Vol. 40 (2009) ACTA PHYSICA POLONICA B No 5

LÉVY FLIGHTS, DYNAMICAL DUALITY AND FRACTIONAL QUANTUM MECHANICS*

Piotr Garbaczewski

Institute of Physics, University of Opole
Luboszycka 3, 45-052 Opole, Poland

(Received March 27, 2009)

We discuss *dual* time evolution scenarios which, albeit running according to the same *real time* clock, in each considered case may be mapped among each other by means of a suitable analytic continuation in time procedure. This dynamical duality is a generic feature of diffusion-type processes. Technically that involves a familiar transformation from a non-Hermitian Fokker–Planck operator to the Hermitian operator (*e.g.* Schrödinger Hamiltonian), whose negative is known to generate a dynamical semigroup. Under suitable restrictions upon the generator, the semigroup admits an analytic continuation in time and ultimately yields dual motions. We analyze an extension of the duality concept to Lévy flights, free and with an external forcing, while presuming that the corresponding evolution rule (fractional dynamical semigroup) is a dual counterpart of the quantum motion (fractional unitary dynamics).

PACS numbers: 02.50.Ey, 05.20.–y, 05.40.Jc

1. Brownian motion inspirations

1.1. Diffusion-type processes and dynamical semigroups

The Langevin equation for a one-dimensional stochastic diffusion process in an external conservative force field $F = -(\nabla V)$: $\dot{x} = F(x) + \sqrt{2D}b(t)$, where $b(t)$ stands for the normalized white noise $\langle b(t)\rangle = 0$, $\langle b(t')b(t)\rangle = \delta(t-t')$, gives rise to the corresponding Fokker–Planck equation for the probability density $\overline{\rho}(x,t)$:

$$\partial_t\overline{\rho} = D\Delta\overline{\rho} - \nabla(F\overline{\rho})\,. \tag{1}$$

* Presented at the XXI Marian Smoluchowski Symposium on Statistical Physics Zakopane, Poland, September 13–18, 2008.

By means of a standard substitution $\overline{\rho}(x,t) = \Psi(x,t)\exp[-V(x)/2D]$, [1], we pass to a generalized diffusion equation for an auxiliary function $\Psi(x,t)$:

$$\partial_t \Psi = D\Delta\Psi - \mathcal{V}(x)\Psi\,, \tag{2}$$

where a compatibility condition $\mathcal{V}(x) = (1/2)[(F^2/2D) + \nabla F]$ needs to be respected. This transformation assigns the role of the dynamics generator to the Hermitian (eventually self-adjoint) operator $-\hat{H} = D\Delta - \mathcal{V}$.

Under suitable restrictions upon $V(x)$, $-\hat{H}$ becomes a legitimate generator of a contractive dynamical semigroup $\exp(-\hat{H}t)$, $t \geq 0$. If additionally the dynamical semigroup is amenable to an analytic continuation in time, the contractive semigroup operator $\exp(-\hat{H}t)$ can be related with the unitary operator $\exp(-i\hat{H}t)$ via so-called Wick rotation $t \to it$. This duality observation underlies our forthcoming discussion and generalizations to Lévy flights framework.

1.2. Free propagation and its analytic continuation in time

The standard theory of Gaussian diffusion-type processes takes the Wiener process as the "free noise" model, with the Laplacian as the "noise" generator. It is an element of folk lore that the related dissipative semigroup dynamics $\exp(tD\Delta) = \exp(-t\hat{H}_0)$ (and thus the heat equation) can be mapped into the unitary dynamics $\exp(itD\Delta) = \exp(-it\hat{H}_0)$ (and thus the free Schrödinger equation), by means of an analytic continuation in time procedure, [2]. A parameter D may be interpreted dimensionally as $D = \hbar/2m$, or $D = k_{\rm B}T/m\beta$ (Einstein's fluctuation-dissipation statement).

Quite often, this mapping is represented by a formal $it \to t$ time transformation of the free Schrödinger picture dynamics (one should be aware that to execute a mapping for concrete solutions, a proper adjustment of the time interval boundaries is necessary):

$$i\partial_t \psi = -D\triangle\psi \longrightarrow \partial_t \theta_* = D\triangle\theta_*\,, \tag{3}$$

where the notation θ_* for solutions of the heat equation has been adopted, to stay in conformity with the forthcoming more general discussion, where $\theta_*(x,t)$ needs not to be a probability density, [2–4].

The mapping is usually exemplified in terms of integral kernels g and k as follows, *c.f.* also [5]:

$$\begin{aligned} \psi(x,t) &= \int dx' g(x-x',t)\psi(x',0)\,, \\ g(x-x',t) &\doteq k(x-x',it) = (4\pi iDt)^{-1/2}\exp\left[-\frac{(x-x')^2}{4iDt}\right] \end{aligned} \tag{4}$$

and
$$\theta_*(x,t) \; = \; \int dx' k(x-x',t)\theta_*(x',0)\,,$$
$$k(x-x',t) \; \doteq \; g(x-x',-it) = (4\pi Dt)^{1/2} \exp\left[-\frac{(x-x')^2}{4Dt}\right]\,, \tag{5}$$

where the initial $t=0$ data need to be properly adjusted. Here, $g(x-x',t)$ is an integral kernel of the unitary evolution operator: $[\exp(iDt\Delta)\,\psi](x,0) = \psi(x,t)$. The heat kernel $k(x-x',y)$ plays the same role with respect to the contractive semigroup operator: $[\exp(Dt\Delta)\,\theta_*](x,0)$.

The special choice of

$$\psi(x,0) = (\pi\alpha^2)^{-1/4} \exp\left(-\frac{x^2}{2\alpha^2}\right) \tag{6}$$

implies

$$\psi(x,t) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} (\alpha^2 + 2iDt)^{-1/2} \exp\left[-\frac{x^2}{2(\alpha^2+2iDt)}\right] \tag{7}$$

and

$$\theta_*(x,t) \doteq \psi(x,-it) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} (\alpha^2 + 2Dt)^{-1/2} \exp\left[-\frac{x^2}{2(\alpha^2+2Dt)}\right]' \tag{8}$$

with $\theta_*(x,0) = \psi(x,0)$.

We note that $\rho = |\psi|^2 = \psi\psi^*$ is a quantum mechanical probability density on R for all times

$$\rho(x,t) = \left[\frac{\alpha^2}{\pi(\alpha^4 + 4D^2t^2)}\right]^{1/2} \exp\left[-\frac{\alpha^2 x^2}{\alpha^4 + 4D^2t^2}\right]\,. \tag{9}$$

The real solution $\theta_*(x,t)$ of the heat equation is not a probability density $\overline{\rho}(x,t) = \theta_*(x,t)\theta(x,t)$, unless multiplied by an appropriate real function $\theta(x,t)$ which solves the time adjoint heat equation (that becomes an ill-posed dynamical problem if considered carelessly).

Case 1: Since $\overline{\rho}(x,t) = [2\pi(\alpha^2+2Dt)]^{-1/2}\exp[-x^2/2(\alpha^2+2Dt)]$ actually is an example of the free Brownian motion probability density for all $t \geq 0$, we infer

$$\overline{\rho}(x,t) = (4\pi\alpha^2)^{1/4}\,\theta_*(x,t) \doteq (\theta\,\theta_*)(x,t)\,, \tag{10}$$

where $\theta(x,t) \equiv \theta = (4\pi\alpha^2)^{1/4}$ is interpreted as a trivial (constant) solution of the time adjoint heat equation $\partial_t\theta = -D\Delta\theta$. We stress that $\theta_* = (4\pi\alpha^2)^{-1/4}\overline{\rho} \sim \overline{\rho}$. This, looking redundant observation, will prove quite useful in a more general framework to be introduced in below.

Case 2: A complex conjugate $\psi^*(x,t)=\psi(x,-t)$ of $\psi(x,t)$, Eq. (7), solves the time-adjoint Schrödinger equation $i\partial_t\psi^* = D\Delta\psi^*$. Hence a time-symmetric approach to the analytic continuation in time might look more compelling. Indeed

$$\theta(x,t) \doteq \psi^*(x,it) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} (\alpha^2 - 2Dt)^{-1/2} \exp\left[-\frac{x^2}{2(\alpha^2-2Dt)}\right] \tag{11}$$

is a legitimate solution of the time-adjoint heat equation $\partial_t\theta = -D\Delta\theta$ as long as $t \in [-T/2, +T/2]$ where $T = \alpha^2/D$.

In the present case, both time adjoint equations set well defined Cauchy problems (at least in the just defined time interval). The subtle point is that the would-be "initial" data for the backward in time evolution, in fact need to be the terminal data, given at the end-point $T/2$ of the considered time-interval.

The only propagation tool, we have in hands, is the heat kernel (3): $k(x-x', t \to t-t')$ with $t \geq t'$. There holds $\theta_*(x,t) = \int k(x-x', t-t')\, \theta_*(x',t')\, dx'$ and $\theta(x',t') = \int \theta(x,t)\, k(x-x', t-t')\, dx$ for any $t' < t \in [-T/2, +T/2]$.

The original quantum mechanical probability density $\rho = |\psi|^2 = \psi\psi^*$, Eq. (7), is mapped into a Brownian bridge (pinned Brownian motion) probability density:

$$\rho(x, \pm it) \doteq \overline{\rho}(x,t) = (\theta\theta_*)(x,t) = \left[\frac{\alpha^2}{\pi(\alpha^4 - 4D^2t^2)}\right]^{1/2} \exp\left[-\frac{\alpha^2 x^2}{\alpha^4 - 4D^2t^2}\right]. \tag{12}$$

The price paid for the time-symmetric appearance of this formula is a limitation of the admissible time span to a finite time-interval of length $T = \alpha^2/D$.

Case 3: To make a direct comparison of *Case 2* with the previous *Case 1*, let us confine the time interval of *Case 2* to $[0, +T/2]$. Now, a conditional Brownian motion connects $\overline{\rho}(x,0) = \rho(x,0) = (\alpha^2\pi)^{-1/2}\exp(-x^2/\alpha^2)$ with $\overline{\rho}(x, t \to +T/2)$ of Eq. (10). Because of $T = \alpha^2/D$, as $t \to T/2$, instead of a regular function we arrive at the linear functional (generalized function), here represented by the Dirac delta $\delta(x)$. Note that $\delta(x-x')$ is a standard initial $t = 0$ value of the heat kernel $k(x-x', t)$.

This behavior is faithfully reproduced by the time evolution of $\theta_*(x,t)$ and $\theta(x,t)$ that compose $\overline{\rho}(x,t) = (\theta_*\,\theta)(x,t)$ for $t \in [0, T/2]$. The initial value of $\theta_*(x,0) = \psi(x,0)$, Eq. (6), is propagated *forward* in accordance with Eq. (8) to $\theta_*(x,T/2) = (4\pi\alpha^2)^{-1/4}\exp(-x^2/4\alpha^2)$.

In parallel, $\theta(x,t)$ of (11) interpolates *backwards* between $\theta(x,T/2) \equiv (4\pi\alpha^2)^{1/4}\,\delta(x)$ and $\theta(x,0) = \theta_*(x,0)$. We have here employed an identity $\delta(ax) = (1/|a|)\delta(x)$. Because of $f(x)\delta(x) \equiv f(0)\delta(x)$, we arrive at $\overline{\rho}(x,T/2) = (\theta_*\,\theta)(x,T/2) \equiv \delta(x)$.

1.3. Schrödinger's boundary data problem

The above discussion provides particular solutions to so-called Schrödinger boundary data problem, under an assumption that a Markov stochastic process which interpolates between two *a priori* given probability densities $\rho(x,0)$ and $\rho(x,T/2)$ can be modeled by means of the Gauss probability law (*e.g.* in terms of the heat kernel). That incorporates the free Brownian motion (Wiener process) and all its conditional variants, Brownian bridges being included, [3,4] and [6–8], *c.f.* also [2].

For our purposes the relevant information is that, if the interpolating process is to display the Markov property, then it has to be specified by the joint probability measure (A and B are Borel sets in R):

$$m(A,B) = \int\limits_A dx \int\limits_B dy\, m(x,y)\,, \tag{13}$$

where $\int_R m(x,y)dy = \rho(x,0)$, and $\int_R m(x,y)dx = \rho(y,T/2)$. From the start, we assign densities to all measures to be dealt with, and we assume the functional form of the density $m(x,y)$

$$m(x,y) = f(x)k(x,0,y,T/2)g(y) \tag{14}$$

to involve two unknown functions $f(x)$ and $g(y)$ which are of the same sign and nonzero, while $k(x,s,y,t)$ is any bounded strictly positive (dynamical semigroup) kernel defined for all times $0 \le s < t \le T/2$. For each concrete choice of the kernel, the above integral equations are known to determine functions $f(x)$, $g(y)$ uniquely (up to constant factors).

By denoting $\theta_*(x,t) = \int f(z)k(z,0,x,t)dz$ and $\theta(x,t) = \int k(x,t,z,T/2) \times g(z)dz$ it follows that

$$\begin{aligned} \overline{\rho}(x,t) &= \theta(x,t)\theta_*(x,t) = \int p(y,s,x,t)\overline{\rho}(y,s)\,dy\,, \\ p(y,s,x,t) &= \frac{k(y,s,x,t)\,\theta(x,t)}{\theta(y,s)}\,, \end{aligned} \tag{15}$$

for all $0 \le s < t \le T/2$. The above $p(y,s,x,t)$ is the transition probability density of the pertinent Markov process that interpolates between $\overline{\rho}(x,0)$ and $\overline{\rho}(x,T/2)$. *Cases 1* through *3* are particular examples of the above reasoning, once $k(x,s,y,t)$ is specified to be the heat kernel (3) and the corresponding boundary density data are chosen. Clearly, $\theta^*(x,0) = f(x)$ while $\theta(x,T/2) = g(x)$.

We recall that in the case of free evolution, by setting $\theta(x,t) = \theta \equiv$ const., as in *Case 1*, we effectively transform an integral kernel k of the $L^1(R)$ norm-preserving semigroup into a transition probability density p of the Markov stochastic process. Then $\theta^* \sim \overline{\rho}$.

2. Free noise models: Lévy flights and fractional (Lévy) semigroups

The Schrödinger boundary data problem is amenable to an immediate generalization to infinitely divisible probability laws which induce contractive semigroups (and their kernels) for general Gaussian and non-Gaussian noise models. They allow for various jump and jump-type stochastic processes instead of a diffusion process.

A subclass of stable probability laws contains a subset that is associated in the literature with the concept of Lévy flights. At this point let us invoke a functional analytic lore, where contractive semigroup operators, their generators and the pertinent integral kernels can be directly deduced from the Lévy–Khitchine formula, compare *e.g.* [8].

Let us consider semigroup generators (Hamiltonians, up to dimensional constants) of the form $\hat{H} = F(\hat{p})$, where $\hat{p} = -i\nabla$ stands for the momentum operator (up to the disregarded $\hbar$ or $2mD$ factor) and for $-\infty < k < +\infty$, the function $F = F(k)$ is real valued, bounded from below and locally integrable. Then,

$$\exp(-t\hat{H}) = \int_{-\infty}^{+\infty} \exp[-tF(k)]\, dE(k)\,, \tag{16}$$

where $t \geq 0$ and $dE(k)$ is the spectral measure of $\hat{p}$.

Because of

$$(E(k)f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{k} \exp(ipx)\tilde{f}(p)\, dp\,, \tag{17}$$

where $\tilde{f}$ is the Fourier transform of f, we learn that

$$\begin{aligned}
[\exp(-t\hat{H})]f(x) &= \left[\int_{-\infty}^{+\infty} \exp(-tF(k))dE(k)f\right](x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp[-tF(k)]\frac{d}{dk}\left[\int_{-\infty}^{k} \exp(ipx)\tilde{f}(p)dp\right] dk \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-tF(k))\exp(ikx)\tilde{f}(k)dk \\
&= \left[\exp(-tF(p))\tilde{f}(p)\right]^{\vee}(x)\,,
\end{aligned} \tag{18}$$

where the superscript $\vee$ denotes the inverse Fourier transform.

Let us set

$$k_t = \frac{1}{\sqrt{2\pi}}[\exp(-tF(p)]^{\vee} . \tag{19}$$

Then the action of $\exp(-t\hat{H})$ can be given in terms of a convolution (*i.e.* by means of an integral kernel $k_t \equiv k(x-y,t) = k(y,0,x,t)$):

$$\exp(-t\hat{H})f = [\exp(-tF(p))\tilde{f}(p)]^{\vee} = f * k_t , \tag{20}$$

where

$$(f*g)(x) := \int_R g(x-z)f(z)dz . \tag{21}$$

We shall restrict considerations only to those $F(p)$ which give rise to positivity preserving semigroups: if $F(p)$ satisfies the celebrated Lévy–Khintchine formula, then k_t is a positive measure for all $t \geq 0$. The most general case refers to a combined contribution from three types of processes: deterministic, Gaussian, and the jump-type process.

We recall that a characteristic function of a random variable X completely determines a probability distribution of that variable. If this distribution admits a density we can write $E[\exp(ipX)] = \int_R \overline{\rho}(x)\exp(ipx)dx$ which, for infinitely divisible probability laws, gives rise to the Lévy–Khintchine formula

$$E[\exp(ipX)] = \exp\{i\alpha p - (\sigma^2/2)p^2 + \int_{-\infty}^{+\infty}\left[\exp(ipy) - 1 - \frac{ipy}{1+y^2}\right]\nu(dy)\} , \tag{22}$$

where $\nu(dy)$ stands for the so-called Lévy measure. In terms of Markov stochastic processes all that amounts to a decomposition of X_t into

$$X_t = \alpha t + \sigma B_t + J_t + M_t , \tag{23}$$

where B_t stands for the free Brownian motion (Wiener process), J_t is a Poisson process while M_t is a general jump-type process (more technically, martingale with jumps).

By disregarding the deterministic and jump-type contributions in the above, we are left with the Wiener process $X_t = \sigma B_t$. For a Gaussian $\overline{\rho}(x) = (2\pi\sigma^2)^{-1/2}\exp(-x^2/2\sigma^2)$ we directly evaluate $E[\exp(ipx)] = \exp(-\sigma^2 p^2/2)$.

Let us set $\sigma^2 = 2Dt$. We get $E[\exp(ipX_t)] = \exp(-tDp^2)$ and subsequently, by employing $p \to \hat{p} = -i\nabla$, we arrive at the contractive semigroup operator $\exp(tD\Delta)$ where the one-dimensional Laplacian $\Delta = d^2/dx^2$ has been introduced. That amounts to choosing a special version of the previously introduced Hamiltonian $\hat{H} = F(\hat{p}) = D\hat{p}^2$. Note that we can get read of the constant D by rescaling the time parameter in the above.

Presently, we shall concentrate on the integral part of the Lévy–Khintchine formula, which is responsible for arbitrary stochastic jump features. By disregarding the deterministic and Brownian motion entries we arrive at:

$$F(p) = -\int_{-\infty}^{+\infty} \left[\exp(ipy) - 1 - \frac{ipy}{1+y^2}\right] \nu(dy)\,, \tag{24}$$

where $\nu(dy)$ stands for the appropriate Lévy measure. The corresponding non-Gaussian Markov process is characterized by

$$E[\exp(ipX_t)] = \exp[-tF(p)] \tag{25}$$

with $F(p)$, (22). Accordingly, the contractive semigroup generator may be defined as follows: $F(\hat{p}) = \hat{H}$.

For concreteness we can mention some explicit examples of non-Gaussian Markov semigroup generators. $F(p) = \gamma|p|^{\mu}$ where $\mu < 2$ and $\gamma > 0$ stands for the intensity parameter of the Lévy process, upon $p \to \hat{p} = -i\nabla$ gives rise to a pseudo-differential operator $\hat{H} = \gamma \Delta^{\mu/2}$ often named the fractional Hamiltonian. Note that, by construction, it is a positive operator (quite alike $-D\Delta$).

The corresponding jump-type dynamics is interpreted in terms of Lévy flights. In particular

$$F(p) = \gamma|p| \to \hat{H} = F(\hat{p}) = \gamma|\nabla| \doteq \gamma(|\Delta|)^{1/2} \tag{26}$$

refers to the Cauchy process.

Since we know that the probability density of the free Brownian motion is a solution of the Fokker–Planck (here, simply — heat) equation

$$\partial_t \overline{\rho} = D\Delta\overline{\rho} \tag{27}$$

it is instructive to set in comparison the pseudo-differential Fokker–Planck equation which corresponds to the fractional Hamiltonian and the fractional semigroup $\exp(-t\hat{H}) = \exp(-\gamma|\Delta|^{\mu/2})$

$$\partial_t \overline{\rho} = -\gamma|\Delta|^{\mu/2}\overline{\rho}\,. \tag{28}$$

As mentioned in the discussion of Case 1, instead of $\overline{\rho}$ in the above we can insert $\theta_* \sim \overline{\rho}$, while remembering that $\theta \equiv \text{const}$.

3. Free fractional Schrödinger equation

Fractional Hamiltonians $\hat{H} = \gamma|\Delta|^{\mu/2}$ with $\mu < 2$ and $\gamma > 0$ are self-adjoint operators in suitable $L^2(R)$ domains. They are also positive operators, so that the respective fractional semigroups are holomorphic, *i.e.* we can replace the time parameter t by a complex one $\sigma = t + is$, $t > 0$ so that

$$\exp(-\sigma\hat{H}) = \int_R \exp(-\sigma F(k))\, dE(k)\,. \tag{29}$$

Its action is defined by

$$[\exp(-\sigma\hat{H})]f = \Big[(\tilde{f}\exp(-\sigma F)\Big]^{\vee} = f * k_\sigma\,. \tag{30}$$

Here, the integral kernel reads $k_\sigma = 1/\sqrt{2\pi}\,[\exp(-\sigma F)]^{\vee}$. Since $\hat{H}$ is self adjoint, the limit $t \downarrow 0$ leaves us with the unitary group $\exp(-is\hat{H})$, acting in the same way: $[\exp(-is\hat{H})]f = [\tilde{f}\exp(-isF)]^{\vee}$, except that now $k_{is} := 1/\sqrt{2\pi}[\exp(-isF)]^{\vee}$ in general is *not* a probability measure.

In view of unitarity, the unit ball in L^2 is an invariant of the dynamics. Hence probability densities, in a standard form $\rho = \psi^*\psi$ can be associated with solutions of the free fractional (pseudodiferential) Schrödinger equations:

$$i\partial_t\psi(x,t) = \gamma|\Delta|^{\mu/2}\psi(x,t) \tag{31}$$

with initial data $\psi(x,0)$. Attempts towards formulating the fractional quantum mechanics can be found in Refs. [8, 11–13].

All that amounts to an analytic continuation in time, in close affinity with the Gaussian pattern (1):

$$i\partial_t\psi = \gamma|\Delta|^{\mu/2}\psi \longleftrightarrow \partial_t\theta^* = -\gamma|\Delta|^{\mu/2}\theta^*\,. \tag{32}$$

We assume that $\theta^* \sim \overline{\rho}$ and thence the corresponding $\theta \equiv \text{const}$.

Stable stochastic processes and their quantum counterparts are plagued by a common disease: it is extremely hard, if possible at all, to produce insightful analytic solutions. To get a flavor of intricacies to be faced and the level of technical difficulties, we shall reproduce some observations in regard to the Cauchy dynamical semigroup and its unitary (quantum) partner. For convenience we scale out a parameter γ.

For the Cauchy process, whose generator is $|\nabla|$, we deal with a probabilistic classics:

$$\overline{\rho}(x,t) = \frac{1}{\pi}\frac{t}{t^2+x^2} \Longrightarrow k(y,s,x,t) = \frac{1}{\pi}\frac{t-s}{(t-s)^2+(x-y)^2}\,. \tag{33}$$

where $0<s<t$. We have $\langle \exp[ipX(t)]\rangle := \int_R \exp(ipx)\overline{\rho}(x,t)\,dx = \exp[-tF(p)]$ $= \exp(-|p|t)$ and

$$\overline{\rho}(x,t) = \int_R k(y,s,x,t)\,\overline{\rho}(y,s)\,dy \tag{34}$$

for all $t > s \geq 0$. We recall that $\lim_{t\downarrow 0} \frac{t}{\pi(x^2+t^2)} \equiv \delta(x)$.

The characteristic function of $k(y,s,x,t)$ for y,s fixed, reads $\exp[ipy -|p|(t-s)]$, and the Lévy measure needed to evaluate the Lévy–Khintchine integral reads:

$$\nu_0(dy) := \lim_{t\downarrow 0}\left[\frac{1}{t}k(0,0,y,t)\right]dy = \frac{dy}{\pi y^2}\,. \tag{35}$$

To pass to a dual Cauchy–Schrödinger dynamics, we need to perform an analytic continuation in time. We deal with a holomorphic fractional semigroup $\exp(-\sigma t|\nabla|)$, $\sigma = t + is$, (27). It is clear that $\exp(-t|\nabla|)$ and $\exp(-is|\nabla|)$ have a common, identity operator limit as $t \downarrow 0$ and $s \equiv t \downarrow 0$.

An analytic continuation of the Cauchy kernel by means of (28) gives rise to:

$$k_t(x) = \frac{1}{\pi}\frac{t}{x^2+t^2} \longrightarrow g_s(x) \doteq k_{is}(x) = \frac{1}{2}[\delta(x-s)+\delta(x+s)] + \frac{1}{\pi}\mathcal{P}\frac{is}{x^2-s^2}\,, \tag{36}$$

where $\mathcal{P}$ indicates that a convolution of the integral kernel with any function should be considered as a principal value of an improper integral, [8]. This should be compared with an almost trivial outcome of the previous mapping (2)$\rightarrow$ (3). Here, we employ the usual notation for the Dirac delta functionals, and the new time label s is a remnant of the limiting procedure $t \downarrow 0$ in $\sigma = t + is$.

The function denoted by $is/\pi(x^2-s^2)$ comes from the inverse Fourier transform of $-i/(\sqrt{2\pi})\,\mathrm{sgn}(p)\sin(sp)$. Because of

$$[\mathrm{sgn}(p)]^\vee = i\sqrt{\frac{2}{\pi}}\mathcal{P}\left(\frac{1}{x}\right), \tag{37}$$

where $\mathcal{P}(\frac{1}{x})$ stands for the functional defined in terms of a principal value of the integral. Using the notation $\delta_{\pm s}$ for the Dirac delta functional $\delta(x \mp s)$:

$$[\sin(sp)]^\vee = i\sqrt{\frac{\pi}{2}}(\delta_s - \delta_{-s}) \tag{38}$$

we realize that

$$\frac{1}{\pi}\frac{is}{x^2-s^2} = \frac{i}{2\pi}(\delta_s - \delta_{-s}) * \mathcal{P}\left(\frac{1}{x}\right) \tag{39}$$

is given in terms of the implicit convolution of two generalized functions. Obviously, a propagation of an initial function $\psi_0(x)$ to time $t > 0$:

$$\psi(x,t) = \int_R g(x - x', t)\psi_0(x')\, dx' \tag{40}$$

gives a solution of the fractional (Cauchy) Schrödinger equation $i\partial_t \psi = -|\nabla|\psi$.

In comparison with the Gaussian case of Sec. 1, one important difference must be emphasized. The improper integrals, which appear while evaluating various convolutions, need to be handled by means of their principal value. Therefore, a simple $it \to t$ transformation recipe no longer works on the level of integral kernels and respective ψ and θ^* functions.

One explicit example is provided by the incongruence of (31) and (34) with respect to the formal $t \to -it$ mapping. Another is provided by considering specific solutions of pseudo-differential equations (30).

To that end, let us consider $\theta_{*0}(x) = (2/\pi)^{1/2} 1/(1 + x^2)$, together with $\theta = (2\pi)^{-1/2}$. Then, $\theta\, \theta_*(x, 0) = 1/(\pi(1 + x^2))$ is an $L(R)$ normalized Cauchy density, while $\theta_{*0}(x)$ itself is the $L^2(R)$ normalized function. Clearly:

$$\begin{aligned} \theta_*(x,t) &= [\exp(-t|\nabla|)\theta_{*0}](x) = \int k(y,0,x,t)\theta_*(y,0)dy \\ &= \left(\frac{2}{\pi}\right)^{1/2} \frac{1+t}{x^2 + (1+t)^2} \end{aligned} \tag{41}$$

while the corresponding $\psi(x,t)$ with $\psi_0(x) = \theta_{*0}(x)$ reads (for details see *e.g.* [8]):

$$\begin{aligned} &\psi(x,s) = [\exp(-is|\nabla|)\, \psi_0](x) \\ &= \frac{1}{2}[\psi_0(x+s) + \psi_0(x-s)] + \frac{i}{2}[(x-s)\psi_0(x-s) - (x+s)\psi_0(x+s)]\,. \end{aligned} \tag{42}$$

4. Dynamical duality in external potentials: fractional Schrödinger semigroups and Lévy flights

4.1. Schrödinger semigroups for Smoluchowski processes

Considerations of Sec. 1, where the free quantum dynamics and free Brownian motion were considered as dual dynamical scenarios, can be generalized to an externally perturbed dynamics, [2]. Namely, one knows that the Schrödinger equation for a quantum particle in an external potential $V(x)$, and the generalized heat equation are connected by analytic continuation in time, known to take the Feynman–Kac (holomorphic semigroup) kernel into the Green function of the corresponding quantum mechanical problem

$$i\partial_t \psi = -D\Delta\psi + \mathcal{V}\psi \longleftrightarrow \partial_t \theta_* = D\Delta\theta_* - \mathcal{V}\theta_* \,. \tag{43}$$

Here $\mathcal{V} \doteq V(x)/2mD$.

For $V = V(x), x \in R$, bounded from below, the generator $\hat{H} = -2mD^2\triangle +V$ is essentially self adjoint on a natural dense subset of L^2, and the kernel $k(x,s,y,t) = [\exp[-(t-s)\hat{H}]](x,y)$ of the related dynamical semigroup $\exp(-t\hat{H})$ is strictly positive. The quantum unitary dynamics $\exp(-i\hat{H}t)$ is an obvious result of the analytic continuation in time of a dynamical semigroup.

Assumptions concerning the admissible potential may be relaxed. The necessary demands are that $\hat{H}$ is self-adjoint and bounded from below. Then the respective dynamical semigroup is holomorphic.

The key role of an integral kernel of the dynamical semigroup operator has been elucidated in formulas (11)–(13), where an explicit form of a transition probability density of the Markov diffusion process was given. We have determined as well the time development of $\theta_*(x,t)$ and $\theta(x,t)$, so that $\overline{\rho}(x,t) = (\theta\theta_*)(x,t)$ is a probability density of the pertinent process.

If we *a priori* consider $\theta(x,t)$ in the functional form $\theta(x,t) \doteq \exp \Phi(x,t)$, so that $\theta_*(x,t) \doteq \overline{\rho}(x,t)\exp[-\Phi(x)]$, and properly define the forward drift $b(x,t) \doteq 2D\nabla\Phi(x,t)$ in the pertinent Fokker–Planck equation:

$$\partial_t\overline{\rho} = D\Delta\overline{\rho} - \nabla(b\,\overline{\rho}) \tag{44}$$

we can recast a diffusion problem in terms of a pair of time adjoint generalized heat equations

$$\partial_t\theta_* = D\Delta\theta_* - \mathcal{V}\theta_* \tag{45}$$

and

$$\partial_t\theta = -D\Delta\theta + \mathcal{V}\theta \,, \tag{46}$$

i.e. as the Schrödinger boundary data problem, where an interpolating stochastic process is uniquely determined by a continuous and positive Feynman–Kac kernel of the Schrödinger semigroup $\exp(-t\hat{H})$, where $\hat{H} = -D\Delta + \mathcal{V}$.

If our departure point is the Fokker–Planck (or Langevin) equation with the *a priori* prescribed potential function $\Phi(x,t)$ for the forward drift $b(x,t)$, then the backward equation (44) becomes an identity from which $\mathcal{V}$ directly follows, in terms of Φ and its derivatives, [6, 7]:

$$\mathcal{V}(x,t) = \left[\partial_t\Phi + \frac{1}{2}\left(\frac{b^2}{2D} + \nabla b\right)\right] \,. \tag{47}$$

For the time-independent drift potential, which is the case for standard Smoluchowski diffusion processes, we get (*c.f.* also [1], where the transformation of the Fokker–Planck equations (42) into an associated Hermitian problem (43) is described in detail):

$$\mathcal{V}(x) = \left[\frac{1}{2}\left(\frac{b^2}{2D} + \nabla b\right)\right] . \tag{48}$$

Notice that $\Phi(x)$ is defined up to an additive constant.

To give an example of a pedestrian reasoning based on the above procedure in case of a concrete Smoluchowski diffusion processes, let us begin from the Langevin equation for the one-dimensional stochastic process in the external conservative force field $F(x) = -(\nabla V)(x)$ (to keep in touch with the previous notations, note that $\Phi \equiv -V$):

$$\frac{dx}{dt} = F(x) + \sqrt{2D}B(t) , \tag{49}$$

where $B(t)$ stands for the normalized white noise: $\langle B(t)\rangle = 0$, $\langle B(t')B(t)\rangle = \delta(t - t')$.

The corresponding Fokker–Planck equation for the probability density $\rho(x,t)$ reads:

$$\partial_t \overline{\rho} = D\Delta\overline{\rho} - \nabla(F\overline{\rho}) \tag{50}$$

and by means of a substitution $\overline{\rho}(x,t) = \theta_*(x,t)\exp[-V(x)/2D]$, [1], can be transformed into the generalized diffusion equation for an auxiliary function $\theta^*(x,t)$:

$$\partial_t \theta_* = D\Delta\theta_* - \mathcal{V}\theta_* , \tag{51}$$

where the consistency condition (reconciling the functional form of $\mathcal{V}$ with this for F)

$$\mathcal{V} = \frac{1}{2}\left(\frac{F^2}{2D} + \nabla F\right) \tag{52}$$

directly comes from the time-adjoint equation

$$\partial_t \theta \equiv 0 = -D\Delta\theta + \mathcal{V}\theta \tag{53}$$

with $\theta(x) = \exp[-V(x)/2D]$.

For the Ornstein–Uhlenbeck process $b(x) = F(x) = -\kappa x$ and accordingly

$$\mathcal{V}(x) = \frac{\kappa^2 x^2}{4D} - \frac{\kappa}{2} \tag{54}$$

is an explicit functional form of the potential $\mathcal{V}$, present in previous formulas (41)–(44).

4.2. Fractional semigroups and perturbed Lévy flights

External perturbations in the additive form:

$$i\partial_t\psi(x,t) = \gamma|\Delta|^{\mu/2}\psi(x,t) + \mathcal{V}(x)\psi(x,t)\,, \tag{55}$$

were considered in the framework of fractional quantum mechanics, [11–13], *c.f.* also [8, 9]. With the dual dynamics concept in mind, Eq. (30), we expect that an analytic continuation in time (if admitted) takes us from the fractional Schrödinger equation to the fractional analog of the generalized diffusion equation:

$$\partial_t\theta^* = -\gamma|\Delta|^{\mu/2}\theta^* - \mathcal{V}\theta^*\,. \tag{56}$$

The time-adjoint equation has the form

$$\partial_t\theta = \gamma|\Delta|^{\mu/2}\theta + \mathcal{V}\theta\,. \tag{57}$$

We shall be particularly interested in the time-independent $\theta(x,t) \equiv \theta(x)$, an assumption affine to that involved in the passage from (46)–(48).

Hermitian fractional problems of the form (48) and/or (49) have also been studied in Refs. [14–16]. However, the major (albeit implicit, never openly stated) assumption of Refs. [14–16] was to consider the so-called step Lévy process instead of the jump-type Lévy process proper.

This amounts to introducing a lower bound on the length of admissible jumps: arbitrarily small jumps are then excluded. That allows to by-pass a serious technical obstacle. Indeed, for a pseudo-differential operator $\gamma\Delta^{\mu/2}$, the action on a function from its domain can be greatly simplified by disregarding jumps of length *not* exceeding a fixed $\epsilon > 0$, see *e.g.* Refs. [8, 9]:

$$\gamma|\Delta|^{\mu/2}f)(x) \;=\; -\int_R \left[f(x+y) - f(x) - \frac{y\,\nabla f(x)}{1+y^2}\right]\,\nu(dy)$$

$$\Downarrow$$

$$\gamma|\Delta|^{\mu/2}_{\epsilon}f)(x) \;=\; -\int_{|y|>\epsilon} [f(x+y) - f(x)]\nu(dy)\,. \tag{58}$$

Compare *e.g.* Eq. (2) in [15] and Eq. (6) in [16].

As a side comment, let us point out that the principal integral value issues of Sec. 3 would not arise in our previous discussion of Cauchy flights and their generators, if arbitrarily small jumps were eliminated from the start. Nonetheless, if the $\epsilon \downarrow 0$ limit is under control, the step process can be considered as a meaningful approximation of the fully-fledged (perturbed) jump-type Lévy process. This approximation problem has been investigated in detail, in the construction of the perturbed Cauchy process, governed by the Hermitian dynamical problem (53), with the input (55), under suitable restrictions on the behavior of $\mathcal{V}$, [9].

Let us come back to time-adjoint fractional equations (54) and (55). We have $\overline{\rho}(x,t) = (\theta\,\theta^*)(x,t)$ and employ the trial ansatz of Sec. 4.2:

$$\begin{aligned} \theta(x,t) &\equiv \theta(x) = \exp[\Phi(x)]\,, \\ \theta^*(x,t) &= \overline{\rho}(x,t)\,\exp[-\Phi(x)]\,. \end{aligned} \tag{59}$$

Accordingly (55) implies, compare *e.g.* [14] for an independent argument:

$$\mathcal{V} = -\gamma \exp(-\Phi)|\Delta|^{\mu/2}\exp(\Phi) \tag{60}$$

to be compared with Eq. (8) in Ref. [15]. In view of (54) we have

$$\partial_t\overline{\rho} = \theta\partial_t\theta^* = -\gamma\exp(\phi)[|\Delta|^{\mu/2}\exp(-\Phi)\overline{\rho}] + \mathcal{V}\overline{\rho} \doteq -\nabla j\,. \tag{61}$$

Langevin-style description of perturbed Lévy flights (deterministic component plus the Lévy noise contribution) are known, [17–19], to generate fractional Fokker–Planck equations of the form

$$\partial_t\overline{\rho} = -\nabla(F\,\overline{\rho}) - \gamma|\Delta|^{\mu/2}\overline{\rho} \doteq -\nabla j\,. \tag{62}$$

Thus we face problems which are left unsettled at the present stage of our investigation:
(i) May the stochastic processes driving (59) and/or (61) coincide under any circumstances, or basically not at all?
(ii) Give an insightful/useful definition of the probability current $j(x,t)$ in both considered cases, while remembering that for fractional derivatives the composition rule for consecutive (Riesz) derivatives typically breaks down.

Both problems *(i)* and *(ii)* have an immediate resolution in the case of diffusion-type processes, where by departing from the Langevin equation one infers Fokker–Planck and continuity equations. In turn, these equations can be alternatively derived by means of the Schrödinger boundary data problem, provided its integral kernel stems from the Schrödinger semigroup, both in the free and perturbed cases. The stochastic diffusion process (corresponding to that associated with the Langevin equation) is then reconstructed as well. Thence, the Schrödinger loop gets closed.

While passing to Lévy processes, we have demonstrated that, with suitable reservations, this Schrödinger "loop" can be completed in the case of free Lévy flights. However, the "loop" remains incomplete (neither definitely proved or disproved) for perturbed Lévy flights.

At this point we should mention clear indications [14] that, once discussing Lévy flights, we actually encounter two different classes of processes with incompatible dynamical properties. One class is related to the Langevin equation, another — termed topological — relies on the "potential landscape" provided by the effective potential $\mathcal{V}(x)$. An extended discussion of the latter problem has been postponed to the forthcoming paper, *c.f.* [23].

Partial support from the Laboratory for Physical Foundations of Information Processing is gratefully acknowledged.

REFERENCES

[1] H. Risken, *The Fokker–Planck Equation*, Sringer-Verlag, Berlin 1989.
[2] P. Garbaczewski, *Phys. Rev.* **E78**, 031101 (2008).
[3] J-C. Zambrini, *J. Math. Phys.* **27**, 2307 (1986).
[4] J.C. Zambrini, *Phys. Rev.* **A35**, 3631 (1987).
[5] M.S. Wang, *Phys. Rev.* **A37**, 1036 (1988).
[6] Ph. Blanchard, P. Garbaczewski, *Phys. Rev.* **E49**, 3815 (1994).
[7] P. Garbaczewski, R. Olkiewicz, *J. Math. Phys.* **37**, 732 (1996).
[8] P. Garbaczewski, J.R. Klauder, R. Olkiewicz, *Phys. Rev.* **E51**, 4114 (1995).
[9] P. Garbaczewski, R. Olkiewicz, *J. Math. Phys.* **40**, 1057 (1999).
[10] P. Garbaczewski, R. Olkiewicz, *J. Math. Phys.* **41**, 6843 (2000).
[11] N. Laskin, *Phys. Rev.* **E62**, 3135 (2000).
[12] N. Laskin, *Phys. Rev.* **E66**, 056108 (2002).
[13] N. Cufaro Petroni, M. Pusterla, *Physica A* **388**, 824 (2009).
[14] D. Brockmann, I. Sokolov, *Chem. Phys.* **284**, 409 (2002).
[15] D. Brockmann, T. Geisel, *Phys. Rev. Lett.* **90**, 170601 (2003).
[16] D. Brockmann, T. Geisel, *Phys. Rev. Lett.* **91**, 048303 (2003).
[17] S. Jespersen, R. Metzler, H.C. Fogedby, *Phys. Rev.* **E59**, 2736 (1999).
[18] P.D. Ditlevsen, *Phys. Rev.* **E60**, 172 (1999).
[19] V.V. Janovsky *et al., Physica A* **282**, 13 (2000).
[20] A. Chechkin *et al., Chem. Phys.* **284**, 233 (2002).
[21] A.A. Dubkov, B. Spagnolo, *Acta Phys. Pol. A* **38**, 1745 (2007).
[22] A.A. Dubkov, B. Spagnolo, V.V. Uchaikin, *Int. J. Bifurcations and Chaos,* **18**, 2549 (2008).
[23] P. Garbaczewski, `arXiv:0902.3536`.