a}_k \;=\; \frac{1}{\sqrt{2}}\left(\frac{\hat{p}_k}{\sqrt{m\omega}} - i\sqrt{m\omega}\hat{x}_k\right)\,, \tag{50}$$

then, in analogy to Section 2, one gets:

$$E_{f\,n_+,n_-,n_k}(t) = \Omega_{f+}(t)\left(n_+ + \frac{1}{2}\right) + \Omega_{f-}(t)\left(n_- + \frac{1}{2}\right) + \omega\left(n_k + \frac{1}{2}\right), \tag{51}$$

with time-dependent coefficients

$$\Omega_{f\pm}(t) = \Omega_f(t) \mp \frac{f_\kappa(t)m\omega^2}{2} \quad \text{and} \quad \Omega_f(t) = \omega\sqrt{1 + \frac{m^2\omega^2}{4}f_\kappa^2(t)}\,. \tag{52}$$

Additionally, one can observe that

$$\Delta E_f(t) := E_{f\,n_++1,n_-+1,n_k+1}(t) - E_{f\,n_+,n_-,n_k}(t) = 2\Omega_f(t) + \omega\,. \tag{53}$$

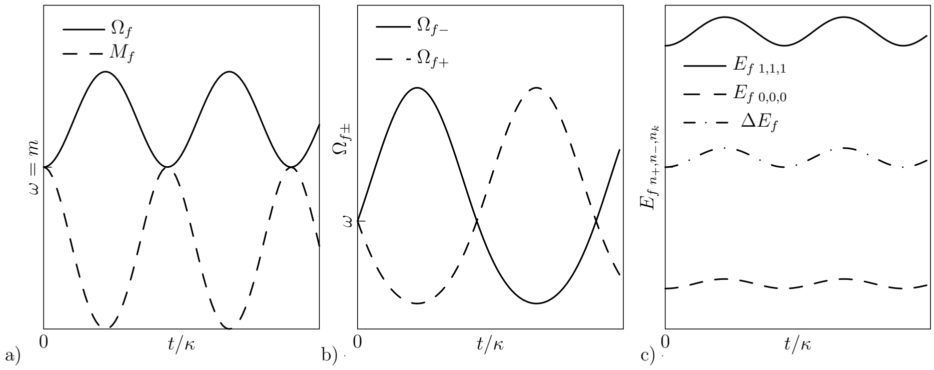

Fig. 5. The time evolution of functions $\Omega_f(t)$, $M_f(t)$, $\Omega_{f\pm}(t)$, $E_{f\,0,0,0}(t)$, $E_{f\,1,1,1}(t)$ and difference $\Delta E_f(t) = E_{f\,1,1,1}(t) - E_{f\,0,0,0}(t)$ for function $f_\kappa(t) = \sin(t/\kappa)$ with fixed parameter κ and $m = \omega = 1$.

Finally, let us note that in the case of function (48a) with fixed parameter κ, the formula (52) takes the form:

$$\Omega_f(t) = \omega\sqrt{1 + \frac{m^2\omega^2}{4}\sin^2\left(\frac{t}{\kappa}\right)}\,, \tag{54}$$

and difference (53) becomes periodic in time (see Fig. 5). For the second, "exponential" choice of $f_\kappa(t)$, the function

$$\Omega_f(t) = \omega\sqrt{1 + \frac{m^2\omega^2}{4}\left(e^{-t/\kappa} - 1\right)^2}\,, \tag{55}$$

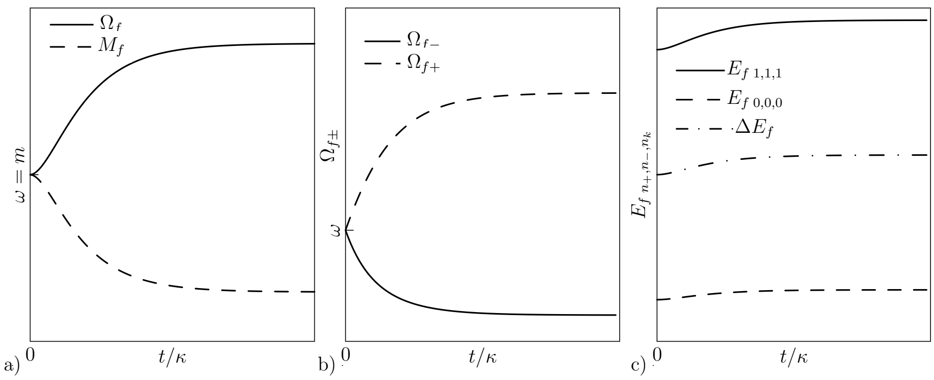

Fig. 6. The time evolution of functions $\Omega_f(t)$, $M_f(t)$, $\Omega_{f\pm}(t)$, $E_{f\,0,0,0}(t)$, $E_{f\,1,1,1}(t)$ and difference $\Delta E_f(t) = E_{f\,1,1,1}(t) - E_{f\,0,0,0}(t)$ for function $f_\kappa(t) = (e^{-t/\kappa} - 1)$ with fixed parameter κ and $m = \omega = 1$.

is smaller than $\Omega_f = \omega\sqrt{1+\frac{m^2\omega^2}{4}}$ for large times, and then, the difference (53) becomes finite (see Fig. 6). The time evolution of $E_{0,0,0}(t)$ and $E_{1,1,1}(t)$ eigenvalues has been presented for two considered cases in Figs. 5 and 6, respectively.

6. Final remarks

In this article we investigate the classical and quantum oscillator model defined on noncommutative space-time (3) and its generalized version (4). The corresponding equations of motion are provided and the time-dependent spectra of both quantum models are analyzed and illustrated.

As we already mentioned, the presented investigations describe the link between noncommutative space-time geometry and oscillator models with time-dependent mass and frequency [42–45]. We mentioned that better understanding of such a connection based on more detailed studies is postponed for future investigations.

It should be noted that the present project can be extended in various ways. First of all, one should consider more complicated system like a particle in a central field potential defined on quantum spaces (3) and (4). Secondly, one can investigate the oscillator model on other deformed space-times such as a fuzzy or twisted quantum spaces [50] and [37]. Finally, one should ask about oscillator system defined on phase space with position noncommutativity (18), (42) supplemented by suitable deformation of the momentum sector. Such a model has been considered recently in [21] in a context of noncommutative space-time (1) with additionally deformed Poisson brackets for momentum coordinates p_μ. The studies in these directions are in progress.

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