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RANDOM WALKS WITH BIVARIATE LÉVY-STABLE JUMPS IN COMPARISON WITH LÉVY FLIGHTS*

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In this paper we compare the Lévy flight model on a plane with the random walk resulting from bivariate Lévy-stable random jumps with the uniform spectral measure. We show that, in general, both processes exhibit similar properties, *i.e.* they are characterized by the presence of the jumps with extremely large lenghts and uniformly distributed directions (reflecting the same heavy-tail behavior and the spherical symmetry of the jump distributions), connecting characteristic clusters of short steps. The bivariate Lévy-stable random walks, belonging to the class of the well investigated stable processes, can enlarge the class of random-walk models for transport phenomena if other than uniform spectral measures are considered.

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1. Introduction

At present, random-walk models provide the basis for many concepts in statistical physics concerning transport phenomena [1–8]. Lévy flights form a special class of random walks with the step lenghts drawn from heavy-tailed probability distribution (typically, from the symmetric non-Gaussian Lévy-stable law). On a plane, they are usually described as two-dimensional random walks with the heavy-tailed step lenghts and the step directions chosen uniformly from a unit circle, independently of the lenghts. Such a definition yields the walker's trajectories characterized by the presence of clusters of short steps that are connected by rare long steps, and hence trajectories

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different than the realizations of the normal diffusive random walk (approximation of the two-dimensional Brownian motion). Due to these properties, Lévy flights have found application to model anomalous diffusion in many physical, chemical, biological and financial systems [1–8]. They have been proposed to describe the distribution of human travel [9, 10] and animal foraging patterns [11–13] as one of the processes representing an optimal solution for searching complex landscapes [14]. Recently [15], new optical materials, called Lévy glasses, have been developed in which light waves perform a Lévy flight to show that it is possible to observe and study this kind of transport processes experimentally.

The aim of this paper is to present a model alternative to the classical Lévy flights. Namely, we study random walks on a plane with bivariate Lévy-stable random steps related to the uniform spectral measure. We show that the proposed stochastic processes exhibit similar properties as the classical Lévy flights. On the other hand, the bivariate Lévy-stable random walks constitute a different class of the two-dimensional walks. This class takes advantage of all well-known properties of Lévy-stable distributions and processes. Moreover, it can lead to new models for transport phenomena with other, not necessarily uniform spectral measures. From the computer-simulation point of view the proposed alternative to Lévy flights requires new numerical methods, already well examined [16–19].

The article is structured as follows: In Section 2 we introduce formal definitions of the Lévy flights and the bivariate Lévy-stable random walks. The similarities and differences between the Lévy flights and their alternative are studied in Section 3. Concluding remarks are given in the last section.

2. Lévy flights and random walks with bivariate stable jumps

One-dimensional Lévy flight is simply a continuous-time random walk

$$R(t) = \sum_{i=1}^{N_t} R_i \,, \tag{1}$$

with non-Gaussian Lévy-stable symmetric jumps R_i and finite-mean-value waiting times T_i defining N_t as $N_t = \max\{n : T_1 + \ldots + T_n \le t\}$, see *e.g.* [20]. Spatio-temporal steps (R_i, T_i), $i \ge 1$, form a sequence of independent and identically distributed random vectors where R_i has a symmetric Lévy-stable distribution such that its characteristic function equals $\varphi_{R_i}(\mathrm{k}) = \mathbb{E}e^{\jmath \mathrm{k} R_i} = e^{-|\sigma \mathrm{k}|^\alpha}$ for some $0 < \alpha < 2$ and $\sigma > 0$, $(\mathrm{k} \in \mathbb{R})$. The mean value of T_i is finite. For simplicity, we assume constant waiting times $T_i = \Delta t$.

Lévy flight on a plane is obtained by multiplying the univariate symmetric Lévy-stable random variables R_i by random vectors $\vec{V}_i$, uniformly distributed over a unit semicircle and independent of R_i. Then, the two-dimensional Lévy flight takes the form:

$$\vec{R}_{Lf}(t) = \sum_{i=1}^{N_t} \vec{X}_i \,, \tag{2}$$

where the two-dimensional steps are given by $\vec{X}_i = R_i \vec{V}_i$.

Another natural generalization of (1) to get the stochastic process on a plane can be easily obtained by replacing univariate jumps R_i by the bivariate Lévy-stable jumps $\vec{R}_i$ with the uniform spectral measure. Such a choice of spectral measure yields all step directions equally probable in analogy to the property of the classical Lévy flights. One obtains then

$$\vec{R}(t) = \sum_{i=1}^{N_t} \vec{R}_i \,, \tag{3}$$

where $\vec{R}_i$ are bivariate Lévy-stable random variables defined by means of the characteristic function [16]

$$\varphi_{\vec{R}_i}(\boldsymbol{k}) = \mathbb{E}\, e^{\jmath\langle \boldsymbol{k}, \vec{R}_i\rangle} = \exp\left(-\int_{\mathbb{S}_2} \psi_\alpha(\langle \boldsymbol{k}, \boldsymbol{s}\rangle) \mathit{\Gamma}(d\boldsymbol{s})\right), \qquad \boldsymbol{k} \in \mathbb{R}^2 \,. \tag{4}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$, $0 < \alpha < 2$ is the index of stability,

$$\psi_\alpha(u) = \begin{cases} |u|^\alpha \left(1 - \operatorname{sgn}(u) \tan \dfrac{\pi\alpha}{2}\right) & 0 < \alpha < 2 \,, \quad \alpha \neq 1 \,, \\ |u| \left(1 + i\dfrac{2}{\pi} \operatorname{sgn}(u) \ln |u|\right) & \alpha = 1 \,. \end{cases}$$

and $\mathit{\Gamma}(\cdot)$ is the uniform spectral measure on the unit circle $\mathbb{S}_2$ such that $\mathit{\Gamma}(\mathbb{S}_2) = M > 0$.

3. Comparison of the bivariate Lévy-stable jumps with the jumps of the Lévy flight

Random unit vector $\vec{V}_i$ can be represented by a random azimuth angle θ_i; namely, $\vec{V}_i = [\cos\theta_i, \sin\theta_i]$, where θ_i is uniformly distributed over interval $[-\pi/2, \pi/2]$. Moreover, $\{\theta_i, i \geq 1\}$ has to be a sequence of independent random variables that is independent of the family of spatio-temporal steps

$\{(R_i, T_i), i \geq 1\}$. The characteristic function of $\vec{X}_i$, the jumps of the classical Lévy flight (2), can be then easily calculated. We get

$$\varphi_{\vec{X}_i}(\boldsymbol{k}) = \frac{1}{\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp(-|\sigma \langle \boldsymbol{k}, \boldsymbol{v}(x) \rangle|^{\alpha})\, dx\,, \qquad \boldsymbol{k} \in \mathbb{R}^2\,, \tag{5}$$

where $\boldsymbol{v}(x) = (\cos x, \sin x)$. Function (5) is obviously different from the bivariate Lévy-stable characteristic function (4) of $\vec{R}_i$. Hence, the bivariate jump distribution of the Lévy flight does not belong to the class of (multivariate) Lévy-stable laws.

However, the distributions of the Lévy-flight jump $\vec{X}_i$ and the bivariate stable random vector $\vec{R}_i$ (with the same α) share some important properties. Namely, they both are spherically symmetric and have heavy tails with the same exponent α. Below in this paper, we shall compare random walks of the considered two kinds for the jump distributions having exactly the same

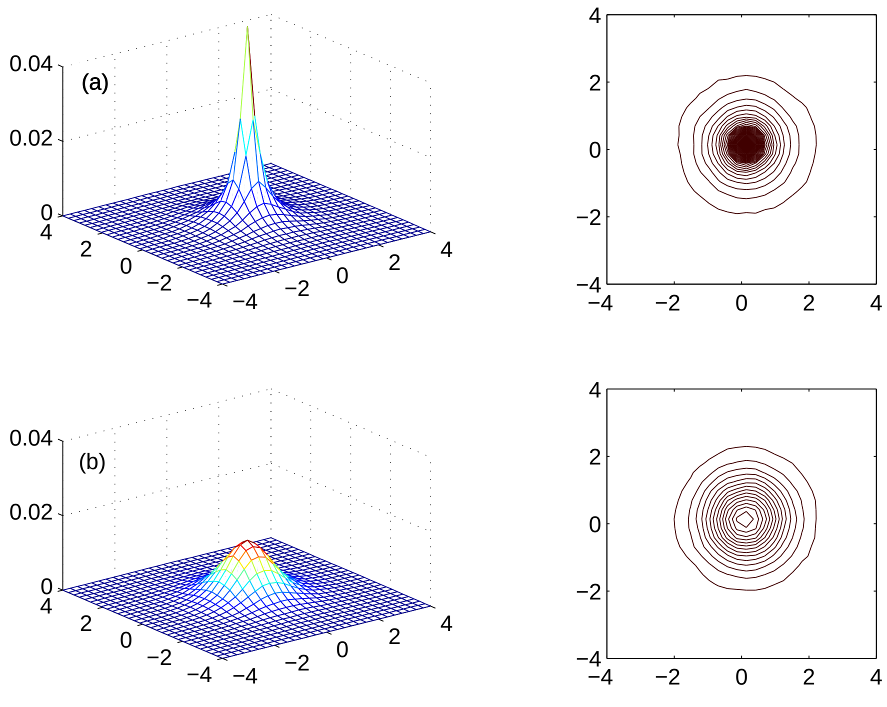

Fig. 1. Two-dimensional probability density functions (together with the respective contour lines) of (a) the Lévy-flight jump and (b) the bivariate stable jumps with the same $\alpha = 1.5$ and the scale parameters $\sigma = M = 1$.

tail asymptotics. For this purpose we only need to choose appropriately the scale parameters σ and M; namely, $\sigma = M = 1$. Two-dimensional probability density functions of the considered Lévy-flight and bivariate stable jumps are shown in Fig. 1. Also the respective contour lines are presented to illustrate spherical symmetry of both considered distributions. The parameters $\alpha = 1.5$ (the same for both jumps) and $\sigma = M = 1$ are chosen to provide the same heavy-tail behavior for both distributions. See Fig. 2, where the log-log plot of the right tails for the distributions of the one-dimensional projections $\langle \boldsymbol{k}, \vec{X}_i \rangle$ of the Lévy-flight jumps and $\langle \boldsymbol{k}, \vec{R}_i \rangle$ of the bivariate stable jumps ($\boldsymbol{k} \in \mathbb{S}_2$) are presented. The plot evidently exhibits identical heavy-tail behavior of the considered distributions. In Fig. 3 the probability density functions of these projections are presented. Notice that the results do not depend on the projection direction given by $\boldsymbol{k}$ since the considered distributions are spherically symmetric. As we see, having the same probability of the occurrence of large jumps — the Lévy flights perform the very small jumps more likely that is reflected by a cusp at point $(0, 0)$ in the respective probability-density graph. The bivariate stable jumps have broader distribution around point $(0, 0)$.

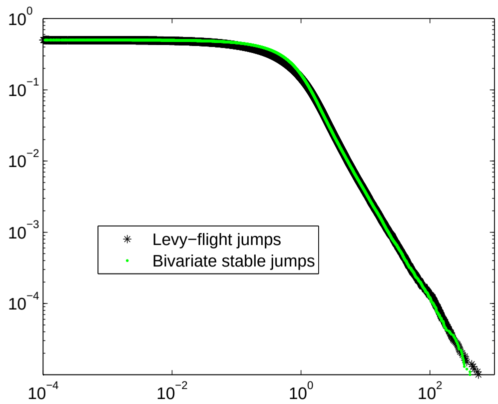

Fig. 2. Log–log plot of the right tails for the distributions of the one-dimensional projections of the Lévy-flight and the bivariate stable jumps. The parameters are chosen as in Fig. 1. Identical heavy-tail behavior for both distributions is evident. (Color on line.)

The plots in Figs. 1–3 have been obtained by Monte Carlo technique from the samples drawn from the corresponding probability laws. Values of the one-dimensional stable random variables have been generated by means of the simulation algorithm described in [21]. Realizations of the bivariate Lévy-stable random vector have been obtained by the simulation method proposed in [19] with the uniform spectral measure approximated by a discrete uniform spectral measure (as justified in [22]).

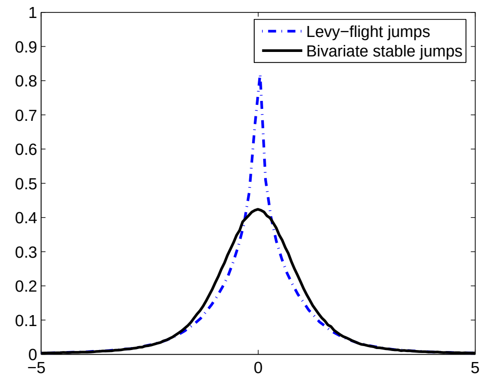

Fig. 3. Probability density functions of the one-dimensional projections of the Lévy-flight and bivariate stable jumps with the same heavy-tail behavior. The shapes of the graphs are independent of the projection direction since both the considered two-dimensional jump distributions are spherically symmetric.

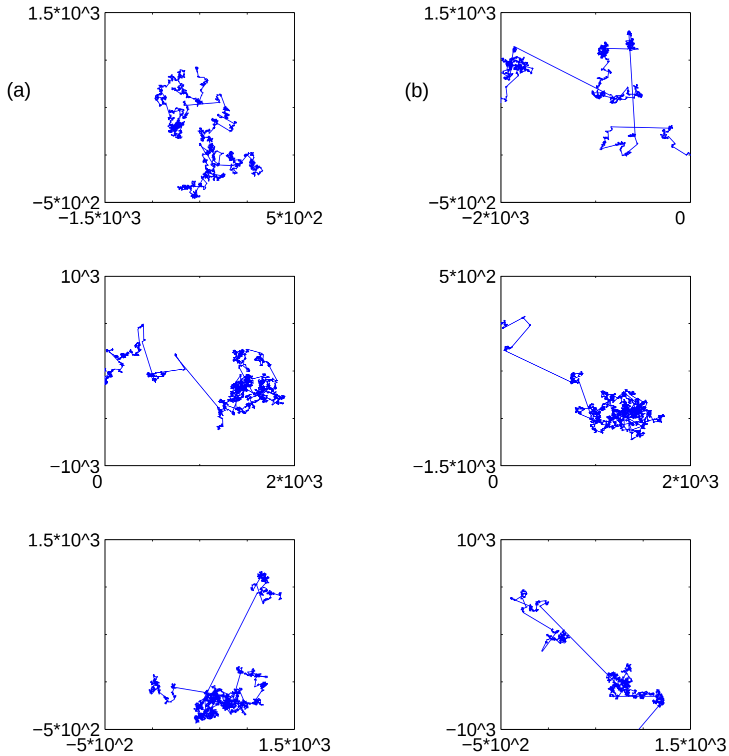

Fig. 4. Exemplary trajectories of (a) the Lévy flight and (b) the random walk with bivariate stable jumps. Parameters as for distributions presented in Fig. 1.

In Fig. 4 exemplary realizations of the Lévy flight and the random walk with bivariate stable jumps (with the same heavy tails) are presented. Observe that both movements are characterized by the presence of extremely large jumps (related to the same heavy-tail behavior of the jump distributions) and equally probable movement directions (related to spherical symmetry of the distributions). Moreover, for both processes characteristic clusters of short steps connected by extremely large jumps appear. In fact, without detailed statistical studies the trajectories of the considered walks are indistinguishable.

4. Concluding remarks

We have compared two kinds of random walks on a plane, sharing the presence of the jumps with extremely large lenghts and uniformly distributed directions that connect clusters of short steps. The first class concerns the classical Lévy flight models on a plane, well investigated and applied to describe many phenomena connected with anomalous diffusion. As an alternative, the random walks resulting from the bivariate Lévy-stable random jumps with the uniform spectral measure have been considered.

In both cases the jump distributions are spherically symmetric and heavy-tailed. We have chosen the parameters to provide exactly the same tail asymptotics, and then we have studied similarities and differences between the respective random walks. We have shown that the trajectories of the considered walks look the same and need detailed statistical methods to be distinguished. Hence, despite evident difference between jump distributions, the bivariate-stable-jump attempt to model anomalous diffusion can be used as an alternative of the Lévy-flight approach.

The new attempt requires new, recently developed techniques for numerical studies of multivariate Lévy-stable distributions. It can enlarge the class of random-walk models for transport phenomena, especially in case of other, nonuniform spectral measures. In further applications of the proposed processes to describe real data, the method of spectral-measure estimation, presented in [23], should be of a great importance.

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