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ITÔ FORMULA FOR SUBORDINATED LANGEVIN EQUATION. A CASE OF TIME DEPENDENT FORCE*

Aleksander Weron

Hugo Steinhaus Center, Institute of Mathematics and Computer Science
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
Aleksander.Weron@pwr.wroc.pl

Sebastian Orzeł

Institute of Mathematics and Computer Science
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
Sebastian.Orzel.mat@gmail.com

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A century after Paul Langevin's landmark paper (1908) we derive here an analog of the Itô formula for subordinated Langevin equation. We show that for any subdiffusion process Y_t with time-dependent force its image $f(t, Y_t)$ by any function $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ is given again by a stochastic differential equation of Langevin type.

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1. Introduction

In 1905 A. Einstein initiated the modern study of random processes with his breaking paper on Brownian motion [1]. Three years later, in 1908, P. Langevin motivated also by a paper of Smoluchowski [2] devised another description of Brownian motion [3] in the form of stochastic differential equations (SDE). Works on Brownian diffusion by Einstein, Smoluchowski and Langevin have inspired physicists and mathematicians over the whole XX century. They still remain current and are widely referenced and discussed [4]. In the large family of stochastic processes used to model different kinds of fluctuations, Brownian motion is beyond any doubt the brightest star. It is the most extensively studied stochastic process and the foundation

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of the modern stochastic analysis. Even earlier, in 1900, Bachelier [5] bound up Brownian motion with financial modeling, by introducing the first model of stock price at the Paris Bourse. It was corrected later by Samuelson and mastered in the famous Black–Scholes–Merton model [6, 7]. Since then, the role of Brownian motion in modeling of various real-live phenomena of random character became absolutely indispensable. The rigorous mathematical treatment of Brownian motion is mostly due to N. Wiener, who proved many properties of the trajectories of the Brownian particle. Further significant contributions were done by Lévy, Kac, Doob, Itô and others [8].

However, in spite of many obvious advantages, models based on Brownian diffusion fail to provide satisfactory description of many dynamical processes [9–11]. The detailed empirical analysis of various complex systems shows that some of the significant properties of such systems cannot be captured by the Brownian diffusion models [10]. One should mention here such properties as: nonlinear in time mean-squared displacement, long-range correlations [12], non exponential relaxation, heavy-tailed and skewed marginal distributions, lack of scale invariance, discontinuity of the trajectories, and many others [9, 13]. To capture such anomalous properties of physical systems, some different mathematical models need to be introduced. In recent years one observes a rapid evolution in this direction, which results in emerging of various alternative models, such as: fractional kinetic equations [9], fractional Fokker–Planck equations (FFPE), fractional Brownian motions, generalized Langevin equations, jump-diffusion models, subordinated Langevin equations [13–17]. The increasing accuracy of the empirical measurements reveals different anomalous properties of the systems, which cannot be satisfactorily described by the models based on Brownian diffusion, so this leads to so called anomalous diffusions [13].

2. Subdiffusive Langevin equation

In this section we consider diffusion process $\{X(\tau)\}_{\tau\geq 0}$ (with internal time τ), given by the following subordinated Itô stochastic differential equation with respect to Brownian motion $\{B(\tau)\}_{\tau\geq 0}$:

$$dX(\tau) = \frac{F(U_\alpha(\tau))}{\eta} d\tau + (2K)^{\frac{1}{2}} dB(\tau)\,, \qquad X(0) = 0\,, \tag{1}$$

where $F(x) = -V'(x)$ ($F(x)$ is force and $V(x)$ is an external potential), $F \in \mathrm{C}[0,\infty)$ and $\{U_\alpha(\tau)\}_{\tau\geq 0}$, $\alpha \in (0,1)$, denotes a strictly increasing α-stable Lévy motion [18], with Laplace transform $\langle e^{-kU(\tau)}\rangle = e^{-\tau k^\alpha}$. The constant K denotes the anomalous diffusion coefficient, whereas η is the generalized friction constant. Subordinated diffusion process (subdiffusion process) is defined as [16, 19]:

$$Y(t) = X(S_t)\,, \tag{2}$$

where $\{S_t\}_{t\geq 0}$ is the inverse-time α-stable subordinator of $\{U_\alpha(\tau)\}_{\tau\geq 0}$, defined as follows [20, 21]:

$$S_t = \inf\{\tau > 0 : U_\alpha(\tau) > t\} \tag{3}$$

and the process $X(\tau)$ is the solution of Eq. (1). Let us note that the p-th moment of subordinator S_t is given by:

$$\langle S_t^p \rangle = \frac{\Gamma(p+1)t^{p\alpha}}{\Gamma(p\alpha+1)}\,, \tag{4}$$

where $p \geq 1 (p \in \mathbb{R})$ and $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$ is well known Gamma function. Density $p(x,t)$ of process (2) is given by the following FFPE [13, 15, 19]:

$$\frac{\partial p(x,t)}{\partial t} = \left[-\frac{F(t)}{\eta}\frac{\partial}{\partial x} + K\frac{\partial^2}{\partial x^2} \right] {}_0D_t^{1-\alpha} p(x,t)\,, \tag{5}$$

where $p(x,0) = \delta(x)$ and ${}_0D_t^{1-\alpha}$, $0 < \alpha < 1$, is the fractional derivative of Riemann–Liouville type [22], which does not commute with Fokker–Planck operator and it is essential that it appears to the right of $F(t)$. Therefore, it does not modify the time-dependent force. Since in every jump moment we have $U(S_t) = t$ (see [23, 24]), process $\{Y_t\}_{t\geq 0}$ satisfies the following subordinated SDE [17]:

$$dY(t) = \frac{F(t)}{\eta}dS_t + (2K)^{\frac{1}{2}}dB(S_t)\,, \qquad Y(0) = 0\,, \tag{6}$$

where $B(S_t)$ is subordinated Brownian motion. Moreover, processes S_t and $B(\tau)$ are assumed to be independent.

3. Itô formula for subdiffusion

Now we derive the analog of Itô formula for the stochastic differential $\{df(Y(t))\}_{t\geq 0}$, where $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ and $Y(t)$ is defined in (2). This is a very useful result for solution of new stochastic differential equations as well as for simulation of subdiffusion processes, and Monte Carlo methods for solving FFPE [19] and [17]. It is definitely a powerful tool for modelling of anomalous diffusion processes.

If $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ and $\{Y(t)\}_{t\geq 0}$ is subdiffusion process defined in (2), then process $\{f(Y_t)\}_{t\geq 0}$ satisfies the following SDE:

$$\begin{aligned} df(t,Y_t) &= f_t(t,Y_t)dt + (2K)^{\frac{1}{2}} f_x(t,Y_t)dB(S_t) \\ &+ \left(\eta^{-1}F(t)f_x(t,Y_t) + Kf_{xx}(t,Y_t)\right) dS_t\,. \end{aligned} \tag{7}$$

The difference between this result and the classical Itô formula is that here we have two new types of stochastic differentials for the inverse-time α-stable subordinator $S(t)$, defined in (3) and for the subordinated Brownian motion $B(S_t)$. Only when we take the deterministic subordinator $S_t = t$ then we obtain the classical Itô formula. For a case of a time-independent function $f(t,x) = f(x)$ and $f \in C^2(\mathbb{R})$, Eq. (7) reduces to the simpler form:

$$df(Y_t) = \Big(\eta^{-1}F(t)f_x(Y_t) + Kf_{xx}(Y_t)\Big)dS_t + (2K)^{\frac{1}{2}} f_x(Y_t)\, dB(S_t)\,. \tag{8}$$

To obtain the above results we will use theory of semimartingales and the Itô lemma for semimartingales [8,25]. Let us notice that the anomalous diffusion process $\{Y(t)\}_{t\geq 0}$ (defined in (2)) is a continuous semimartingale (see [26]). From (6) we obtain the following representation of $\{Y(t)\}_{t\geq 0}$ [16,17]:

$$Y_t = \int_0^t \eta^{-1}F(u)dS_u + (2K)^{\frac{1}{2}} B(S_t)\,, \tag{9}$$

where $\int_0^t \eta^{-1}F(u)dS_u$ is defined in the Stieltjes sense, because $S(t)$ is continuous finite variation process almost surely and $F \in \mathrm{C}[0,\infty)$. Since $B(S_t)$ is continuous (local) martingale [26,27] and $\int_0^t \eta^{-1}F(u)dS_u$ is continuous finite variation process, so $Y(t)$ is semimartingale. Now we use the Itô lemma for semimartingale $Y(t)$ (see [25]) and obtain:

$$df(t,Y_t) = f_t(t,Y_t)\, dt + f_x(t,Y_t)\, dY_t + \tfrac{1}{2}\, f_{xx}(t,Y_t)\, d[Y]_t\,, \tag{10}$$

where $[Y]_t$ is a quadratic variation [25,26] of process $Y(t)$. Moreover, from representation (6) of $Y(t)$, we have:

$$\begin{aligned} [Y]_t &= \left[\int_0^t \eta^{-1}F(u)dS_u + (2K)^{\frac{1}{2}}B(S)\right]_t = \left[\int_0^t \eta^{-1}F(u)dS_u\right]_t \\ &+ [(2K)^{\frac{1}{2}}B(S)]_t + 2\left[\int_0^t \eta^{-1}F(u)dS_u, (2K)^{\frac{1}{2}}B(S)\right]_t\,. \end{aligned} \tag{11}$$

At first let us notice that, for all $0 < t < \infty$, $[\int_0^t \eta^{-1} F(u) dS_u]_t = 0$, because it is continuous finite variation process [25, 26].

Next we show that $[\int_0^t F(u)dS_u, B(S)]_t = 0$. Denote $\int_0^t \eta^{-1}F(u)dS_u = \widehat{S}_t$ and $(2K)^{\frac{1}{2}} B(S_t) = \widehat{B}_t$, let $0 = t_0 < t_1 < \ldots < t_{m_n} = t$ will be n-th partition of the interval $[0, t]$. Let us notice that:

$$\left\langle \sum_{k=1}^{m_n} \big(\widehat{S}_{t_k^{(n)}} - \widehat{S}_{t_{k-1}^{(n)}}\big)\big(\widehat{B}_{t_k^{(n)}} - \widehat{B}_{t_{k-1}^{(n)}}\big)\right\rangle = \sum_{k=1}^{m_n} \left\langle \big(\widehat{S}_{t_k^{(n)}} - \widehat{S}_{t_{k-1}^{(n)}}\big)\big(\widehat{B}_{t_k^{(n)}} - \widehat{B}_{t_{k-1}^{(n)}}\big)\right\rangle$$
$$= \sum_{k=1}^{m_n} \left(\left\langle \widehat{S}_{t_k^{(n)}} \widehat{B}_{t_k^{(n)}} \right\rangle - \left\langle \widehat{S}_{t_k^{(n)}} \widehat{B}_{t_{k-1}^{(n)}} \right\rangle - \left\langle \widehat{S}_{t_{k-1}^{(n)}} \widehat{B}_{t_k^{(n)}} \right\rangle + \left\langle \widehat{S}_{t_{k-1}^{(n)}} \widehat{B}_{t_{k-1}^{(n)}} \right\rangle \right) = 0\,, \tag{12}$$

because for all $0 \le t < \infty$ and $0 \le s < \infty$, $\langle \widehat{S}_t \widehat{B}_u \rangle = 0$ from [17]. Convergence in L^1 implies convergence in probability, therefore finally:

$$[\widehat{S}, \widehat{B}]_t = \lim_{n\to\infty} \sum_{k=1}^{m_n} \big(\widehat{S}_{t_k^{(n)}} - \widehat{S}_{t_{k-1}^{(n)}}\big)\big(\widehat{B}_{t_k^{(n)}} - \widehat{B}_{t_{k-1}^{(n)}}\big) = 0.$$

Hence $[Y]_t$ defined in (11), takes the form:

$$[Y]_t = [(2K)^{\frac{1}{2}} B(S)]_t = 2K[B(S)]_t\,. \tag{13}$$

Now, we show that $[B(S)]_t = S_t$. From Doob's decomposition theorem for submartingale $B^2(S_t)$ [26], we have:

$$B^2(S_t) = M_t + \langle B(S)\rangle_t\,,$$

where M_t is martingale (for all $t \ge 0$, $\langle M_t\rangle = C = \text{const.}$) and $\langle B(S)\rangle_t$ is called the compensating process. Moreover, $\langle B(S)\rangle_t$ is non-decreasing and $\langle B(S)\rangle_0 = 0$ almost surely. Process $B(S_t)$ is continuous (local) martingale and $\langle B^2(S_t)\rangle = \langle S_t\rangle < \infty$ (see [27] and (4)), therefore for all $0 \le t < \infty$, $[B(S)]_t = \langle B(S)\rangle_t$ (see [28]). Hence we obtain:

$$\langle S_t\rangle = \langle B^2(S_t)\rangle = \langle M_t\rangle + \langle\langle B(S)\rangle_t\rangle = C + \langle\langle B(S)\rangle_t\rangle = C + \langle [B(S)]_t\rangle\,.$$

Since we know from the definition and Doob's theorem, that $S_0 = 0$ and $[B(S)]_0 = 0$, hence $C = 0$. Therefore $\langle S_t\rangle = \langle [B(S)]_t\rangle$, so $[B(S)]_t = S_t$ in L^1 and from the Chebyshev inequality [25] we obtain that this equality holds in probability. Thus, we obtain finally that $[Y]_t = 2KS_t$ and Eq. (10) reduces to the following form:

$$df(t, Y_t) = f_t(t, Y_t)dt + f_x(t, Y_t)dY_t + Kf_{xx}(t, Y_t)dS_t\,.$$

The last formula in connection with Eq. (6) gives us the desired SDE (7).

4. Conclusions

We have derived in this paper a new Itô formula for subdiffusion with a time-dependent force. It reduces to the classical Itô formula only when we consider the deterministic subordinator $S_t = t$. An advanced semimartingales techniques was employed here. By stochastic representation for FFPE via the corresponding subordinated Langevin equation [16, 17], one can apply this result as a basic tool for a Monte Carlo solution of more complicated FFPE's.

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