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DIFFUSION OF BROWNIAN PARTICLES AND LIOUVILLE FIELD THEORY

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In this work we review a recently proposed transformation which is useful in order to simplify non-polynomial potential given in the form of an exponential. As an application, it is shown that the Liouville field theory may be mapped into a field theory with a polynomial interaction between two scalar fields and a massive vector field. The used methodology is illustrated with the help of the simple case of a particle performing a random walk in a delta function potentials.

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1. Introduction

The Liouville field theory (LFT) is traditionally related to string theory and to mathematical applications in the complex geometry of Riemann surfaces. It has been for years a longstanding problem whose solution has required outstanding efforts. It would be impossible to cite in this short article all the relevant works on this subject, so we just give here a limited and very incomplete list of them [1–14]. The intense research dedicated to LFT has greatly expanded our knowledge of it. Recently, the so-called three point correlation function of the theory has been explicitly derived. All the other correlation functions can be computed using a recursive algorithm, so that the LFT has finally been proved to be solvable. One of the main difficulties in treating the LFT is the fact that the potential of the theory consists of the exponential $V_{\rm L}(\phi) = \mu e^{-b\phi}$, where ϕ is the Liouville field and μ, b are constants. This kind of nonpolynomial potentials is not a problem in quantum mechanics or in the theory of the random walk, where a limited number of particles are considered. It becomes however awkward in a field theory, where there are infinite degrees of freedom. Despite this fact, it is possible to compute the vacuum expectation values of the observables of LFT using a consistent perturbative approach [14, 15]. The LFT has also applications in statistical physics. For instance, the grand canonical partition function of a particle in a 2-dimensional random potential whose correlation functions grow logarithmically with the distance coincides to the partition function of LFT [17]. Moreover, in [5, 16] the LFT has been associated to quantum chaos in anyon systems. In [18] the identity

$$\frac{(-1)^N}{(N-1)!}\int d\mu\mu^{N-1}e^{d^2x\mu e^{-b\phi}} = \frac{1}{\left(\int d^2xe^{-b\phi}\right)^N}\,,$$

has been used to simplify the normalization of certain vacuum expectation values.

Even if our present understanding of the LFT is excellent, this model continues to inspire new ideas. A fresh look at LFT comes this time from its connection to random walks. It turns out that the LFT may be "linearized", *i.e.* it is possible to map the LFT with its nonpolynomial interaction into another field theory, in which the interaction is polynomial. More specifically, it has been shown that the partition function of LFT is related to the grand canonical partition function of a system of particles performing a random walk in the presence of a disorder vector field [19]. In the equilibrium limit the two partition functions coincide. The strategy used to map the LFT into a statistical model will be explained here starting from a simple example, namely that of a random walk in a delta function potential. Other physical examples will be discussed in the next two sections. Let us note that the opposite strategy, which consists in rewriting the grand canonical partition function of particles in the form of a field theory, is known since a long time and it is now part of classical books on the subject of statistical physics, see for instance the case of particles in the Yukawa potential discussed in Ref. [20], Appendix A.27.

The presented material is divided as follows. In Section 2 we discuss the case of a particle performing a random walk in the presence of a delta function potential. In Section 3 two cases are presented, in which thanks to the introduction of auxiliary fields it has been possible to simplify complicated interactions, allowing in this way to achieve a better knowledge of polymer systems. Section 4 is dedicated to the theory of Liouville. Finally our conclusions are presented in Section 5.

2. Random walk in a delta function potential

We present in this section a simple case, which will be used as a toy model to illustrate the strategy for "linearizing" the LFT.

Let us consider the random walk of a particle in d-dimensions around an infinitely narrow and deep potential well located at the point $\boldsymbol{r}_0$. This kind of obstacles is described by the Dirac delta function potential $V(\boldsymbol{r}(t)) = -v_0\delta(\boldsymbol{r}(t) - \boldsymbol{r}_0)$, $v_0 > 0$, where $\boldsymbol{r}(t)$ is the radius vector which denotes the position of the particle at each instant t. According to the duality between statistical mechanics and quantum mechanics, the partition function $Z_\delta(T; \boldsymbol{r}_2, \boldsymbol{r}_1)$ of our particle may be formally obtained from the partition function of a quantum particle immersed in the repulsive potential $+v_0\delta(\boldsymbol{r}(t) - \boldsymbol{r}_0)$ after continuing the time to imaginary values:

$$Z_\delta(T; \boldsymbol{r}_2, \boldsymbol{r}_1) = \int\limits_{\boldsymbol{r}(0)=\boldsymbol{r}_1}^{\boldsymbol{r}(T)=\boldsymbol{r}_2} \mathcal{D}\boldsymbol{r}(t) \exp\left\{ -\int\limits_0^T dt \left(\dot{\boldsymbol{r}}^2(t) + v_0\delta(\boldsymbol{r}(t) - \boldsymbol{r}_0)\right)\right\} . \quad (1)$$

In the above equation $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the positions of the particle at the initial and final instants $t = 0$ and $t = T$, respectively. It turns out that, after a Laplace transform with respect to the time T, $Z_\delta(T; \boldsymbol{r}_2, \boldsymbol{r}_1)$ satisfies a Schrödinger-like equation which is solvable.

We will now concentrate on the delta function potential. It is not difficult to convince ourselves that this is a complicated, nonlinear potential. To this purpose, it is sufficient to write it using the Fourier representation:

$$\delta(\boldsymbol{r}(t) - \boldsymbol{r}_0) = \int d^d k e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}(t)-\boldsymbol{r}_0)} . \quad (2)$$

It is possible to linearize this potential at the price of introducing fields using the following identity:

$$e^{-\int_0^T dt v_0\delta(\boldsymbol{r}(t)-\boldsymbol{r}_0)} = \int \mathcal{D}\varphi_1(\boldsymbol{x})\mathcal{D}\varphi_2(\boldsymbol{x}) e^{-i\int d^d\boldsymbol{x}\varphi_1\left[\frac{1}{v_0}\varphi_2 - J_1\right]} e^{-\int d^d\boldsymbol{x}\varphi_2 J_2} , \quad (3)$$

where

$$J_1(\boldsymbol{x}) = \int\limits_0^T dt \delta(\boldsymbol{x} - \boldsymbol{r}(t)) \qquad \text{and} \qquad J_2(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{r}_0) . \quad (4)$$

To prove Eq. (3), we integrate over the field φ_1 in the right-hand side of Eq. (3):

$$\int \mathcal{D}\varphi_1\mathcal{D}\varphi_2 e^{-i\int d^d\boldsymbol{x}\varphi_1\left[\frac{1}{v_0}\varphi_2 - J_1\right]} e^{-\int d^d\boldsymbol{x}\varphi_2 J_2} = \int \mathcal{D}\varphi_2 \delta\left(\frac{\varphi_2}{v_0} - J_1\right) e^{-\int d^d\boldsymbol{x}\varphi_2 J_2} . \quad (5)$$

In other words, the field $\varphi_1(\boldsymbol{x})$ in Eq. (3) is just a Lagrange multiplier enforcing the condition $\varphi_2 = v_0 J_1$. A simple integration over the remaining field φ_2 in Eq. (5) gives as a result:

$$\int \mathcal{D}\varphi_2(\boldsymbol{x})\delta\left(\frac{\varphi_2}{v_0} - J_1\right) e^{-\int d^d\boldsymbol{x}\varphi_2 J_2} = e^{-\int d^d\boldsymbol{x} v_0 J_1(\boldsymbol{x})J_2(\boldsymbol{x})} . \tag{6}$$

The proof of identity (3) is completed by noticing that the external currents J_1, J_2 defined in Eq. (4) have been chosen exactly in such a way that: $\int d^d\boldsymbol{x} v_0 J_1(\boldsymbol{x})J_2(\boldsymbol{x}) = v_0 \int_0^T dt\delta(\boldsymbol{r}(t) - \boldsymbol{r}_0)$.

Using Eq. (3), one may write the partition function $Z_\delta(T;\boldsymbol{r}_2,\boldsymbol{r}_1)$ as follows:

$$\begin{aligned} Z_\delta(T;\boldsymbol{r}_2,\boldsymbol{r}_1) \; &= \int\limits_{\boldsymbol{r}(0)=\boldsymbol{r}_1}^{\boldsymbol{r}(T)=\boldsymbol{r}_2} \mathcal{D}\boldsymbol{r}(t)\mathcal{D}\varphi_1(\boldsymbol{x})\mathcal{D}\varphi_2(\boldsymbol{x})\, e^{-\int_0^T dt\left[\dot{\boldsymbol{r}}^2(t) + i\varphi_1(\boldsymbol{r}(t))\right]} \\ &\quad \times e^{-i\int d^d\boldsymbol{x}\varphi_1 \frac{1}{v_0}\varphi_2} e^{-\int d^d\boldsymbol{x}\varphi_2 J_2} . \end{aligned} \tag{7}$$

As it stands, $Z_\delta(T;\boldsymbol{r}_2,\boldsymbol{r}_1)$ describes a random walk in the imaginary random potential $i\varphi_1(\boldsymbol{r}(t))$. A more physical interpretation of $Z_\delta(T;\boldsymbol{r}_2,\boldsymbol{r}_1)$ may be achieved by considering its Laplace transform with respect to T:

$$Z_\delta(E;\boldsymbol{r}_2,\boldsymbol{r}_1) = \int\limits_0^{+\infty} dT e^{-TE} Z_\delta(T;\boldsymbol{r}_2,\boldsymbol{r}_1) . \tag{8}$$

The variable E plays the same role of the energy in quantum mechanics. For this reason, it will be called pseudo-energy. It is convenient to rewrite the expression of $Z_\delta(E;\boldsymbol{r}_2,\boldsymbol{r}_1)$ as follows:

$$\begin{aligned} Z_\delta(E;\boldsymbol{r}_2,\boldsymbol{r}_1) \; &= \int \mathcal{D}\varphi_1 \int \mathcal{D}\varphi_2 \Xi(E;\boldsymbol{r}_2,\boldsymbol{r}_1,[\varphi_1]) \\ &\quad \times e^{-i\int d^d\boldsymbol{x}\varphi_1 \frac{1}{v_0}\varphi_2}\, e^{-\int d^d\boldsymbol{x}\varphi_2 J_2} , \end{aligned} \tag{9}$$

where

$$\Xi(E;\boldsymbol{r}_2,\boldsymbol{r}_1,[\varphi_1]) \; = \int\limits_0^{+\infty} dT e^{-TE} \int\limits_{\boldsymbol{r}(0)=\boldsymbol{r}_1}^{\boldsymbol{r}(T)=\boldsymbol{r}_2} \mathcal{D}\boldsymbol{r}(t)\, e^{-\int_0^T dt\left[\dot{\boldsymbol{r}}^2(t) + i\varphi_1(\boldsymbol{r}(t))\right]} . \tag{10}$$

Now we apply to $\Xi(E;\boldsymbol{r}_2,\boldsymbol{r}_1,[\varphi_1])$ the formula:

$$\begin{aligned} \Xi(E;\boldsymbol{r}_2,\boldsymbol{r}_1,[\varphi_1]) \; &= \lim_{n\to 0} \int \mathcal{D}\vec{\phi}(\boldsymbol{x})\phi_1(\boldsymbol{r}_2)\phi_1(\boldsymbol{r}_1) \\ &\quad \times e^{-\frac{1}{2}\int d^2\boldsymbol{x}\left[(\boldsymbol{\nabla}\vec{\phi})^2 + E\vec{\phi}^2 + i\varphi_1\vec{\phi}^2\right]} . \end{aligned} \tag{11}$$

The above equation is a variant of the so-called replica trick that can be found in classical textbooks [20, 23]. The replica field $\vec{\phi}$ is a n-components vector field $\vec{\phi} = (\phi_1, \ldots, \phi_n)$. The limit $n \longrightarrow 0$ in Eq. (11) has the following meaning. The path integral in the right hand side should be computed for integer values of n such that $n \geq 1$. In this way, the field theory of Eq. (11) becomes a $O(n)$ scalar field theory and its correlation functions may be derived by standard techniques [24]. The result of such calculations is then continued analytically to any value of n. At the end, one should take the limit $n \longrightarrow 0$. Such a limit is nontrivial. It is possible to show for instance that certain discrete spin systems with $O(n)$ symmetry describe a self-avoiding random walk in the limit $n \longrightarrow 0$ [25, 26]. In the continuous case it is possible to see that many Feynman diagrams vanish in a $O(n)$ field theory when n approaches zero, but even in the one-loop approximation some of them are able to survive and deliver a finite expression of the partition function of polymer systems [27, 28].

Substituting Eq. (11) in Eq. (9) the partition function $Z_\delta(E; \boldsymbol{r}_2, \boldsymbol{r}_1)$ becomes:

$$Z_\delta(E; \boldsymbol{r}_2, \boldsymbol{r}_1) = \lim_{n \to 0} \int \mathcal{D}\vec{\phi} \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \, e^{-\frac{1}{2}\int d^d\boldsymbol{x}\left[(\boldsymbol{\nabla}\vec{\phi})^2 + E\vec{\phi}^2 + i\varphi_1\vec{\phi}^2\right]} \times e^{-i\int d^d\boldsymbol{x}\varphi_1 \frac{1}{v_0}\varphi_2} \, e^{-\int d^d\boldsymbol{x}\varphi_2 J_2} \, \phi_1(\boldsymbol{r}_1)\phi_1(\boldsymbol{r}_2) \,. \quad (12)$$

A straightforward integration over the fields φ_1, φ_2 produces the result:

$$Z_\delta(E; \boldsymbol{r}_2, \boldsymbol{r}_1) = \lim_{n \to 0} \int \mathcal{D}\vec{\phi}(\boldsymbol{x}) \, e^{-\frac{1}{2}\int d^d\boldsymbol{x}\left[(\boldsymbol{\nabla}\vec{\phi})^2 + E\vec{\phi}^2\right]} \, e^{\frac{v_0}{2}\vec{\phi}^2(\boldsymbol{r}_0)} \phi_1(\boldsymbol{r}_1)\phi_1(\boldsymbol{r}_2) \,. \quad (13)$$

Concluding, the partition function $Z_\delta(E; \boldsymbol{r}_2, \boldsymbol{r}_1)$ of the initial random walk has been transformed in a path integral describing a Gaussian field theory. It is easy to realize after taking the limit $n \longrightarrow 0$ that $Z_\delta(E; \boldsymbol{r}_2, \boldsymbol{r}_1)$ coincides with the propagator of a scalar field theory with action $S = 1/2 \int d^d\boldsymbol{x} \left[(\boldsymbol{\nabla}\phi_1)^2 + E\phi_1^2 - v_0\delta(\boldsymbol{x} - \boldsymbol{r}_0)\phi_1^2(\boldsymbol{x})\right]$. From this action it is possible to derive the differential equation satisfied by $Z_\delta(E; \boldsymbol{r}_2, \boldsymbol{r}_1)$:

$$\left[E - \varDelta_{\boldsymbol{r}_2} - v_0\delta(\boldsymbol{r}_2 - \boldsymbol{r}_0)\right] Z_\delta(E; \boldsymbol{r}_2, \boldsymbol{r}_1) = \delta(\boldsymbol{r}_2 - \boldsymbol{r}_1) \,, \quad (14)$$

$\varDelta_{\boldsymbol{r}_2}$ is the Laplacian with respect to the coordinates $\boldsymbol{r}_2$. $Z_\delta(E; \boldsymbol{r}_2, \boldsymbol{r}_1)$ satisfies also an analogous equation with $\boldsymbol{r}_2$ replaced by $\boldsymbol{r}_1$. Eq. (14) is exactly the solvable Schrödinger-like equation mentioned after Eq. (1) that appears also in other problems in which particles move in a delta function potential. For completeness, we give the solution of Eq. (14) [29]:

$$Z_\delta(E; \boldsymbol{r}_2, \boldsymbol{r}_1) = Z_0(E; \boldsymbol{r}_2, \boldsymbol{r}_1) + \frac{Z_0(E; \boldsymbol{r}_2, \boldsymbol{r}_0) Z_0(E; \boldsymbol{r}_0, \boldsymbol{r}_1)}{Z_0(E; \boldsymbol{r}_0, \boldsymbol{r}_0) - 1/v_0} \,, \quad (15)$$

where $Z_0(E;\boldsymbol{r}_2,\boldsymbol{r}_1)$ is the Green function satisfying the free equation

$$[E-\Delta_{\boldsymbol{r}_2}]\, Z_0(E;\boldsymbol{r}_2,\boldsymbol{r}_1)=\delta(\boldsymbol{r}_2-\boldsymbol{r}_1)\,. \tag{16}$$

We remark that, as it stands, the above solution is just formal, because the Green function $Z_0(E;\boldsymbol{r}_2,\boldsymbol{r}_1)$ computed at coinciding points $\boldsymbol{r}_2=\boldsymbol{r}_1=\boldsymbol{r}_0$ is singular. To eliminate this singularity a suitable prescription is necessary, see for example [21] for more information on this subject.

3. The case of polymer physics

A random flight chain composed by N segments of equal length a when N becomes large and a becomes small looks like the trajectory of a fluctuating particle whose average free path amounts to a. The random flight chain is the basic model of a polymer. For this reason, polymer physics and the theory of random walks are related. Of course, the trajectory of a particle may intersect itself many times, but this is not the case of a polymer chain due to the strong repulsions between the electron layers of the molecules composing the chain. These repulsive interactions have been modeled by de Gennes and coworkers with the help of the so-called self-avoiding potential. We give here its expression for a continuous chain:

$$V(1)=\int_0^L ds\int_0^L ds'\,\frac{v_0}{2}\delta^{(3)}(\boldsymbol{r}(s)-\boldsymbol{r}(s'))\,. \tag{17}$$

In writing the above equation the chain, which is supposed to be without branches, has been parametrized using its arc-length s. The initial and final points of the random walk $r(0)=\boldsymbol{r}_1$ and $\boldsymbol{r}(T)=\boldsymbol{r}_2$ have been replaced by the ends of the polymers $\boldsymbol{r}(0)$ and $\boldsymbol{r}(L)$. L is the total length of the chain.

As in the previous case, also the self-avoiding potential contains a delta function, which may be "linearized" applying a variant of the Hubbard–Stratonovich transformations:

$$e^{-V(1)}=\int\mathcal{D}\varphi e^{-A_\varphi(1)}\exp\left\{-i\int_0^L ds\varphi(\boldsymbol{r}(s))\right\}\,, \tag{18}$$

where the action of the φ field is given by $A_\varphi(1)=(1/2v_0)\int d^3x\varphi^2(\boldsymbol{x})$.

Another system with a complicated potential which needs simplification to make the theory tractable is that of two closed chains A and B of lengths L_A and L_B, respectively, subjected to topological constraints. The simplest topological invariant that can be used to impose topological constraints in three dimensions is the Gauss linking number $\chi(A,B)$. For two closed curves $\boldsymbol{r}_A(s)$ and $\boldsymbol{r}_B(s)$ representing the polymer chains $\chi(A,B)$ can be expressed as follows:

$$\chi(A,B) = \int_0^{L_1} ds_A \dot{\boldsymbol{r}}_A(s_A) \cdot \boldsymbol{b}_B(\boldsymbol{r}_A(s_A))\,. \tag{19}$$

Here $\boldsymbol{b}_B$ is the magnetic field generated by the fictitious current $\boldsymbol{j}_B(\boldsymbol{x}) = 1/(4\pi)\int_0^{L_B} ds_B \dot{\boldsymbol{r}}_B(s_b)\delta(\boldsymbol{x}-\boldsymbol{r}_B(s_B))$ flowing along the chain B:

$$\boldsymbol{b}_B(\boldsymbol{x}) = \frac{1}{4\pi}\int_0^{L_B} ds_B \dot{\boldsymbol{r}}_B(s_B) \times \frac{(\boldsymbol{x}-\boldsymbol{r}_B(s_B))}{|\boldsymbol{x}-\boldsymbol{r}_B(s_B)|}\,. \tag{20}$$

It is possible to check that $\boldsymbol{\nabla}\times\boldsymbol{b}_B = \boldsymbol{j}_B$ and that $\boldsymbol{\nabla}\cdot\boldsymbol{b}_B = 0$, justifying the interpretation of $\boldsymbol{b}_B$ as a magnetic field.

In the action which describes the fluctuations of the two chains A and B, the effect of the topological conditions imposed on their trajectories is the appearance of the fictitious magnetic interaction term given in equation (19). The analog of the identities (3) and (18), which simplifies in the present case the interaction of topological origin and allows to extract from the theory physical predictions [22], is given by:

$$e^{-i\chi(A,B)} = \int \mathcal{D}\boldsymbol{A}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\, e^{-iA_{CS}+\frac{1}{4\pi}\int d^3\boldsymbol{x}\boldsymbol{J}_A\cdot\boldsymbol{A}+\int d^3\boldsymbol{x}\boldsymbol{J}_B\cdot\boldsymbol{B}}\,, \tag{21}$$

where A_{CS} is the action of the Chern–Simons fields $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$A_{\mathrm{CS}} = i\int d^3\boldsymbol{x}\boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) \tag{22}$$

and

$$\boldsymbol{J}_A(\boldsymbol{x}) = \int_0^{L_A} ds_A\delta(\boldsymbol{x}-\boldsymbol{r}_A(s_A))\,, \qquad \boldsymbol{J}_B(\boldsymbol{x}) = \int_0^{L_B} ds_B\delta(\boldsymbol{x}-\boldsymbol{r}_B(s_B))\,. \tag{23}$$

4. The Liouville field theory

In this section we consider the Liouville action on the Euclidean plane:

$$Z_{\mathrm{L}} = \int \mathcal{D}\phi e^{-\int d^2x\left[\frac{1}{2}(\boldsymbol{\nabla}\phi)^2+\mu\, e^{-b(\phi(\boldsymbol{x})-\phi(\boldsymbol{r}_0))}\right]}\,. \tag{24}$$

Let us note that in the standard LFT the exponential potential is given by $e^{-b\phi(\boldsymbol{x})}$. To simplify the nonpolynomial interaction appearing in Eq. (24) it is possible to use the identity:

$$e^{-\int d^d\boldsymbol{x}\mu e^{-b(\phi(\boldsymbol{x})-\phi(\boldsymbol{r}_0))}} = \lim_{T\to+\infty} \Xi(T,[\phi])\,, \tag{25}$$

where

$$\begin{aligned}\Xi(T,[\phi]) &= \int \mathcal{D}\varphi_1(t,\boldsymbol{x}) \int \mathcal{D}\varphi_2(t,\boldsymbol{x})\, e^{-i\int d^d\boldsymbol{x}dt\left[\varphi_1\left(\frac{\partial}{\partial t}-g(\boldsymbol{\nabla}+b\nabla\phi)^2\right)\varphi_2\right]} \\ &\quad\times e^{-i\int d^d\boldsymbol{x}dt\varphi_1 J_1} e^{-\int d^d\boldsymbol{x}dt\varphi_2 J_2}\end{aligned} \tag{26}$$

and

$$J_1(t,\boldsymbol{x}) = \delta(\boldsymbol{x}-\boldsymbol{r}_0)\delta(t)\,, \qquad J_2(t,\boldsymbol{x}) = (4\pi g T)^{\frac{d}{2}}\mu\delta(T-t)\,. \tag{27}$$

This is actually a general identity which is valid in d-dimensions. The left-hand side of Eq. (25) coincides with the Liouville potential term when $d=2$.

Let us prove Eq. (25). Once again φ_1 is a Lagrange multiplier that can be easily integrated out giving as a result:

$$\Xi(T,[\phi]) = \int \mathcal{D}\varphi_2 \delta\left[\left(\frac{\partial}{\partial t} - g(\boldsymbol{\nabla}+b\boldsymbol{\nabla}\phi)^2\right)\varphi_2 - J_1\right] e^{-\int d^d\boldsymbol{x}dt\varphi_2 J_2}\,. \tag{28}$$

The functional delta function in Eq. (28) forces the condition:

$$\left[\frac{\partial}{\partial t} - g(\nabla + b\boldsymbol{\nabla}\phi)^2\right]\varphi_2 = J_1\,. \tag{29}$$

To derive the explicit expression of φ_2 we compute the Green function of the associated point source equation:

$$\left[\frac{\partial}{\partial t} - g(\nabla + b\boldsymbol{\nabla}\phi)^2\right] G(t-t';\boldsymbol{x},\boldsymbol{x}',[\phi]) = \delta(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{30}$$

The solution of Eq. (30) is:

$$G(t-t';\boldsymbol{x},\boldsymbol{x}',[\phi]) = e^{-b(\phi(\boldsymbol{x})-\phi(\boldsymbol{x}'))}\frac{\theta(t-t')}{(4\pi(t-t')g)^{\frac{d}{2}}}\, e^{-\frac{(\boldsymbol{x}-\boldsymbol{x}')^2}{4(t-t')g}}\,, \tag{31}$$

$\theta(t)$ is the Heaviside theta function. $G(t-t';\boldsymbol{x},\boldsymbol{x}',[\phi])$ is the Green function of a stochastic process describing the diffusion of a particle in a random potential $\boldsymbol{\nabla}\phi$. Knowing $G(t-t';\boldsymbol{x},\boldsymbol{x}',[\phi])$, it is possible to solve Eq. (29) as follows:

$$\varphi_2(t,\boldsymbol{x}) = \int d^d\boldsymbol{x}'dt'G(t-t';\boldsymbol{x},\boldsymbol{x}',[\phi])J_1(t',\boldsymbol{x}') \tag{32}$$

As a consequence, after the elimination of the remaining field φ_2 in Eq. (28), we obtain:

$$\Xi(T,[\phi]) = e^{-\int d^d\boldsymbol{x}dtd^d\boldsymbol{x}'dt'J_2(t,\boldsymbol{x})\,G(t-t';\boldsymbol{x},\boldsymbol{x}',[\phi])J_1(t',\boldsymbol{x}')}\,. \tag{33}$$

Recalling the expression of the Green function $G(t-t';\boldsymbol{x},\boldsymbol{x}',[\phi])$ given in Eq. (31) and the special choice of currents (27), we get:

$$\Xi(T,[\phi]) = \exp\left\{-\int d^d\boldsymbol{x}\;\, e^{-b(\phi(\boldsymbol{x})-\phi(\boldsymbol{x}'))}\,e^{-\frac{(\boldsymbol{x}-\boldsymbol{x}')^2}{4Tg}}\right\}\,. \tag{34}$$

In the limit $T \longrightarrow +\infty$ the functional $\Xi(T,[\phi])$ becomes exactly equal to the left-hand side of Eq. (25). This completes our proof.

Applying the identity in Eq. (25) for $d=2$ in order to simplify the potential of the Liouville action, the partition function of the LFT may be written as follows:

$$\begin{aligned} Z_{\mathrm{L}} \;=\; & \lim_{T\to+\infty}\int \mathcal{D}\phi(\boldsymbol{x})\mathcal{D}\varphi_1(t,\boldsymbol{x})\mathcal{D}\varphi_2(t,\boldsymbol{x})\,e^{-\frac{1}{2}d^2\boldsymbol{x}(\nabla\phi)^2} \\ & e^{-i\int d^2\boldsymbol{x}dt\left[\varphi_1\left(\frac{\partial}{\partial t}-g(\boldsymbol{\nabla}+b\nabla\phi)^2\right)\varphi_2\right]}e^{-i\int d^d\boldsymbol{x}dt\varphi_1J_1}e^{-\int d^d\boldsymbol{x}dt\varphi_2J_2}\,. \end{aligned} \tag{35}$$

This is our final result. It is possible to note that the above field theory has only polynomial interactions and it is very similar to scalar electrodynamics.

5. Conclusions

In Section 2 we have discussed in details the simple example of the random walk of a particle in a delta function potential. In Eq. (13) the partition function $Z_\delta(E;\boldsymbol{r}_2,\boldsymbol{r}_1)$ of this system has been rewritten in the form of a Gaussian field theory using the identity (3). At this point, to derive $Z_\delta(E;\boldsymbol{r}_2,\boldsymbol{r}_1)$ it is sufficient to solve the pseudo-Schrödinger equation (14). This result is in agreement with previous research on the subject, see for instance [21].

In Section 3 we have presented two other identities, Eqs. (18) and (21), in which highly nonlinear potentials have been transformed in polynomial interactions. The application of the identity of Eq. (18) together with the replica trick has brought astonishing advances in our understanding of polymer solutions and melts. The advantages of Eq. (21) in the case of polymer systems with topological constraints have been illustrated in Ref. [22].

Finally, Section 5 is dedicated to LFT. Eq. (25) is the generalization to d-dimensions of the two dimensional formula of Ref. [19]. Using the identity (25), it is possible to rewrite the LFT as a field theory with polynomial interactions which closely resembles scalar electrodynamics.

It is interesting to note the analogies and differences of our work on LFT with that of Ref. [17]. In [17] the statistical mechanics of a particle moving in a Gaussian scalar random potential ϕ has been studied. If the two point function of this random potential grows logarithmically with the distance, the grand canonical partition function of the system coincides with LFT. Eq. (35) can be regarded instead as the average over the vector random potential $\nabla\phi$ of the functional $\Xi(T,[\phi])$. As it has been shown in Ref. [19], $\Xi(T,[\phi])$ is the grand canonical partition function of a system of particles performing a random walk starting from the point $\boldsymbol{r}_0$. It is quite remarkable that, in the limit $T \longrightarrow +\infty$, the LFT is recovered. In other words, the grand canonical partition function of Ref. [17] represents the equilibrium limit of $\Xi(T,[\phi])$.

The aim of this article and of [19] was mainly to establish the identity (25). This identity has also the effect of "linearizing" the exponential potential of LFT. In this way, it becomes possible to exploit the standard techniques of field theory in the study of the Liouville model. Of course, as already mentioned in the Introduction, the LFT has been already solved with the method of conformal bootstrap. Moreover, a perturbative approach to LFT has been developed in [14, 15]. Despite that, the form of LFT obtained here makes it possible to use other techniques, such as for instance the loop expansion successfully applied to scalar electrodynamics [30]. What is even more important, the present approach is open to several extensions. While the method of conformal bootstrap is limited to two dimensions, here we are able to map in principle also the d-dimensional Liouville model into a polynomial vector field theory which the generalization of Eq. (35) to d-dimensions. Of course, one should be careful in doing that, because the Liouville field theory is nonrenormalizable in more than two dimensions. Moreover, there are also other field theories which are not exactly solvable, but have potentials of the exponential form. In that case our technique may provide an useful tool to investigate such theories.

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