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SERIES DECOMPOSITION OF FRACTIONAL BROWNIAN MOTION AND ITS LAMPERTI TRANSFORM*

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The Lamperti transformation of a self-similar process is a stationary process. In particular, the fractional Brownian motion transforms to the second order stationary Gaussian process. This process is represented as a series of independent processes. The terms of this series are Ornstein–Uhlenbeck processes if $H < 1/2$, and linear combinations of two dependent Ornstein–Uhlenbeck processes whose two dimensional structure is Markovian if $H > 1/2$. From the representation effective approximations of the process are derived. The corresponding results for the fractional Brownian motion are obtained by applying the inverse Lamperti transformation. Implications for simulating the fractional Brownian motion are discussed.

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1. Introduction

1.1. Fractional generalizations of Ornstein–Uhlenbeck process

The Ornstein–Uhlenbeck process has been introduced in [1] as the velocity process of Brownian motion following a successful application of the theory of stochastic processes in the explanation of this phenomenum. The

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progress was initiated by the ground breaking work of Einstein, Smoluchowski and Langevin that put foundation for the modern statistical mechanics, see [2–4]. The process is a solution to Langevin's equation

$$dX = -\lambda X + c^{1/2} dB ,$$

where dB is a Gaussian noise (so B is the Wiener process that nowadays is typically referred to as the Brownian motion). Recently, there is increasing interest to consider the above equation with Brownian motion B replaced by fractional Brownian motion. The resulting process could be thus naturally called fractional Ornstein–Uhlenbeck process. However, there are at least two different stochastic processes that can be called with this name. For example the Lamperti transformation of fractional Brownian motion is also occasionally referred to as the fractional Ornstein–Uhlenbeck process (see [5–7]). We refer to [6, 8–10] for more discussion of so defined fractional Ornstein–Uhlenbeck processes. We just note here that although in the special case of $H = 1/2$ these processes coincide, in general they differ. For example, the covariance function of the properly defined solutions to the Langevin equation has the asymptotics of increments of fBM given by (3) and it can be used as an argument for using the name of *fractional Ornstein–Uhlenbeck processes*. Therefore to avoid confusion of terminology, we have opted for a more descriptive name: the Lamperti transformation of fractional Brownian motion (Lt–fBm). We note also that in [11] the Lt–fBm's are termed as the stationary generators of fBm. Our main focus in this work is the representation of the Lt–fBm and fBm and although these representations can be used to analyze other versions of fractional type of Ornstein–Uhlenbeck processes we do not consider this issue.

The relation between self-similar and stationary processes suggests the theoretical utility of working with a self-similar process through its Lamperti transform by applying to the latter well-developed methods for stationary processes. Such an approach has been taken to obtain linear estimation of self-similar processes in [8]. This paper utilizes the same approach to obtain a convenient series representation of fractional Brownian motion. The obtained series representation differ in their structure from the ones of [12] or [13], where the representations are in terms of deterministic functions with random independent coefficients. In contrast, the terms in our representation are independent Markovian processes that are rescaled Brownian motions in the case $H < 1/2$ or combinations of two dependent and rescaled Brownian motions in the case $H > 1/2$.

1.2. Fractional Brownian motion

Fractional Brownian motions are extensions of Brownian motion that preserve homogeneity of increments, self-similarity and normality of distributions — the properties that uniquely characterize them — but have more complex dependence structure, including long-range dependence [see [14]]. These properties and the rich mathematical theory that is built upon them incited a growing interest in applications of fractional Brownian motion to stochastic modeling in hydrology, optics and theory of turbulence, finance, and telecommunication networks [see, for example, [15, 16]]. Self-similarity and dependence structure are the prominent features of all these models and this work provides another look at both. We represent fractional Brownian motion as series of independent self-similar processes that have simpler dependence structure that is directly derived from the independent increments of the standard Brownian motion. We arrive to this by studying and analyzing stationary Gaussian processes that are obtained by the Lamperti transformation of fractional Brownian motions. These processes are interesting on their own terms and the obtained series representation "disects" their dependence structure into simpler ones of the independent terms. Resulting Gaussian approximations converge fast in the mean square error and at the same time preserve the dependence rate.

Let us recall that a real valued Gaussian process $\{B_H(t), t \geq 0\}$, with $0 < H \leq 1$ is called *fractional Brownian motion (fBm)* if $B_H(0) = 0$, $E(B_H(t)) = 0$ and

$$\mathbb{E}(B_H(t)B_H(s)) = \frac{\sigma^2}{2}\left(t^{2H} + s^{2H} - (t-s)^{2H}\right), \qquad t, s > 0\,. \tag{1}$$

The fBm has stationary increments and is self-similar with self-similarity (Hurst) parameter H (H-ss). Recall that a stochastic process $\{X(t), t \geq 0\}$ is H-ss if for each $a \geq 0$:

$$X(at) \stackrel{\mathrm{d}}{=} a^H X(t), t \geq 0\,, \tag{2}$$

in the sense that all the finite dimensional distributions of the two processes are the same. In the special case of $H = 1/2$ the fBm reduces to the classical Brownian motion. Another important property of fBm is that its increments exhibit *long-range dependence*. We say that a discrete-time, zero mean stationary process $Z_k, k \in \mathbb{Z}_+$ exhibits long-range dependence if the covariance function $\gamma(k) = \mathbb{E}(Z_i Z_{i+k})$ tends to zero so slowly that the series $\sum_{k=1}^{\infty} \gamma(k)$ diverges. In the case of the fBm, the increment process $\{Z_k = B_H(k+1) - B_H(k), k \in \mathbb{Z}_+\}$, called the fractional Gaussian noise, exhibits long-range dependence for $H \in (1/2, 1)$, since its covariance function satisfies

$$\gamma(k) \sim \sigma^2 H(2H-1)k^{2H-2} \text{ as } k \to \infty\,. \tag{3}$$

For more details we refer to [17], where a comprehensive survey of results on the fBm is presented.

1.3. Lamperti transform

There is a one-to-one correspondence between H-ss processes and strictly stationary processes through a nonlinear deterministic time-scale change called the Lamperti transformation. More specifically, if $\{X(t), t \in \mathbb{R}\}$ is a strictly stationary process and if for some $H > 0$, we let

$$Y(t) = t^H X(\log t), \text{for } t > 0\,; \qquad Y(0) = 0\,,$$

then $\{Y(t), t \geq 0\}$ is H-ss. Conversely, if $\{Y(t), t \geq 0\}$ is H-ss and if we let

$$X(t) = e^{-tH} Y(e^t)\,, \qquad t \in \mathbb{R}\,,$$

then $\{X(t), t \in \mathbb{R}\}$ is strictly stationary [see [14] for more on the theory of self-similar processes]. We refer to $X(t)$ as the *Lamperti transform* of $Y(t)$.

Let $\{B_H(t), t \geq 0\}$ be a fBm, which is H-ss. Then its Lamperti transform $\{L_H(t), t \in \mathbb{R}\}$ is a zero mean Gaussian strictly stationary process, and its covariance is

$$\begin{aligned} R_H(t,s) &= \sigma^2 \left(\cosh((t-s)H) - 2^{2H-1} \sinh^{2H} \left(\frac{t-s}{2} \right) \right) \\ &= \sigma^2 \left(\cosh((t-s)H) - 2^{H-1} |\cosh(t-s) - 1|^H \right), \end{aligned}$$

with the spectral density

$$S_H(\omega) = \frac{\sigma^2 \sin(\pi H) \Gamma(2H+1) |\Gamma(1-H+i\omega)|^2}{2\pi |\Gamma(\frac{1}{2}+i\omega)|^2 (H^2+\omega^2)}\,.$$

[See [8].] Following our terminology this stationary process is referred to as the Lamperti transform of fBm (Lt–fBm). The rate of dependence measured by $R_H(k) = R_H(t, t+k)$ is exponential and proportional to $e^{-k(H \wedge (1-H))}$. We observe that the Lt–fBm does not inherit the dependence rate exhibited by increments of the corresponding fBm. In particular, we do not have long-range dependence of the Lt–fBm for any value of H. If $H = 1/2$, then $R_H(t,s) = \sigma^2 \exp(-|t-s|/2)$, *i.e.*, the covariance of the Ornstein–Uhlenbeck (O–U) process.

2. Series decomposition of Lt–fBm

2.1. Lt–fBm as a series of O–U processes

We call a stationary process $X(t)$ a series of independent moving averages

$$U_n(t) = \int_{-\infty}^{\infty} f_n(s-t)dB_n(s)\,,$$

where $f_n(s) \in \mathbb{L}_2(\mathbb{R})$, $\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f_n^2(s)ds < \infty$ and $B_n(s)$'s are independently scattered Gaussian measures controlled by the Lebesgue measure (called from now on standard Brownian measures), if

$$X(t) \stackrel{\mathrm{d}}{=} \sum_{n=1}^{\infty} U_n(t)\,, \qquad t > 0\,.$$

Note that convergence of the above series is in the quadratic mean sense and the resulting process is Gaussian with the covariance

$$R_X(t) = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f_n(s-t)f_n(s)ds\,.$$

In particular, the variance is given by $\mathbb{V}\mathrm{ar}\,(X(t)) = \sum_{n=1}^{\infty}\mathbb{V}\mathrm{ar}\,(U_n(t)) = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f_n^2(s)ds$.

Example 1 *A special case of moving averages are the classical Ornstein–Uhlenbeck (O–U) processes which for $\alpha \in \mathbb{R}$, $\beta > 0$ can be expressed as the following moving averages*

$$\begin{aligned} U(t;\alpha,\beta) &\stackrel{\mathrm{d}}{=} \alpha\int_{t}^{\infty} e^{\beta(t-s)}dB(s) \\ &\stackrel{\mathrm{d}}{=} \alpha\int_{-\infty}^{t} e^{-\beta(t-s)}dB(s)\,. \end{aligned} \tag{4}$$

The series $X(t) = \sum_{n=1}^{\infty} U_n(t;\alpha_n,\beta_n)$ of independent O–U processes is well defined if the numerical series $\sum_{n=1}^{\infty}\alpha_n^2/\beta_n$ is convergent. Moreover, a stationary Gaussian process is uniquely identified as a series of the O–U processes if its covariance has the form

$$R(t) = \sum_{n=1}^{\infty}\frac{\alpha_n^2}{2\beta_n}e^{-\beta_n|t|}\,. \tag{5}$$

If there is a minimal value β among β_n's then the rate of decay of covariance is proportional to $e^{-\beta t}$. Further, we have the following

$$\sum_{n=1}^{\infty} U_n(t;\alpha_n,\beta) \stackrel{\mathrm{d}}{=} U(t;\alpha,\beta)\,, \tag{6}$$

where $\alpha^2 = \sum_{n=1}^{\infty} \alpha_n^2$.

Note that the covariance function of the Lt–fBm, for $t > 0$ can be written as

$$\begin{aligned} R(t) &= \frac{\sigma^2}{2}\left[e^{tH} + e^{-tH} - \left(e^{t/2} - e^{-t/2}\right)^{2H}\right] \\ &= \frac{\sigma^2}{2}\left[e^{tH}\left(1 + e^{-2tH} - \left(1 - e^{-t}\right)^{2H}\right)\right] \\ &= \frac{\sigma^2}{2}\left[e^{tH}\left(1 + e^{-2tH} - \sum_{n=0}^{\infty}\binom{2H}{n}\left(-e^{-t}\right)^n\right)\right] \\ &= \frac{\sigma^2}{2}\left[e^{tH}\left(1 + e^{-2tH} - 1 - \sum_{n=1}^{\infty}\binom{2H}{n}\left(-e^{-t}\right)^n\right)\right] \\ &= \frac{\sigma^2}{2}\left[e^{-tH} + \sum_{n=1}^{\infty}\binom{2H}{n}(-1)^{n-1}e^{-t(n-H)}\right], \end{aligned} \tag{7}$$

where $\binom{\alpha}{n} \stackrel{\text{def}}{=} \frac{\alpha\cdot(\alpha-1)\cdots(\alpha-n+1)}{n!}$ for $n \in \mathbb{N}$ and $\binom{\alpha}{0} \stackrel{\text{def}}{=} 1$. For $H \in (0, 1/2]$, the terms in the above series are all positive so the Lt–fBm is a series of O–U processes as specified in the following result.

Theorem 1 *If $H \in (0, 1/2]$ a Lt–fBm $X(t)$ can be represented as the following series*

$$X(t) = \sum_{n=1}^{\infty} U_n(t;\alpha_n,\beta_n)\,, \qquad t > 0\,, \tag{8}$$

where $U_n(t;\alpha_n,\beta_n)$ are O–U processes with

$$\alpha_n^2 = \sigma^2(-1)^n\binom{2H}{n-1}(n-H-1)\,, \qquad \beta_n = |n-H-1|\,.$$

Proof. Notice if $H \in (0, 1/2]$, then

$$\begin{aligned} \binom{2H}{n}(-1)^{n-1} &= \frac{2H(2H-1)\cdots(2H-n+1)}{n!}(-1)^{n-1} \\ &= 2H\left(1 - \frac{2H+1}{2}\right)\left(1 - \frac{2H+1}{3}\right)\cdots\left(1 - \frac{2H+1}{n}\right) \end{aligned}$$

with all factors non-negative (positive if $H < 1/2$). Thus the result follows from (5) and from the fact that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\binom{2H}{n} = 1 < \infty.$$

$\square$

Remark 1 *It follows from the general theory of stochastic processes that the convergence of the series is also uniform with probability one on any interval* $[0,T]$ *[see [18, 19]].*

Remark 2 *The following simple recurrent relations are useful in numerical computations of the coefficients in the above series*

$$\begin{aligned} \alpha_1^2 &= \sigma^2 H\,, \quad \alpha_{n+1}^2 = \left(1-\frac{H}{n}\right)\left(1-\frac{H}{n-H-1}\right)\alpha_n^2\,, \quad n \geq 1\,,\\ \beta_1 &= H\,, \qquad\quad \beta_2 = 1-H\,, \quad \beta_{n+1} = \beta_n + 1\,, \quad n \geq 2\,. \end{aligned}$$

We notice that the sequence of β_n's is increasing and thus the first term in the representation (8) holds the rate of dependence of $X(t)$, *i.e.*, $e^{-H\tau}$, the rate for the second term is $e^{-(1-H)\tau}$, and the rate for each subsequent term is decreasing by $e^{-\tau}$. We conclude that the initial terms of the representation are representing most of the dependence structure of the Lt–fBm.

2.2. Series decomposition in the case $\boldsymbol{H > 1/2}$

In the case $H \in (1/2, 1)$, the terms in (7) for $n \geq 2$ are negative and thus a moving average series representation is less straightforward.

We start with a discussion of the structure of a pair of dependent O–U processes. In what follows, vectors will be identified with one-column matrices so

$$(x_1, x_2) \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Also without losing generality, the time parameters will be assumed all non-negative.

Lemma 1 *Let* $\boldsymbol{U}(t) = (U(t;1,\beta), U(t;1,\gamma\beta))$, $\beta > 0$, $\gamma > 0$ *and* $U(t;1,\beta)$ *is an O–U process defined by (1). Then the process* $\boldsymbol{U}$ *is stationary and its covariance matrix function* $\boldsymbol{R_U}(h) = \mathbb{E}\left(\boldsymbol{U}(t)\boldsymbol{U}^T(t+h)\right)$ *has, for* $h > 0$, *the form*

$$\boldsymbol{R_U}(h) = \frac{e^{-\beta h}}{2\beta}\begin{bmatrix} 1 & \frac{2}{1+\gamma}e^{-(\gamma-1)\beta h} \\ \frac{2}{1+\gamma} & \frac{1}{\gamma}e^{-(\gamma-1)\beta h} \end{bmatrix}. \tag{9}$$

Moreover, $\boldsymbol{U}$ is a Markov process such that $\boldsymbol{U}(t+h)$ given $\boldsymbol{U}(t) = \boldsymbol{u}_0$ can be represented in distribution as the process in $h > 0$:

$$\boldsymbol{U}_{\boldsymbol{u}_0}(h) = \boldsymbol{A}(h)\big(\boldsymbol{u}_0 + \boldsymbol{B}(h)\boldsymbol{Y}(h)\big)\,, \tag{10}$$

where $\boldsymbol{A}(h)$ and $\boldsymbol{B}(h)$ are non-random matrices given by

$$\begin{aligned} \boldsymbol{A}(h) &= \begin{bmatrix} e^{-\beta h} & 0 \\ 0 & e^{-\gamma\beta h} \end{bmatrix}, \\ \boldsymbol{B}(h) &= \begin{bmatrix} 1 & 0 \\ \frac{2}{1+\gamma}\frac{e^{\beta(1+\gamma)h}-1}{e^{2\beta h}-1} & 1 \end{bmatrix} \end{aligned}$$

and the vector Gaussian process $\boldsymbol{Y}(h) = (Y_1(h), Y_2(h))$ has for each fixed $h > 0$ independent coordinates and is represented in distribution as

$$\boldsymbol{Y}(h) = \left(\int_0^h e^{\beta z} dB(z), \int_0^h e^{\beta\gamma z} - \frac{2}{1+\gamma}\frac{e^{\beta(1+\gamma)h} - 1}{e^{2\beta h} - 1} e^{\beta z} dB(z) \right).$$

Proof. For $h \geq 0$

$$\mathbb{E}\left[U_1(t+h)U_2(t)\right] = \int_{t+h}^{\infty} e^{\beta(1+\gamma)t + \beta h - \beta(1+\gamma)s} ds = \frac{1}{\beta(1+\gamma)} e^{-\gamma\beta h}\,. \tag{11}$$

Similarly, we find that

$$\mathbb{E}\left[U_1(t)U_2(t+h)\right] = \frac{1}{\beta(1+\gamma)} e^{-\beta h}\,. \tag{12}$$

The formulas for $\mathbb{E}[U_1(t+h)U_1(t)]$ and $\mathbb{E}[U_2(t+h)U_2(t)]$ follow from (11) and (12) for $\gamma = 1$.

The process $\boldsymbol{U}(t)$ can be equivalently represented as

$$\left(\int_{-\infty}^{t} e^{-\beta(t-s)} dB(s), \int_{-\infty}^{t} e^{-\beta\gamma(t-s)} dB(s) \right)$$

which shows more naturally the dependence on its past and this representation is used throughout the rest of the proof.

The Markov property follows easily from independence of integrals with respect to the independently scattered measure over disjoint sets and the following representation

$$\boldsymbol{U}(t+h) = \begin{bmatrix} e^{-\beta h} & 0 \\ 0 & e^{-\gamma\beta h} \end{bmatrix} \left(\begin{bmatrix} \int_t^{t+h} e^{-\beta(t-s)} dB(s) \\ \int_t^{t+h} e^{-\gamma\beta(t-s)} dB(s) \end{bmatrix} + \boldsymbol{U}(t) \right).$$

Then (10) can be derived from the above by applying orthogonalization to the two variables $\int_0^h e^{\beta s} dB(s)$ and $\int_0^h e^{\gamma\beta s} dB(s)$ and using the independence of the vector Gaussian process

$$\boldsymbol{U}(t, t+h) = \left(\int_t^{t+h} e^{-\beta(t-s)} dB(s), \int_t^{t+h} e^{-\gamma\beta(t-s)} dB(s) \right)$$

to $\boldsymbol{U}(t) = \boldsymbol{u}_0$. This leads to the following equality in distribution

$$\begin{bmatrix} \int_t^{t+h} e^{-\beta(t-s)} dB(s) \\ \int_t^{t+h} e^{-\gamma\beta(t-s)} dB(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{1+\gamma} \frac{e^{\beta(1+\gamma)h}-1}{e^{2\beta h}-1} & 1 \end{bmatrix} \boldsymbol{Y}(h),$$

where

$$\boldsymbol{Y}(h) = \left(\int_0^h e^{\beta z} dB(z), \int_0^h e^{\beta\gamma z} - \frac{2}{1+\gamma} \frac{e^{\beta(1+\gamma)h} - 1}{e^{2\beta h} - 1} e^{\beta z} dB(z) \right).$$

This concludes the proof.

□

Example 2 *Let us consider the following difference of the O–U processes (1):*

$$D(t;\alpha,\beta,\gamma) = U(t;\alpha,\beta) - U(t;\alpha,\gamma\beta), \qquad \gamma > 0. \tag{13}$$

Then this process is a stationary Gaussian moving average

$$D(t;\alpha,\beta,\gamma) = \alpha \int_t^{\infty} e^{\beta(t-s)} - e^{\gamma\beta(t-s)} dB(s)$$

and its covariance function follows directly from Lemma 1.

Corollary 1 *The covariance function of the process $D(\cdot;\alpha,\beta,\gamma)$ defined in (13) is given by*

$$R_D(t;\alpha,\beta,\gamma) = \frac{\alpha^2}{2\beta}e^{-\beta t}\frac{\gamma-1}{\gamma+1}\left(1-\frac{1}{\gamma}e^{-\beta(\gamma-1)t}\right).$$

Proof. It is easy to notice that the covariance function will be of the form

$$R_D(t;\alpha,\beta,\gamma) = \boldsymbol{v}^T\boldsymbol{R_U}(t)\boldsymbol{v}\,,$$

where $\boldsymbol{v} = (\alpha,-\alpha)$.

□

In particular, $\mathbb{V}\mathrm{ar}\,D(t;\alpha,\beta,\gamma) = \frac{\alpha^2}{2\beta}\frac{(\gamma-1)^2}{\gamma(\gamma+1)}$. Consequently, the mixture of such processes $X(t) = \sum_{n=1}^{\infty} D_n(t;\alpha_n,\beta_n,\gamma_n)$ is well defined if

$$\sum_{n=1}^{\infty}\frac{\alpha_n^2}{\beta_n(1\wedge\gamma_n)} < \infty\,. \tag{14}$$

The next result shows that for $H > 1/2$ the Lt–fBm can be represented as a series of moving averages that are differences of dependent O–U processes as in (13).

Theorem 2 *For $H \in (1/2,1)$ a Lt–fBm $X(t)$ can be represented as the following series of the moving averages:*

$$X(t) = \sum_{n=1}^{\infty} D_n(t;\alpha_n,H,\gamma_n) + \sum_{n=1}^{\infty} D'_n(t;\alpha'_n,1-H,\gamma'_n)\,, \tag{15}$$

where $D_n(t;\alpha_n,H,\gamma_n)$ and $D'_n(t;\alpha'_n,1-H,\gamma'_n)$, are mutually independent differences of dependent O–U processes as defined in (13) with $\gamma_n = \frac{n+1-H}{H}$, $\gamma'_n = \frac{n+1-H}{1-H}$ and

$$\begin{aligned}
\alpha_n^2 &= \frac{\sigma^2}{3-2H}(-1)^{n+1}\binom{2H}{n+1}\frac{(n+1)(n+1-H)}{n+1-2H}\,,\\
{\alpha'_n}^2 &= \frac{2\sigma^2(1-H)}{3-2H}(-1)^{n+1}\binom{2H}{n+1}\frac{(n+2-2H)(n+1-H)}{n}\,.
\end{aligned}$$

Proof. Let us start by noticing that $\gamma_n > 1$ and

$$\max(\alpha_n^2,{\alpha'_n}^2) \le 16\sigma^2(-1)^{n-1}\binom{2H-1}{n}$$

thus by (14) both series of moving averages are well-defined and almost surely convergent in the supremum norm over compact sets, [see [18] and [19]]. We need to show that

$$R_X(t) = \sum_{n=1}^{\infty} \left(R_{D_n}(t) + R_{D'_n}(t) \right) . \tag{16}$$

Notice also that the covariance function of X may be rewritten as

$$R_X(t) = \frac{\sigma^2}{2} \left[e^{-Ht} + 2He^{-(1-H)t} + \sum_{n=1}^{\infty} \binom{2H}{n+1} (-1)^n e^{-(n+1-H)t} \right] . \tag{17}$$

The third term in (17), may be written as

$$\begin{aligned} &\sum_{n=1}^{\infty} \binom{2H}{n+1} (-1)^n e^{-(n+1-H)t} \\ &= \sum_{n=1}^{\infty} \frac{\binom{2H}{n+1}(-1)^n}{\left(\frac{1}{2\gamma_n} - \frac{1}{\gamma_n+1}\right)} \left[\left(\frac{1}{2\gamma_n} - \frac{1}{\gamma_n + 1} \right) e^{-(n+1-H)t} \right. \\ &+ \left. \left(\frac{1}{2} - \frac{1}{\gamma_n + 1} \right) e^{-Ht} - \left(\frac{1}{2} - \frac{1}{\gamma_n + 1} \right) e^{-Ht} \right] \\ &= \sum_{n=1}^{\infty} \frac{\binom{2H}{n+1}(-1)^n}{\frac{\alpha_n^2}{H}\left(\frac{1}{2\gamma_n} - \frac{1}{\gamma_n+1}\right)} \left\{ R_{D_n}(t) - \frac{\alpha_n^2}{H} \left(\frac{1}{2} - \frac{1}{\gamma_n + 1} \right) e^{-Ht} \right\} . \end{aligned} \tag{18}$$

By the same argument with $\alpha_n, \beta_n = H, \gamma_n$ replaced by $\alpha'_n, \beta'_n = 1 - H$ and γ'_n we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} \binom{2H}{n+1} (-1)^n e^{-(n+1-H)t} \\ &= \sum_{n=1}^{\infty} \frac{\binom{2H}{n+1}(-1)^n}{\frac{\alpha_n'^2}{1-H}\left(\frac{1}{2\gamma'_n} - \frac{1}{\gamma'_n+1}\right)} \left\{ R_{D'_n}(t) - \frac{\alpha_n'^2}{1-H} \left(\frac{1}{2} - \frac{1}{\gamma'_n + 1} \right) e^{-(1-H)t} \right\} . \end{aligned} \tag{19}$$

Now, consider the following expression

$$\sum_{n=1}^{\infty}\binom{2H}{n+1}(-1)^n e^{-(n+1-H)t}$$
$$=\frac{1}{3-2H}\sum_{n=1}^{\infty}\frac{\binom{2H}{n+1}(-1)^n}{\frac{\alpha_n^2}{H}\left(\frac{1}{2\gamma_n}-\frac{1}{\gamma_n+1}\right)}\left[R_{D_n}(t)-\frac{\alpha_n^2}{H}\left(\frac{1}{2}-\frac{1}{\gamma_n+1}\right)e^{-Ht}\right]$$
$$+\frac{2(1-H)}{3-2H}\sum_{n=1}^{\infty}\frac{\binom{2H}{n+1}(-1)^n}{\frac{\alpha_n'^2}{1-H}\left(\frac{1}{2\gamma_n'}-\frac{1}{\gamma_n'+1}\right)}$$
$$\times\left[R_{D_n'}(t)-\frac{\alpha_n'^2}{1-H}\left(\frac{1}{2}-\frac{1}{\gamma_n'+1}\right)e^{-(1-H)t}\right]. \tag{20}$$

Substitution of $\alpha_n, \alpha_n', \gamma_n$ and γ_n' in (20) and the following identities

$$\sum_{n=1}^{\infty}\binom{2H}{n+1}(-1)^n = -(2H-1)\,,$$

$$\sum_{n=1}^{\infty}\binom{2H}{n+1}(-1)^n(n+1) = 2H\sum_{n=1}^{\infty}(-1)^n\binom{2H-1}{n} = -2H$$

gives us

$$\sum_{n=1}^{\infty}\binom{2H}{n+1}(-1)^n e^{-(n+1-H)t} = \frac{2}{\sigma^2}\sum_{n=1}^{\infty}\left(R_{D_n}(t)+R_{D_n'}(t)\right) -e^{-Ht}-2He^{-(1-H)t}\,,$$

which is equivalent to (16).

□

2.3. *Approximations of Lt–fBm*

We start with a discussion of the case $H < 1/2$. We have seen in Subsection 2.1 that the Lt–fBm $X(t)$ can be approximated by $X_N(t) = \sum_{n=1}^{N} U_n(t;\alpha_n,\beta_n)$, $t > 0$. Here we investigate how accuracy of this approximation depends on N. The O–U process $U(t) = U(t;\alpha,\beta)$, the only one-dimensional stationary Gaussian and Markov process, has the transition distribution of $U(t+h)$, $h > 0$, given past up to time t and $U(t) = u_0$ represented as

$$U_{u_0}(h) = e^{-\beta h}(u_0 + Y(h))\,, \tag{21}$$

where $Y(h) = \alpha \int_0^h e^{\beta s} dB(s)$. From that the simulation of $U(t)$ at the discrete time points $0 < t_1 < \cdots < t_n$ is straightforward. Let $h_i = t_{i+1} - t_i$, $i = 1, \ldots, n-1$ and $z_1, z_2, \ldots, z_n$ be simulated values of independent standard normal variables. Define $u_1 = \sqrt{\frac{\alpha^2}{2\beta}} z_1$ and, for $i = 2, \ldots, n$, $y_i = \sigma(h_{i-1}) z_i$, where $\sigma(h_i) = \sqrt{\frac{\alpha^2}{2\beta}(e^{2\beta h_i} - 1)}$. By the recursive formula that follows from (21), the values $(u_1, \ldots, u_n)$ defined by

$$u_i = e^{-\beta h_{i-1}} (u_{i-1} + y_i) \ , \qquad i = 2, \cdots, n \, ,$$

constitute a sample from the distribution of $(U(t_1), \ldots, U(t_n))$. Samples generated using this algorithm evaluated at 500 equally spaced points over the interval $[0, 1]$ are presented in figure 1.

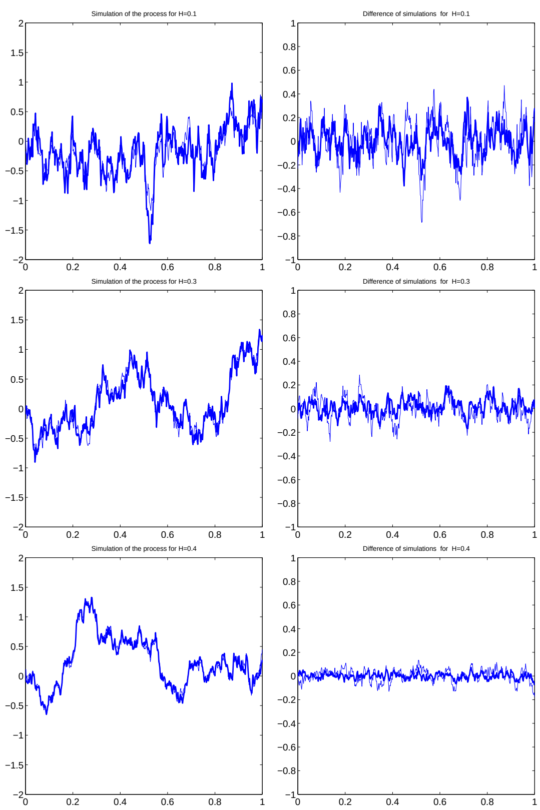

Fig. 1. Trajectories of approximations X_N, for $H = 0.1, 0.3, 0.4$ (from top to bottom) at 500 equally spaced points. Left: approximations for $N = 50$ (thin line) $N = 150$ (thick line); Right: the differences between approximation for $N = 150$ and the ones for: $N = 50$ (thin line) and $N = 100$ (thick line).

We see that the accuracy of approximation is improving as H approaches $1/2$ and generally is very good as the trajectories for $N = 50$ (thin line) and $N = 150$ (thick line) are hardly distinguishable. A better insight into it can be obtained by studying the mean square error of $X(t) - X_N(t)$. By the hypergeometric Gauss identity (see the Appendix) we have

$$\begin{aligned} MSE_N(H) &\stackrel{\text{def}}{=} \mathbb{E}\,(X(t) - X_N(t))^2 \\ &= \frac{\sigma^2}{2} \sum_{n=N+1}^{\infty} (-1)^n \binom{2H}{n-1} \\ &= \frac{\sigma^2}{2} \sum_{k=0}^{\infty} (-1)^{k+N-1} \binom{2H}{k+N} \\ &= (-1)^{N-1} \frac{\sigma^2}{2} {}_2F_1 \left[\begin{matrix} N-2H \;\; 1 \\ N+1 \end{matrix} ; 1 \right] \\ &= (-1)^{N-1} \frac{\sigma^2}{2} \binom{2H-1}{N-1}. \end{aligned}$$

The dependence of the rate of convergence on the value of $H < 1/2$ can be seen better from the following inequalities (see Lemma 4 of the Appendix)

$$\begin{aligned} \left(1 - \frac{2H}{N}\right)^{N-2H} \frac{(1-2H)^{2H}}{2N^{2H}} &\leq \frac{MSE_{N+1}(H)}{\sigma^2} \\ &\leq \left(1 - \frac{2H}{N}\right)^{N-2H} \frac{(1-2H)^{2H-1}}{2(N+1)^{2H}}. \quad (22) \end{aligned}$$

Thus the rate of convergence is of the order of the power function N^{-2H}. This sort of dependence of the convergence rate on the Hurst parameter should be expected as the smaller H, the more negatively correlated is the underlying fractional Brownian motion, consequently, trajectories are more variable and thus harder to approximate. In figure 2, we have plotted the mean square error (MSE) for different values of H and N.

We turn now to the case of $H > 1/2$. First notice that the variances of the Gaussian variables $Y_1(h)$ and $Y_2(h)$ in Lemma 1 are given by

$$\sigma_1^2(h) = \frac{1}{2\beta}(e^{2\beta h} - 1) = \frac{e^{\beta h}}{\beta} \sinh(\beta h), \quad (23)$$

$$\begin{aligned} \sigma_2^2(h) &= \frac{1}{2\beta} \left(\frac{e^{2\beta\gamma h} - 1}{\gamma} - \frac{4}{(1+\gamma)^2} \frac{\left(e^{\beta(1+\gamma)h} - 1\right)^2}{e^{2\beta h} - 1} \right) \\ &= \frac{e^{\beta\gamma h}}{\beta} \left(\frac{\sinh \beta\gamma h}{\gamma} - \frac{4}{(1+\gamma)^2} \frac{\sinh^2 \beta(1+\gamma)h/2}{\sinh \beta h} \right). \quad (24) \end{aligned}$$

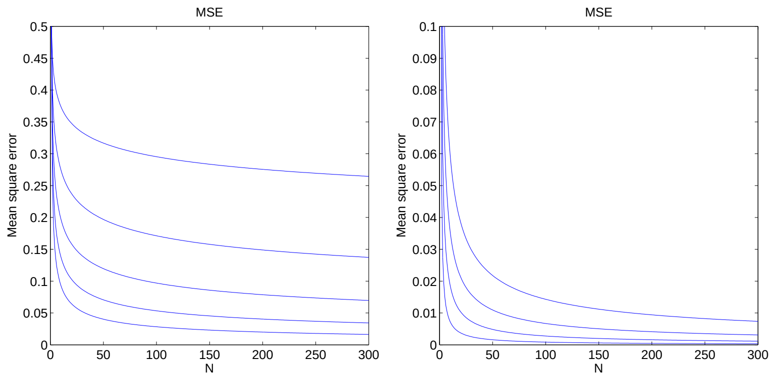

Fig. 2. The mean square error $MSE_N(H)$ for $H = 0.05, 0.1, 0.15, 0.2, 0.25$ (left) and $0.3, 0.35, 0.4, 0.45$ (right), $N = 1, \dots, 300$. Larger MSE values correspond to smaller H.

For arbitrarily chosen points $t_0 = 0 < t_1 < \cdots < t_n$ we describe a method of obtaining a sample $(\boldsymbol{u}_i; i = 0, \dots, n)$ from the distribution of $(\boldsymbol{U}(t_i); i = 0, \dots, n)$, where $\boldsymbol{U}(t)$ is the process of Lemma 1.

Let $h_i = t_i - t_{i-1}$, $i = 1, \dots, n$. First generate a vector value $\boldsymbol{u}_0$ from the two-dimensional Gaussian distribution with mean zero and covariance matrix

$$\frac{1}{\beta} \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{1+\gamma} \\ \frac{1}{1+\gamma} & \frac{1}{2\gamma} \end{array} \right].$$

Then let $z_{11}, \dots, z_{1n}$ and $z_{21}, \dots, z_{2n}$ be simulated values of mutually independent standard normal variables. Define the vector values $\boldsymbol{y}_i$ by

$$\boldsymbol{y}_i = (\sigma_1(h_i) z_{1i}, \sigma_2(h_i) z_{2i}) ,$$

i.e. obtain independent values from the distributions of $\boldsymbol{Y}(t_i - t_{i-1})$, $i = 1, \dots, n$.

Then define the value $\boldsymbol{u}_1$ that corresponds to value of $\boldsymbol{U}(t_1)$ conditionally on $\boldsymbol{U}(t_0) = \boldsymbol{u}_0$ as

$$\boldsymbol{u}_1 = \boldsymbol{A}(h_1) \left(\boldsymbol{u}_0 + \boldsymbol{B}(h_1) \boldsymbol{y}_1 \right) ,$$

and by the recursion that follows from the Markov property

$$\boldsymbol{u}_i = \boldsymbol{A}(h_i) \left(\boldsymbol{u}_{i-1} + \boldsymbol{B}(h_i) \boldsymbol{y}_i \right) , \qquad i = 2, \dots, n ,$$

where the matrices $\boldsymbol{A}(h)$, $\boldsymbol{B}(h)$ are defined in Lemma 1. Samples of $\boldsymbol{U}(t_i)$, generated using this algorithm are presented in figure 3. This time the trajectories are over the interval $[0, 10]$ to see better the difference in the time dependence structures. In these graphs we have used the relation

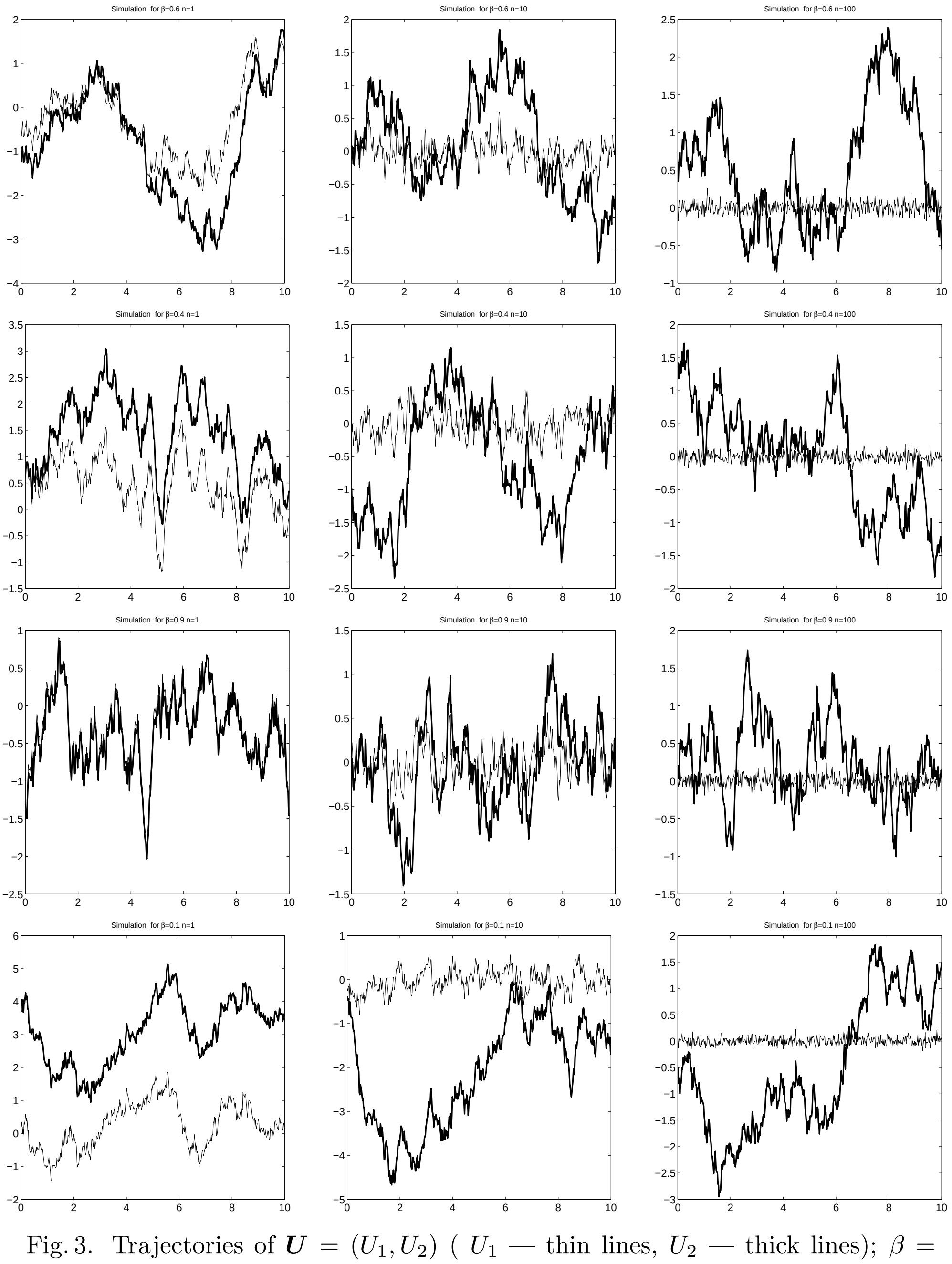

Fig. 3. Trajectories of $\boldsymbol{U} = (U_1, U_2)$ (U_1 — thin lines, U_2 — thick lines); $\beta = 0.6, 0.4, 0.9, 0.1$ (from top to bottom), γ_n, $n = 1, 10, 100$ (from left to right).

$\gamma_n = (n+1)/\beta - 1$, if $\beta \in (0.5, 1)$ and $\gamma_n = n/\beta + 1$, if $\beta \in (0, 0.5)$ that is valid for the terms of the representation (15). We observe that when n and thus γ_n increases the correlation between coordinates of $\boldsymbol{U}(t; \beta, \gamma) = \boldsymbol{U}(t)$ decreases as does the value of the second coordinate relatively to the first one. This can be also clearly seen in the correlation (9), where we also observe less dependence in time with β increasing from 0 to 1. For example, the correlations for U_1 evaluated at $t = 1$ for $\beta = 0.1, 0.4, 0.6, 0.9$ are roughly equal to $0.9, 0.7, 0.6, 0.4$, respectively.

The Lt–fBm is represented in (15) as a series of independent processes that are obtained as differences of the coordinates $\boldsymbol{U}$. Thus the partial sum processes of this series

$$S_N(t) = \sum_{n=1}^{N} D_n(t; \alpha_n, H, \gamma_n) + \sum_{n=1}^{N} D'_n(t; \alpha'_n, 1-H, \gamma'_n) \tag{25}$$

can be used for approximating the Lt–fBm. The accuracy of approximation can be measured by studying the mean square error $MSE_N(H)$ of $X(t) - S_N(t)$. Proceeding in an analogous way as in the case $H < 1/2$ we obtain the explicit value for this error [see also (A.7,A.8) in the Appendix]

$$\begin{aligned} MSE_N(H) &= \sum_{n=N+1}^{\infty} \mathbb{E}(D_n^2(t; \alpha_n, H, \gamma_n)) + \sum_{n=N+1}^{\infty} \mathbb{E}(D_n'^2(t; \alpha'_n, 1-H, \gamma'_n)) \\ &= \frac{\sigma^2}{3-2H} \sum_{n=N+1}^{\infty} \left\{ (-1)^{n-1} \binom{2H}{n+1} (n+1) \frac{2H+1}{2H} - 2(-1)^{n-1} \binom{2H}{n+1} \right\} \\ &= \sigma^2 (-1)^N \binom{2H-2}{N} \frac{1+2H-2(2H-1)/(N+1)}{3-2H} . \end{aligned}$$

The rate of approximation for $H > 1/2$ can be best examined through the following inequalities that are simple consequences of Lemma 4 in the Appendix

$$2(1-H)\frac{C_H(N)}{N^{2H-1}} \le \frac{MSE_N(H)}{\sigma^2} \le \frac{C_H(N)}{(N+1)^{2H-1}} , \tag{26}$$

where

$$C_H(N) = \frac{(2-2H)^{2H-2}}{3-2H} \left(1 + 2H - \frac{4H-2}{N+1}\right) \left(1 - \frac{2H-1}{N}\right)^{N+1-2H}$$

is $\mathcal{O}(N)$ as $N \to \infty$.

We note the power rate N^{1-2H} of the approximation that is more accurate the further H is from 1/2 (the classical O–U process). This rate of approximation can be improved by a certain modification of the approximation as described below.

The terms in the series representation (15) decrease in the mean square. However, they are still characterized by the constant (in N) time dependence rates: e^{-Ht} for the first and $e^{-(1-H)t}$ for the second series. These rates are due to the first coordinates of the corresponding vector Markov processes $\boldsymbol{U}_n$ and $\boldsymbol{U}'_n$, *i.e.*, $U_n(t;\alpha_n,H)$ and $U'_n(t;\alpha'_n,1-H)$. In the following result we present an approximation for which this time dependence rate is removed from the remainder while simultaneously the rate of approximation is improved. Here we use the notation of Theorem 2 as well as the terminology and results for hypergeometric functions and series described in the Appendix.

Theorem 3 *Let us define*

$$
\begin{aligned}
V_N(t) &= \sum_{n=N+1}^{\infty} U_n(t;\alpha_n,H)\,,\\
V'_N(t) &= \sum_{n=N+1}^{\infty} U'_n(t;\alpha'_n,1-H)\,,
\end{aligned}
$$

where U_n *and* U'_n *are the O–U processes used in Theorem 2 to define* D_n *and* D'_n *through (13). Then* V_N *and* V'_N *are independent O–U processes that are also independent of* S_N *given by (25). They are represented in distribution as*

$$
V_N(t) \stackrel{\mathrm{d}}{=} U(t;r_N,H) \quad V'_N(t) \stackrel{\mathrm{d}}{=} U(t;r'_N,1-H)\,,
$$

where U *is given by (1) and*

$$
\begin{aligned}
r_N^2 &= K_H(N)H\left(2\frac{N+1}{H-1/2}+1+\frac{4H^2}{(N+2)(N+2-2H)^2}f_{N,H}(1)\right),\\
{r'_N}^2 &= K_H(N)H(2-2H)\left(\frac{N+1}{H-1/2}+\frac{2}{H}-3+\frac{(2-2H)^2}{(N+1)(N+2)}g_{N,H}(1)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
K_H(N) &= \frac{\sigma^2}{3-2H}(-1)^N\binom{2H-1}{N+1},\\
f_{N,H}(x) &= {}_3F_2\left[\begin{matrix}1 \quad N+2-2H \quad N+2-2H\\ N+3 \quad N+3-2H\end{matrix}\;;x\right],\\
g_{N,H}(x) &= {}_3F_2\left[\begin{matrix}1 \quad N+1 \quad N+2-2H\\ N+2 \quad N+3\end{matrix}\;;x\right].
\end{aligned}
$$

The approximation $X_N = S_N + V_N + V'_N$ *of a Lt–fBm* X *has the mean square error*

$$MSE_H(N) = K_H(N)\left(\frac{3}{2}+\frac{2H}{N+2}\left(\frac{H}{N+2-2H}f_{N,H}(1)+\frac{2-2H}{N+1}g_{N,H}(1)\right)\right)$$

and the covariance function of the approximation error $X - X_N$ *is given by*

$$\begin{aligned} R_H(t;N) &= K_H(N)e^{-(N+2-H)t}\frac{2H}{N+2} \\ &\times \left(\frac{3}{2}h_{N,H}(e^{-t}) + \frac{H}{N+2-2H}f_{N,H}(e^{-t}) + \frac{2-2H}{N+1}g_{N,H}(e^{-t})\right), \end{aligned}$$

where

$$h_{N,H}(x) = {}_2F_1\left[\begin{matrix} 1 \quad N+2-2H \\ N+3 \end{matrix}\; ; x\right].$$

Proof. By (A.5), (A.7)–(A.10) in the Appendix we have $r_N^2 = \sum_{n=N+1}^{\infty}\alpha_n^2$ and ${r'_N}^2 = \sum_{n=N+1}^{\infty}{\alpha'_n}^2$ thus it follows from (6) that V_N and V'_N are O–U processes with the properties as stated.

Further we note that the approximation error is given by

$$X_N - X = \sum_{n=N+1}^{\infty} U_n(t;\alpha_n, H\gamma_n) + \sum_{n=N+1}^{\infty} U'_n(t;\alpha'_n,(1-H)\gamma'_n)$$

and consequently

$$\begin{aligned} MSE_H(N) &= \frac{1}{2}\left(\frac{1}{H}\sum_{n=N+1}^{\infty}\frac{\alpha_n^2}{\gamma_n} + \frac{1}{1-H}\sum_{n=N+1}^{\infty}\frac{{\alpha'_n}^2}{\gamma'_n}\right), \\ R_H(t;N) &= \frac{1}{2}\left(\frac{1}{H}\sum_{n=N+1}^{\infty}\frac{\alpha_n^2}{\gamma_n}e^{-H\gamma_n t} + \frac{1}{1-H}\sum_{n=N+1}^{\infty}\frac{{\alpha'_n}^2}{\gamma'_n}e^{-(1-H)\gamma'_n}\right). \end{aligned}$$

The final formulas for $MSE_H(N)$ and $R_H(t;N)$ are following directly from (A.6), (A.8)–(A.10) shown in the Appendix.

□

The above result can be utilized in numerical approximations of trajectories of the Lt–fBm for $H > 1/2$ as the method of generating samples of the terms of S_N and independent O–U processes V_N and V'_N have been discussed above. Let us only mention that the scales r_N and r'_N of V_N and V'_N, respectively can be computed using the values of appropriate hypergeometric functions or in the case of r_N one can utilize the explicit formula (A.4). At present we do not know how to obtain values of r'_N without explicitly using hypergeometric functions.

Remark 3 *The advantages of using X_N instead of S_N to approximate the Lt–fBm are clear. It is seen from the formula for $R_H(t;N)$ in Theorem 3 that the error of approximation has the dependence in time rate in covariance decreasing exponentially with N as is given by $e^{-(N+2-H)t}$ (as opposed to the fixed rate $e^{-(1-H)t}$ of X_N). Moreover, the mean square error $MSE_H(N)$ is improved over the one for S_N as it is seen from the following argument. Note first that*

$$
{}_3F_2\left[\begin{matrix} 1\ N+2-2H\ N+2-2H \\ N+3\ N+3-2H \end{matrix} ;1\right] < {}_3F_2\left[\begin{matrix} 1\ N+1\ N+2-2H \\ N+2\ N+3 \end{matrix} ;1\right] < {}_2F_1\left[\begin{matrix} 1\ N+1 \\ N+3 \end{matrix} ;1\right] = N+2\,.
$$

This leads, after some algebra, to

$$
\tfrac{3}{2}K_H(N) < MSE_H(N) < \tfrac{1}{5}2K_H(N)\,.
$$

Since $\binom{2H-1}{N+1} = \frac{2H-1}{N+1}\binom{2H-2}{N}$, we conclude that the MSE rate N^{-2H} (a significant improvement over (26)). Illustration of the mean square errors is presented in figure 4, where we see that the approximation by X_N is better not only because of the asymptotics with respect to N but also by being exact for each N for H in the neighborhood of 0.5.

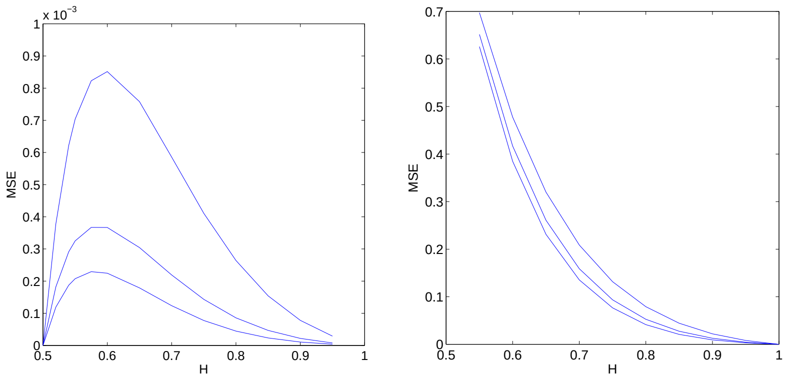

Fig. 4. Mean square error as the function H for X_N (left) and for S_N (right), $N = 50, 100, 150$.

The results of simulations of $X_N = S_N + V_N + V_N'$ for $N = 50, 100, 150$ and $H = 0.6, 0.7, 0.8, 0.9$ over the interval $[0, 10]$ are presented in figure 5. We see on the left hand side graphs that the trajectories for $N = 50$ and $N = 150$ are almost indistinguishable due to the thickness of the used lines and the quality of the approximation. The differences between the approximations are seen on the right hand side graphs.

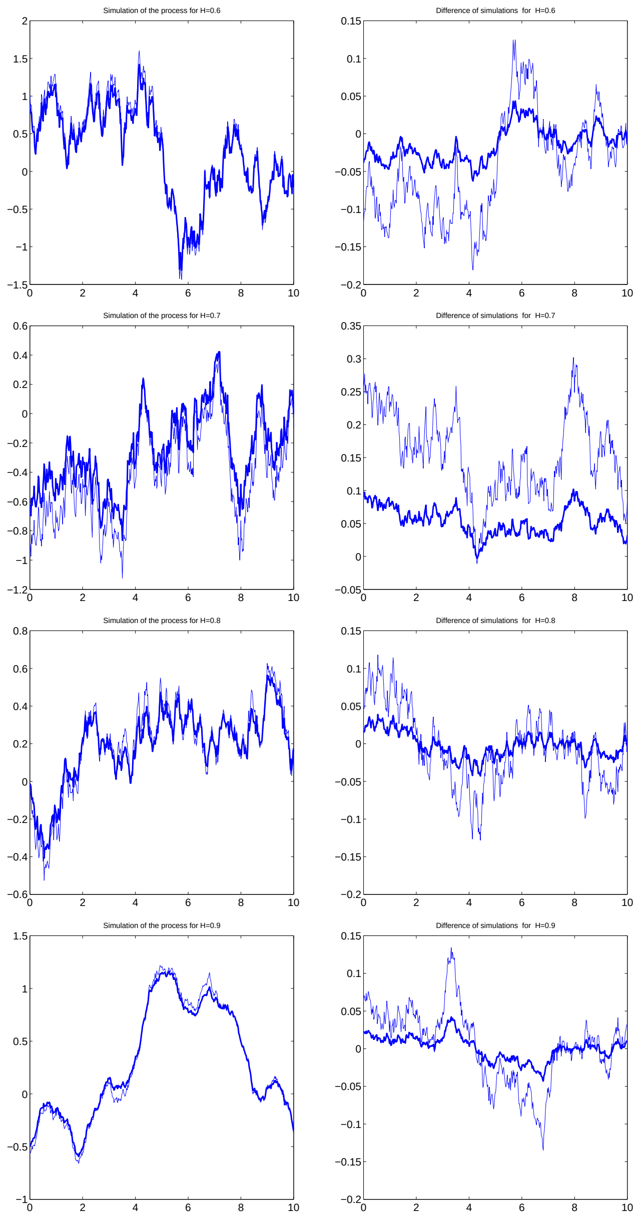

Fig. 5. Trajectories of approximations X_N, for $H = 0.6, 0.7, 0.8, 0.9$ (from top to bottom) over $[0, 10]$. Left: approximations for $N = 50$ (thin line), $N = 150$ (thick line); Right: the differences between approximation for $N = 150$ and the ones for $N = 50$ (thin line) and $N = 100$ (thick line).

3. Decomposition of fBm

3.1. fBm as a series of transformed Brownian motions

In the case $H \leq 1/2$, the representation of Lt–fBm of Theorem 1 and application of the inverse Lamperti transform (iLt) lead directly to the corresponding representation of fBm in terms of series of transformed Brownian motions.

Theorem 4 *For $H \in (0, 1/2]$, a fBm $Y(t)$ can be represented as the following series*

$$Y(t) = t^H \sum_{n=1}^{\infty} \frac{\alpha_n}{\sqrt{2\beta_n}} \frac{B_n(t^{2\beta_n})}{t^{\beta_n}}, \qquad t > 0,$$

where $B_n(t)$ are independent standard Brownian motions and α_n^2, β_n are as in Theorem 1.

Proof.
We first notice that the fBm $Y(t)$ is iLt of the Lt–fBm process $X(t)$.

If $H \in (0, \frac{1}{2}]$, then according to Theorem 1, the Lt–fBm can be represented as a series of independent O–U processes. Thus

$$Y(t) = t^H X(\log t) = \sum_{n=1}^{\infty} t^H U_n(\log t; \alpha_n, \beta_n), \qquad t > 0, \tag{27}$$

where $U_n(t; \alpha_n, \beta_n)$ are independent O–U processes with α_n, β_n as given in Theorem 1.

Moreover, the process $t \mapsto t^H U_n(\log t; \alpha, \beta)$ has the same distribution as the Gaussian process $t \mapsto \frac{\alpha}{\sqrt{2\beta}} t^{H-\beta} B_n(t^{2\beta})$, where $B_n(t)$ is a standard Brownian motion and the result follows.

□

Remark 4 *Let us define a sequence of Gaussian processes*

$$Y_n(t) = \frac{\alpha_n}{\sqrt{2\beta_n}} t^{H-\beta_n} B_n(t^{2\beta_n}).$$

Then $Y_n(t) = t^H U_n(\log t; \alpha_n, \beta_n)$ and Y_n's are H-ss processes. They can be represented as the following integrals with respect to independent standard Brownian measures B_n's

$$Y_n(t) = \alpha_n \int_{-\infty}^{\log t} t^{H-\beta_n} e^{\beta_n x} dB_n(x).$$

Moreover, Y_n's are Markovian and the distribution of $Y_n(t)$ is zero mean normal with variance $\frac{\alpha_n^2}{2\beta_n}t^{2H}$ and for $s > t$ the distribution of $Y_n(s)$ given $Y_n(t) = y$ is normal with the mean $(s/t)^{H-\beta_n}y$ and variance $\frac{\alpha_n^2}{2\beta_n}s^{2H}(1 - (t/s)^{2\beta_n})$. Therefore the representation of Theorem 4 can be viewed also as the representation of the fBm as series of independent, H-ss, Markovian and Gaussian processes.

3.2. Decomposition of fBm for $\boldsymbol{H} > 1/2$

A similar approach through the iLt as used in the case $H \leq 1/2$ leads to series representation of fBm in terms of differences of dependent deterministically transformed Brownian motions when $H \in (1/2, 1)$. First, we investigate the structure of a two dimensional vector of dependent standard Brownian motions that will be used for this representation.

Lemma 2 *Let $\boldsymbol{B}(t) = (B_1(t), B_2(t))$ be the Gaussian vector process defined through*

$$\begin{aligned} B_1(t) &= \sqrt{2\beta t} \cdot U\left(\log t^{\frac{1}{2\beta}}; 1, \beta\right), \\ B_2(t) &= \sqrt{2\beta\gamma t} \cdot U\left(\log t^{\frac{1}{2\beta\gamma}}; 1, \gamma\beta\right), \end{aligned}$$

where an O–U process U is defined by (1). Then its covariance function for $s \leq t$ is given by

$$\boldsymbol{R_B}(s,t) = \mathbb{E}\left(\boldsymbol{B}(s)\boldsymbol{B}^T(t)\right) = \begin{bmatrix} s & \frac{2\sqrt{\gamma ts}}{1+\gamma}\left(\frac{s^{\beta\gamma}\wedge t^{\beta}}{s^{\beta\gamma}\vee t^{\beta}}\right)^{\frac{1}{2\beta\gamma}} \\ \frac{2\sqrt{\gamma ts}}{1+\gamma}\left(\frac{s^{\beta}\wedge t^{\beta\gamma}}{s^{\beta}\vee t^{\beta\gamma}}\right)^{\frac{1}{2\beta}} & s \end{bmatrix}.$$

Proof. Let $\tilde{\boldsymbol{U}}(t) = (\tilde{U}_1(t), \tilde{U}_2(t))$ be the Gaussian vector process defined through

$$\begin{aligned} \tilde{U}_1(t) &= U\left(\log t^{\frac{1}{2\beta}}; 1, \beta\right), \\ \tilde{U}_2(t) &= U\left(\log t^{\frac{1}{2\beta\gamma}}; 1, \gamma\beta\right), \end{aligned}$$

so

$$\boldsymbol{B}(t) = \begin{bmatrix} \sqrt{2\beta t} & 0 \\ 0 & \sqrt{2\beta\gamma t} \end{bmatrix} \tilde{\boldsymbol{U}}(t).$$

Then

$$\boldsymbol{R_B}(s,t) = \begin{bmatrix} \sqrt{2\beta s} & 0 \\ 0 & \sqrt{2\beta\gamma s} \end{bmatrix} \mathbb{E}\left(\tilde{\boldsymbol{U}}(s)\tilde{\boldsymbol{U}}^T(t)\right) \begin{bmatrix} \sqrt{2\beta t} & 0 \\ 0 & \sqrt{2\beta\gamma t} \end{bmatrix}.$$

The expectation $\mathbb{E}\left(\tilde{\boldsymbol{U}}(s)\tilde{\boldsymbol{U}}^T(t)\right)$ can be obtained from Lemma 1 as follows. Let first consider $s \le t$, $s^\beta < t^{\beta\gamma}$ and $s^{\beta\gamma} > t^\beta$, then

$$\begin{aligned}
\mathbb{E}\left(\tilde{\boldsymbol{U}}(s)\tilde{\boldsymbol{U}}^T(t)\right) &= \begin{bmatrix} R_{11}\left(\log(\frac{s}{t})^{-\frac{1}{2\beta}}\right) & R_{21}\left(\log(\frac{s^{\frac{1}{2\beta}}}{t^{\frac{1}{2\beta\gamma}}})\right) \\ R_{21}\left(\log(\frac{t^{\frac{1}{2\beta}}}{s^{\frac{1}{2\beta\gamma}}})\right) & R_{22}\left(\log(\frac{s}{t})^{-\frac{1}{2\beta\gamma}}\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2\beta}\sqrt{\frac{s}{t}} & \frac{\sqrt{st}}{\beta(1+\gamma)} s^{-\frac{1+\gamma}{2}} \\ \frac{\sqrt{st}}{\beta(1+\gamma)} t^{-\frac{1+\gamma}{2}} & \frac{1}{2\beta\gamma}\sqrt{\frac{s}{t}} \end{bmatrix},
\end{aligned}$$

where $R_{ij}(h)$ are the terms of $\boldsymbol{R_U}(h)$ from Lemma 1. The case $s \le t$, $s^\beta < t^{\beta\gamma}$ and $s^{\beta\gamma} \le t^\beta$ can be obtained in a similar manner. The final conclusion then follows from multiplying appropriate matrices and using the obvious properties of covariance matrix.

□

The series representation presented below uses the independent differences of deterministically transformed Brownian motions as in Lemma 2 to represent a fBm with $H > 1/2$.

Theorem 5 *Let $Y(t)$ be a fBm with the Hurst parameter $H \in (1/2, 1)$. Then it has the following representation in distribution*

$$\begin{aligned}
Y(t) &= t^H \sum_{n=1}^{\infty} \left(\frac{\alpha_n}{\sqrt{2H}} \left(\frac{B_{1,n}(t^{2H})}{t^H} - \frac{B_{2,n}(t^{2H\gamma_n})}{\sqrt{\gamma_n}\cdot t^{H\gamma_n}} \right) \right. \qquad (28) \\
&\quad \left. + \frac{\alpha_n'}{\sqrt{2\bar{H}}} \left(\frac{B_{1,n}'(t^{2\bar{H}})}{t^{\bar{H}}} - \frac{B_{2,n}'(t^{2\bar{H}\gamma_n'})}{\sqrt{\gamma_n'}\cdot t^{\bar{H}\gamma_n'}} \right) \right),
\end{aligned}$$

where $\bar{H} = 1–H$ while $\boldsymbol{B}_n(t) = (B_{1,n}(t), B_{2,n}(t))$ and $\boldsymbol{B}_n'(t) = (B_{1,n}'(t), B_{2,n}'(t))$ are independent sequences of mutually independent vector Brownian motions having the covariance structure given in Lemma 2 with the parameters β_n and β_n' replaced by H, $\bar{H}$, respectively, while α_n, γ_n and α_n', γ_n' are given in Theorem 2.

Proof. If $H \in (\frac{1}{2}, 1)$, then according to Theorem 2, the Lt–fBm can be represented as a mixture of moving averages. Thus,

$$Y(t) = t^H X(\log t) = \sum_{n=1}^{\infty} t^H D_n(\log t; \alpha_n, H, \gamma_n) + \sum_{n=1}^{\infty} t^H D_n'(\log t; \alpha_n', \bar{H}, \gamma_n'), \qquad (29)$$

where $D_n(t;\alpha_n,H,\gamma_n)$ and $D'_n(t;\alpha'_n,\bar{H},\gamma'_n)$ are mutually independent differences of dependent O–U processes as defined in (13) with $\alpha_n,\alpha'_n,\gamma_n$ and γ'_n as in Theorem 2.

Moreover, for each n the process

$$\begin{aligned} t\mapsto t^H D_n(\log t;\alpha_n,H,\gamma_n) &= t^H U_n(\log t;\alpha_n,H)-t^H U_n(\log t;\alpha_n,H\gamma_n)\\ &= t^H\alpha_n\big(U_n(\log t;1,H)-U_n(\log t;1,H\gamma_n)\big)\end{aligned}$$

has the same distribution as the Gaussian process

$$\Delta_n(t)=t^H\alpha_n\left(\frac{B_{1,n}(t^{2H})}{\sqrt{2H}\cdot t^H}-\frac{B_{2,n}(t^{2H\gamma_n})}{\sqrt{2H\gamma_n}\cdot t^{H\gamma_n}}\right). \tag{30}$$

The structure of Gaussian processes $\boldsymbol{B}_n(t)=(B_{1,n}(t),B_{2,n}(t))$ and $\boldsymbol{B}'_n(t)=(B'_{1,n}(t),B'_{2,n}(t))$ is given in Lemma 2, with β taking the values H and $\bar{H}$, respectively, and the remaining parameters being α_n,γ_n and α'_n,γ'_n, respectively. Combining it with (29) we obtain (28).

□

Remark 5 *The covariance of the processes $\Delta_n(t)$ defined in (30) that appear as independent terms in the above representation is given for $s>t$ by*

$$R_\Delta(t,s)=\frac{\alpha_n^2}{2H}s^{2H}\left(\frac{\gamma_n-1}{\gamma_n+1}-\frac{2}{\gamma_n+1}\left(\frac{t}{s}\right)^{H(1+\gamma_n)}+\frac{1}{\gamma_n}\left(\frac{t}{s}\right)^{H(1-\gamma_n)}\right).$$

In the above results we expressed the fractional Brownian motion in terms of differences of dependent and deterministically transformed Brownian motions. Similarly as in the case $H\le 1/2$, this decomposition can be equivalently represented using Gaussian and Markovian self-similar processes. The main difference is that in the present case these processes are two dimensional in values. In the following lemma their general structure is described.

Lemma 3 *Let $\boldsymbol{\Gamma}(t)=(\Gamma_1(t),\Gamma_2(t))$ be the vector Gaussian process defined for $\beta>0$ and $\gamma>0$ through*

$$\begin{aligned}\Gamma_1(t) &= \frac{B_1(t^{2\beta})}{\sqrt{2\beta}},\\ \Gamma_2(t) &= t^{\beta(1-\gamma)}\frac{B_2(t^{2\beta\gamma})}{\sqrt{2\beta\gamma}},\end{aligned}$$

where B_1 and B_2 are as in Lemma 2. Then $\boldsymbol{\Gamma}(t)$ is a β-ss two dimensional Gaussian process that has covariance matrix given for $s > t$ by

$$\boldsymbol{R}_{\boldsymbol{\Gamma}}(t,s) = \frac{s^{2\beta}}{2\beta}\begin{bmatrix} \left(\frac{t}{s}\right)^{2\beta} & \frac{2}{1+\gamma}\left(\frac{t}{s}\right)^{\beta(\gamma+1)} \\ \frac{2}{1+\gamma}\left(\frac{t}{s}\right)^{2\beta} & \frac{1}{\gamma}\left(\frac{t}{s}\right)^{\beta(\gamma+1)} \end{bmatrix}.$$

Moreover, $\boldsymbol{\Gamma}$ is a Markov process such that for $s > t$ the distribution of $\boldsymbol{\Gamma}(s)$ given that $\boldsymbol{\Gamma}(t) = \boldsymbol{\gamma}$ is bivariate normal with mean

$$\boldsymbol{\mu}(\boldsymbol{\gamma}) = \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{s}{t}\right)^{\beta(1-\gamma)} \end{bmatrix}\boldsymbol{\gamma}$$

and covariance

$$\boldsymbol{\Sigma} = \frac{s^{2\beta}}{2\beta}\begin{bmatrix} 1-\left(\frac{t}{s}\right)^{2\beta} & \frac{2}{1+\gamma}\left(1-\left(\frac{t}{s}\right)^{\beta(1+\gamma)}\right) \\ \frac{2}{1+\gamma}\left(1-\left(\frac{t}{s}\right)^{\beta(1+\gamma)}\right) & \frac{1}{\gamma}\left(1-\left(\frac{t}{s}\right)^{2\beta\gamma}\right) \end{bmatrix}.$$

Proof. Note that $\boldsymbol{\Gamma}(t) = t^{\beta}\boldsymbol{U}(\log t)$, where $\boldsymbol{U}$ is the stationary process defined in Lemma 1 and self-similarity follows from the properties of the Lamperti transform. Moreover since $\boldsymbol{U}$ has been Markovian and the logarithmic time change preserves the Markov property, it is enough to carry out standard algebra to obtain the means and covariances of the distributions.

To this end observe the following relation between the covariances of $\boldsymbol{U}$ and $\boldsymbol{\Gamma}$:

$$\boldsymbol{R}_{\boldsymbol{\Gamma}}(t,s) = t^{\beta}s^{\beta}\boldsymbol{R}_{\boldsymbol{U}}\left(\log\frac{t}{s}\right).$$

This formula for $\boldsymbol{R}_{\boldsymbol{\Gamma}}$ follows easily from the formula for $\boldsymbol{R}_{\boldsymbol{U}}$ of Lemma 1.

Next assume that $s > t$ and notice that the Markovian structure of $\boldsymbol{U}$ gives the following structure for the conditional distribution of $\boldsymbol{\Gamma}(s)$ given that $\boldsymbol{\Gamma}(t) = \boldsymbol{\gamma}$:

$$s^{\beta}\boldsymbol{A}\left(\log\frac{s}{t}\right)\left(t^{-\beta}\boldsymbol{\gamma} + \boldsymbol{B}\left(\log\frac{s}{t}\right)\boldsymbol{Y}\left(\log\frac{s}{t}\right)\right),$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{Y}$ are defined in Lemma 1. To see the formula for variance and the mean of conditional distribution note the following relations between matrices that can be checked by direct algebra

$$\begin{aligned}
\boldsymbol{A} &\stackrel{\text{def}}{=} \boldsymbol{A}\left(\log\frac{s}{t}\right) = \begin{bmatrix} \left(\frac{t}{s}\right)^{\beta} & 0 \\ 0 & \left(\frac{t}{s}\right)^{\beta\gamma} \end{bmatrix}, \\
\boldsymbol{B} &\stackrel{\text{def}}{=} \boldsymbol{B}\left(\log\frac{s}{t}\right) = \begin{bmatrix} 1 & 0 \\ \frac{2}{1+\gamma}\frac{\left(\frac{s}{t}\right)^{\beta(1+\gamma)}-1}{\left(\frac{s}{t}\right)^{2\beta}-1} & 1 \end{bmatrix}, \\
\boldsymbol{\Sigma} &= \boldsymbol{A}\boldsymbol{B}\begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\boldsymbol{B}^T\boldsymbol{A}^T,
\end{aligned}$$

where σ_1^2 and σ_2^2 are variances of the independent coordinates of $\boldsymbol{Y}\left(\log \frac{s}{t}\right)$. It follows from (23) in Section 2 that

$$\begin{aligned}\sigma_1^2 &= \frac{1}{2\beta}\left(\left(\frac{s}{t}\right)^{2\beta}-1\right),\\ \sigma_2^2 &= \frac{1}{2\beta}\left(\frac{1}{\gamma}\left(\left(\frac{s}{t}\right)^{2\beta\gamma}-1\right)-\frac{4}{(1+\gamma)^2}\frac{\left(\left(\frac{s}{t}\right)^{\beta(1+\gamma)}-1\right)^2}{\left(\frac{s}{t}\right)^{2\beta}-1}\right).\end{aligned}$$

The rest of the proof follows from straightforward matrix algebra. □

Remark 6 *For any β-ss process $X(t)$, the process $X'(t) = t^{\beta'-\beta}X(t)$ is β'-ss. In particular, considering the process $\boldsymbol{\Gamma}'(t) = t^{\beta'-\beta}\boldsymbol{\Gamma}(t)$, ere $\boldsymbol{\Gamma}$ is as in Lemma 3, it is easy to argue that it is Gaussian, Markovian, and the corresponding parameters are as follows*

$$\begin{aligned}\boldsymbol{R}_{\boldsymbol{\Gamma}'}(t,s) &= (ts)^{\beta'-\beta}\boldsymbol{R}_{\boldsymbol{\Gamma}}(t,s),\\ \boldsymbol{\mu}'(\boldsymbol{\gamma}) &= (s/t)^{\beta'-\beta}\boldsymbol{\mu}(\boldsymbol{\gamma}),\\ \boldsymbol{\Sigma}' &= s^{2\beta'-2\beta}\boldsymbol{\Sigma}.\end{aligned}$$

Remark 7 *Using the representation from Example 2, the process of differences $\Delta(t) = \Gamma_1(t) - \Gamma_2(t)$ can be also written as*

$$\Delta(t) = \int_{-\infty}^{\log t} e^{\beta s} - t^{\beta(1-\gamma)}e^{\gamma\beta s}dB(s),$$

where B is a standard Brownian measure.

In order to formulate the result that summarizes our discussion of the series representations of fractional Brownian motion, we need the following notation. Let us fix $H \in (1/2, 1)$, $\alpha \in \mathbb{R}$, and $\gamma > 0$. The H-ss process $\boldsymbol{\Gamma}$ defined in Lemma 3 with $\beta = H$ is denoted as $\boldsymbol{\Gamma}(t;\gamma)$. Additionally, if $\boldsymbol{\Gamma}$ from Lemma 3 has $\beta = \bar{H} = 1 - H$, then the H-ss process $\boldsymbol{\Gamma}'(t) = t^{2H-1}\boldsymbol{\Gamma}(t)$ is denoted as $\boldsymbol{\Gamma}'(t;\gamma)$. With this notation we can write the decomposition of fBm as a series of H-ss, Gaussian, and Markovian processes.

Proposition 1 *Let $H \in (1/2, 1)$, and α_n, α'_n, γ_n, γ'_n are defined in Theorem 2. Let $\boldsymbol{\Gamma}_n(t;\gamma_n)$ and $\boldsymbol{\Gamma}'_n(t;\gamma'_n)$ be mutually independent sequences of independent, Gaussian and Markovian processes as defined above. Then*

$$\boldsymbol{Y}(t) = \sum_{n=1}^{\infty}\alpha_n\boldsymbol{\Gamma}_n(t;\gamma_n) + \sum_{n=1}^{\infty}\alpha'_n\boldsymbol{\Gamma}'_n(t;\gamma'_n)$$

is an H-ss bivariate Gaussian process and the fBm $Y(t)$ with the Hurst index H can be represented as

$$Y(t) = \boldsymbol{v}^T \boldsymbol{Y}(t)\,,$$

where $\boldsymbol{v} = (1, -1)$.

Proof. The result is a direct consequence of Theorem 5 and Lemma 3. □

Remark 8 *Using Remark 7 we can also write*

$$\begin{aligned} Y(t) \;=\; & \sum_{n=1}^{\infty} \alpha_n \int_{-\infty}^{\log t} e^{Hs} - t^{H(1-\gamma_n)} e^{\gamma_n H s} dB_n(s) \\ & + \sum_{n=1}^{\infty} \alpha'_n \int_{-\infty}^{\log t} t^{H-\bar{H}} e^{\bar{H}s} - t^{H-\bar{H}\gamma'_n} e^{\gamma'_n \bar{H} s} dB'_n(s)\,, \end{aligned}$$

where B_n and B'_n are mutually independent standard Brownian measures.

3.3. Approximations of fBm

The series representation of Lt–fBm can provide effective approximations of fBm that can be utilized, for example, in simulations. The results are direct consequences of the previously discussed simulations of Lt–fBm, so we limit ourselves to their very brief overview. For simplicity of notation in this section we assume that the scale parameter σ is equal to one. Let us start with the case of $H < 1/2$ that is simpler.

By Theorem 4, the fBm $Y(t)$ can be approximated by the partial sum $S_N(t) = \sum_{n=1}^{N} Y(t)$. Simulation of the $Y_n(t)$ process at the discrete time points $0 < t_1 < \cdots < t_m$ is straightforward since the process is Gaussian and Markov (see Remark 4).

Indeed, for a sample $(y_1, \cdots, y_m)$ from the distribution of random vector $(Y_n(t_1), \cdots, Y_n(t_m))$, let y_1 be simulated as a normal variable with mean zero and variance $\frac{\alpha_n^2}{2\beta_n} t_1^{2H}$. Then by recursive formula the values $(y_2, \cdots, y_m)$ are simulated from the distributions

$$N\left(\left(\frac{t_i}{t_{i-1}}\right)^{H-\beta_n} y_{i-1}, \frac{\alpha_n^2}{2\beta_n} t_i^{2H} \left(1 - \left(\frac{t_{i-1}}{t_i}\right)^{2\beta_n}\right)\right), \quad i = 2, \ldots, m\,.$$

Samples generated using this algorithm evaluated at 500 equally spaced points over the interval $[0, 1]$ are presented in figure 6.

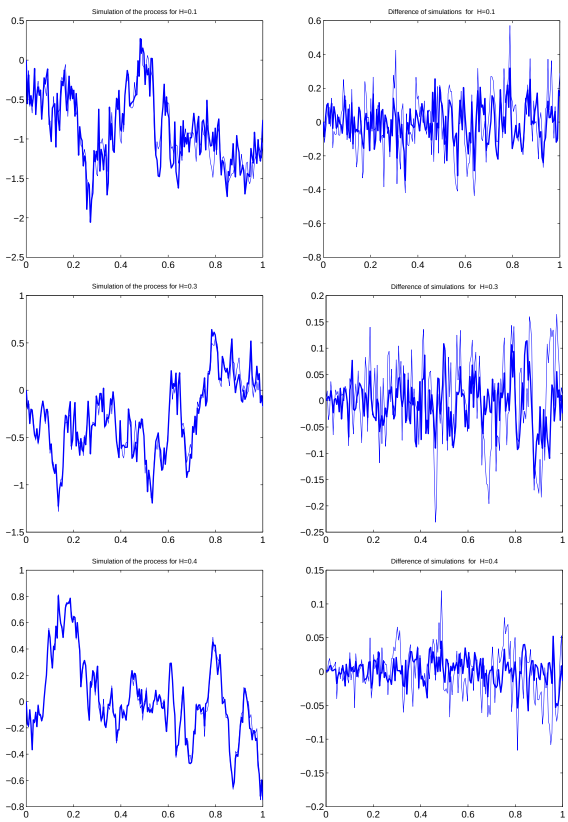

Fig. 6. Trajectories of approximations X_N, for $H = 0.1, 0.3, 0.4$ evaluated at 500 equally spaced points of $[0, 1]$. Left: approximations for $N = 50$ (thin line), $N = 150$ (thick line); Right: the differences between approximation for $N = 150$ and the ones for $N = 50$ (thin line) and $N = 100$ (thick line).

Using the results of Subsection 2.3 we can obtain a straightforward assessment of the MSE of $Y(t) - S_N(t)$. By the representation

$$Y(t) = t^H X(\log(t)),$$

where the process $X(t)$ is the Lt–fBm and the MSE formula for the Lt–fBm is given in Section 2.3 we have

$$\begin{aligned} MSE_N(t;H) &\stackrel{\text{def}}{=} \mathbb{E}\,(Y(t) - S_N(t))^2 \\ &= \mathbb{E}\left(t^H(X(\log(t)) - X_N(\log(t)))\right)^2 \\ &= t^{2H}(-1)^{N-1}\frac{\sigma^2}{2}\binom{2H-1}{N-1}. \end{aligned}$$

Thus for fixed t, the rate of convergence of the MSE is of the order of the power function N^{-2H}. The MSE for $t = 2$ can be seen in figure 7.

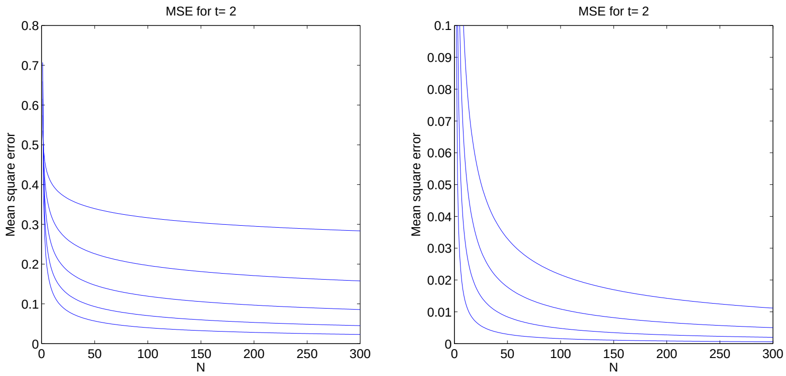

Fig. 7. The mean square error $MSE_{N,t}(H)$ for $H = 0.05, 0.1, 0.15, 0.2, 0.25$ (left) and $0.3, 0.35, 0.4, 0.45$ (right), $N = 1, \dots, 300$ and $t = 2$. Larger MSE values correspond to smaller H.

While the case $H \le 1/2$ has been straightforward, the structure of the two dimensional Brownian motions in the case $H > 1/2$ must be taken into account to effectively express the terms of series representation. It has been seen in Subsection 2.3 that an effective approximation is of the form

$$
\begin{aligned}
Y_N(t) \; &= \sum_{n=1}^{N} \alpha_n \boldsymbol{v}^T \boldsymbol{\Gamma}_n(t;\gamma_n) + \sum_{n=1}^{N} \alpha_n' \boldsymbol{v}^T \boldsymbol{\Gamma}_n'(t;\gamma_n') \\
&\quad + \frac{r_N^2}{2H} B(t^{2H}) + \frac{{r'_N}^2}{2(1-H)} B'(t^{2-2H}),
\end{aligned}
$$

where all the terms are mutually independent, $\boldsymbol{\Gamma}_n$ and $\boldsymbol{\Gamma}_n'$ are generated using their Markovian structure as described below, B, B' are standard Brownian motions, and all other constants are defined either in Theorem 2 or in Theorem 3.

To obtain a sample on the grid $t_0 = 0 < t_1 < \cdots < t_{I-1} < t_I$, one can generate independently samples from the standard normal distribution: z_i, z_i', and the bivariate samples: $\boldsymbol{\gamma}_{n,i}$, $\boldsymbol{\gamma}_{n,i}'$, $i = 1, \dots, I$, $n \in 1, \dots, N$ that are defined recursively as described next. First, $\boldsymbol{\gamma}_{n,1}$ and $\boldsymbol{\gamma}_{n,1}'$ are drawn from the zero mean normal bivariate distributions with the covariances

$$
\frac{t_1^{2H}}{2H} \begin{bmatrix} 1 & \frac{2}{1+\gamma_n} \\ \frac{2}{1+\gamma_n} & \frac{1}{\gamma_n} \end{bmatrix}, \qquad \frac{t_1^{2H}}{2(1-H)} \begin{bmatrix} 1 & \frac{2}{1+\gamma_n'} \\ \frac{2}{1+\gamma_n'} & \frac{1}{\gamma_n'} \end{bmatrix},
$$

respectively. en $\boldsymbol{\gamma}_{n,i}$, $\boldsymbol{\gamma}'_{n,i}$, $i = 2, \dots, n$, are obtained recursively

$$\boldsymbol{\gamma}_{n,i} = \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{t_i}{t_{i-1}}\right)^{H(1-\gamma_n)} \end{bmatrix} \boldsymbol{\gamma}_{n,i-1} + \boldsymbol{x}_{n,i} \,,$$

$$\boldsymbol{\gamma}'_{n,i} = \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{t_i}{t_{i-1}}\right)^{1-2H-(1-H)\gamma'_n} \end{bmatrix} \boldsymbol{\gamma}'_{n,i-1} + \boldsymbol{x}'_{n,i} \,,$$

where $\boldsymbol{x}_{n,i}$'s and $\boldsymbol{x}'_{n,i}$ are samples from the zero mean bivariate normal distributions with covariances

$$\frac{t_i^{2H}}{2H} \begin{bmatrix} 1-\left(\frac{t_{i-1}}{t_i}\right)^{2H} & \frac{2}{1+\gamma_n}\left(1-\left(\frac{t_{i-1}}{t_i}\right)^{H(1+\gamma_n)}\right) \\ \frac{2}{1+\gamma_n}\left(1-\left(\frac{t_{i-1}}{t_i}\right)^{H(1+\gamma_n)}\right) & \frac{1}{\gamma_n}\left(1-\left(\frac{t_{i-1}}{t_i}\right)^{2H\gamma_n}\right) \end{bmatrix},$$

$$\frac{t_i^{2H}}{2(1-H)} \begin{bmatrix} 1-\left(\frac{t_{i-1}}{t_i}\right)^{2(1-H)} & \frac{2}{1+\gamma'_n}\left(1-\left(\frac{t_{i-1}}{t_i}\right)^{(1-H)(1+\gamma'_n)}\right) \\ \frac{2}{1+\gamma'_n}\left(1-\left(\frac{t_{i-1}}{t_i}\right)^{(1-H)(1+\gamma'_n)}\right) & \frac{1}{\gamma'_n}\left(1-\left(\frac{t_{i-1}}{t_i}\right)^{2(1-H)\gamma'_n}\right) \end{bmatrix},$$

respectively. Then a sample trajectory of fBm at t_i is obtained as

$$\begin{aligned} y_i &= \sum_{n=1}^{N} \alpha_n \boldsymbol{v}^T \boldsymbol{\gamma}_{n,i} + \sum_{n=1}^{N} \alpha'_n \boldsymbol{v}^T \boldsymbol{\gamma}'_{n,i} \\ &\quad + \frac{r_N^2}{2H} \sum_{k=1}^{i} (t_k^{2H} - t_{k-1}^{2H})^{1/2} z_k + \frac{{r'}_N^2}{2(1-H)} \sum_{k=1}^{i} (t_k^{2-2H} - t_{k-1}^{2-2H})^{1/2} z'_k \,. \end{aligned}$$

Examples of trajectories generated by this method are seen in figure 8. We have also an explicit form of the MSE of the above approximation following easily from Theorem 3

$$\begin{aligned} MSE_N(t;H) &= t^{2H} \frac{\sigma^2(-1)^N}{3-2H} \binom{2H-1}{N+1} \\ &\quad \times \left(\frac{3}{2} + \frac{2H}{N+2} \left(\frac{2\bar{H}}{N+1} {}_3F_2 \left[\begin{matrix} 1 \; N+1 \; N+2\bar{H} \\ N+2 \; N+3 \end{matrix} ;1 \right] \right. \right. \\ &\quad \left. \left. + \frac{H}{N+2\bar{H}} {}_3F_2 \left[\begin{matrix} 1 \; N+2\bar{H} \; N+2\bar{H} \\ N+3 \; N+3-2H \end{matrix} ;1 \right] \right) \right), \end{aligned}$$

where $\bar{H} = 1 - H$.

The graph illustrating the above error is presented in figure 9.

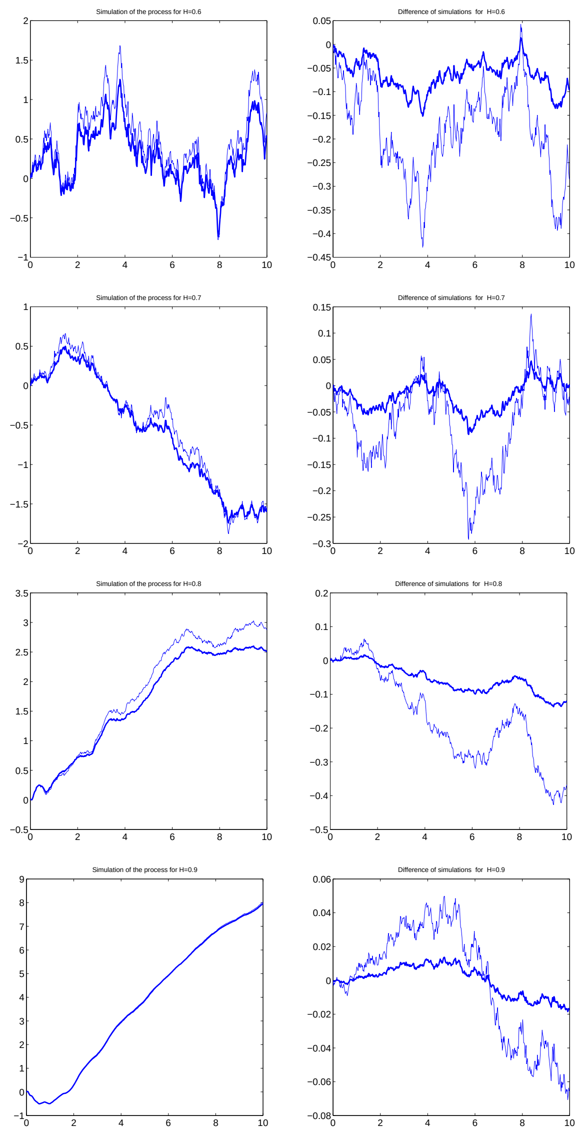

Fig. 8. Trajectories of approximations Y_N, for $H = 0.6, 0.7, 0.8, 0.9$ (from top to bottom) over $[0, 10]$. Left: approximations for $N = 50$ (thin line), $N = 150$ (thick line); Right: the differences between approximation for $N = 150$ and the ones for $N = 50$ (thin line) and $N = 100$ (thick line).

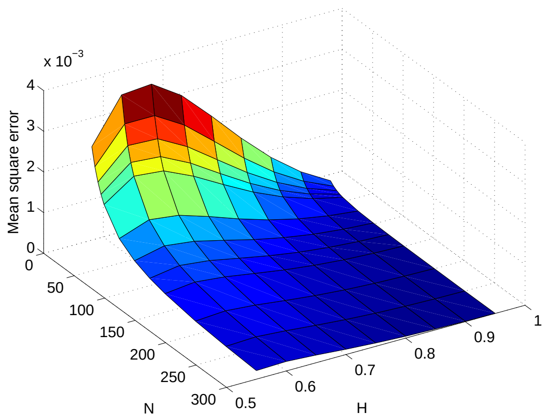

Fig. 9. The mean square error $MSE_{N,t}(H)$ for $H = 0.55, 0.6, 0.65, 0.7, 0.75$ (left) and $0.8, 0.85, 0.9, 0.95$ (right), $N = 30, \ldots, 300$ and $t = 2$. Larger MSE values correspond to smaller H.

4. Long-range dependence

Beside having simple Markovian and Gaussian structure, the subsequent self-similar components in our series decomposition of the fBm exhibit also decreasing rates of time dependence with the first term having the rate equal to the fBm rate. In this sense, it can be roughly said that the time dependence of the fBm is represented by the first term in the decomposition. It maybe particularly interesting (although slightly more complex) for the long range dependence of the case $H > 1/2$ as it may reduce studying the more complex fractional Brownian motion to the simpler deterministically transformed regular Brownian motion. Here we list several simple results that illustrate this feature of the presented series decomposition.

There are many possible ways of measuring the dependence rate for stochastic processes. For stationary processes is measured by the rate of decay of covariance function and in particular, a stationary sequence is declared to have long-range dependence if its covariance function is not summable. This terminology is carried over to homogeneous increment processes, where long-range dependence is defined through long-range dependence of the (stationary) process of increments. Alternative although related definitions of the long-range dependence for non-stationary and non-homogeneous increment processes has been discussed in the literature [see for example [20]]. Here we would like to point on the fairly simple dependence properties of the components in the discussed decompositions leaving a more thorough study for future research. Namely, we consider the autocorrelation functions of the processes.

For the fractional Brownian motion $Y(t)$, the correlation

$$\rho_Y(t,h) = \mathbb{C}\mathrm{ov}\,(Y(t), Y(t+h))/\sqrt{\mathbb{V}\mathrm{ar}\,(Y(t))\mathbb{V}\mathrm{ar}\,(Y(t+s))}$$

has the following tail behavior for fixed $t > 0$ and $h \to \infty$:

$$\rho_Y(t,h) \sim \begin{cases} \frac{1}{2}\left(\frac{t}{h}\right)^H & : \quad H \in (0, 1/2]\,, \\ \frac{1}{4H}\left(\frac{t}{h}\right)^{1-H} & : \quad H \in (1/2, 1)\,. \end{cases}$$

For $H \leq 1/2$, the terms of the decomposition have the correlation function given by

$$\rho_{Y_n}(t,h) = \left(1 + \frac{h}{t}\right)^{-\beta_n},$$

where $\beta_n = |n - H - 1|$ and thus the rate is the slowest for the first term, *i.e.*

$$\rho_{Y_1}(t,h) \sim \left(\frac{t}{h}\right)^H,$$

which is the same as for the fBm $Y(t)$.

For $H > 1/2$, the picture is slightly more complex. As it was discussed in Section 2, it is not enough to consider simply the terms in the series decomposition and in order to remove the longer rate of dependence from the error of approximations of the fBm one has to consider

$$Y_N(t) = t^H(S_N(\log t) + V_N(\log t) + V_N'(\log t))\,.$$

For the error $E_N = Y - Y_N$ of approximation, the correlation can be directly computed and its rate for large h is summarized in the following result.

Proposition 2 *For each $N \in \mathbb{N}$ and $t > 0$ the autocorrelation $\rho_{E_N}(t,h)$ has the asymptotic for $h \to \infty$ given by*

$$\rho_{E_N}(t,h) \sim A(N;H)\left(\frac{t}{h}\right)^{N+2-H},$$

where

$$A(N;H) = \frac{2H\left(3(N+1)(N+2\bar{H}) + 2H(N+1) + 4\bar{H}(N+2\bar{H})\right)}{3(N+1)(N+2)(N+2\bar{H}) + 4HN(N+1)f_{N,H}(1) + 8H\bar{H}(N+2\bar{H})g_{N,H}(1)},$$

where $\bar{H} = 1 - H$. Moreover, the autocorrelation $\rho_{Y_N}(t,h)$ of the approximation Y_N is asymptotically given by

$$\rho_{Y_N}(t,h) \sim B(N;H)\left(\frac{t}{H}\right)^{1-H},$$

where

$$B(N;H) = \frac{\sum_{n=1}^{N} \frac{\alpha'_n}{2(1-H)} \frac{\gamma'_n-1}{\gamma'_n+1} + \frac{{r'_N}^2}{2(1-H)}}{\sum_{n=1}^{N} \frac{\alpha_n}{2H} \frac{(\gamma_n-1)^2}{\gamma_n(\gamma_n+1)} + \frac{{r_N}^2}{2H} + \sum_{n=1}^{N} \frac{\alpha'_n}{2(1-H)} \frac{(\gamma'_n-1)^2}{\gamma'_n(\gamma'_n+1)} + \frac{{r'_N}^2}{2(1-H)}}.$$

Proof. The result is a combination of tedious but straightforward calculations of covariances that for ρ_{E_N} uses formulas given in Theorem 3, while in computation of the asymptotic for ρ_{Y_N} uses independence of the approximation terms and the covariance formula of Corollary 1.

□

In the above result, we see that the errors of the subsequent approximations have correlation of the decreasing order $1/h^{N+2-H}$ in h that is lower that the one corresponding to the fBm which is $1/h^{1-H}$. Thus we may conclude that the long range dependence of the fBm Y, for $H > 1/2$, is represented well by the approximation Y_N which has the same rate $1/h^{1-H}$ as the decay of autocorrelation. It should be also noticed that this low rate of decay is only represented by

$$Y'_N = \sum_{n=1}^{N} \alpha'_n \boldsymbol{v}^T \boldsymbol{\Gamma}'_n(t;\gamma'_n) + {r'_N}^2 B'(t^{2-2H})/(2-2H)\,,$$

while the time dependence of the other terms is shorter as it is of a higher order.

The authors would like to thank Igor Rychlik for pointing out the Ornstein–Uhlenbeck covariances in the decomposition of the covariance of the Lt–fBm — this remark eventually led us to the presented representation of the fBm, as well as Patrik Albin for valuable discussion.

Appendix A

In this appendix, we have collected some technical facts about the coefficients of the series representations that have been used throughout the paper.

Lemma 4 *For* $A \in (0,1)$ *define*

$$a_N = (-1)^N \binom{A-1}{N}.$$

Then the following inequalities hold

$$\frac{(1-A)^A(1-A/N)^{N-A}}{N^A} \le a_N < \frac{(1-A)^{A-1}(1-A/N)^{N-A}}{(N+1)^A}\,.$$

Proof 1 *Note that a_N is positive and take $b_N = \log a_N$. Then we have*

$$\begin{aligned} b_N &= \sum_{k=1}^{N} \ln(1-A/k) \\ &= -\sum_{i=1}^{\infty} \frac{A^i}{i} \sum_{k=1}^{N} k^{-i}\,. \end{aligned}$$

Since

$$\int_1^{N+1} x^{-i}dx < \sum_{k=1}^{N} k^{-i} \le 1 + \int_1^{N} x^{-i}dx \tag{A.1}$$

we obtain

$$\begin{aligned} b_N &< -A\ln(N+1) - \sum_{i=2}^{\infty} \frac{A^i}{i(i-1)}\left(1-(N+1)^{-i+1}\right) \\ &= \ln\frac{1}{(N+1)^A} - A\left(\sum_{i=1}^{\infty}\frac{A^i}{i(i+1)} - \sum_{i=1}^{\infty}\frac{(A/(N+1))^i}{i(i+1)}\right). \end{aligned}$$

We note the following identity for $x \in (-1,1)$:

$$\begin{aligned} -\sum_{i=1}^{\infty}\frac{x^i}{i(i+1)} &= -\sum_{i=1}^{\infty}\frac{x^i}{i} + \left(\sum_{i=2}^{\infty}\frac{x^i}{i}\right)/x \\ &= \ln(1-x) - \ln(1-x)^{1/x} - 1 \\ &= \ln(1-x)^{1-1/x} - 1\,. \end{aligned}$$

This leads to

$$\begin{aligned} b_N &< \ln\frac{1}{(N+1)^A} + \ln(1-A)^{A-1} - \ln(1-A/N)^{A-N} \\ &= \ln\frac{(1-A)^{A-1}(1-A/N)^{N-A}}{(N+1)^A}\,, \end{aligned}$$

which proves the upper bound for a_N. In a similar manner, by using the lower bound in (A.1), we obtain

$$
\begin{aligned}
b_N &\geq -A(1+\ln N) - \sum_{i=2}^{\infty} \frac{A^i}{i}\left(1 + \frac{1-N^{-i+1}}{i-1}\right) \\
&= \ln(1-A) + \ln\frac{1}{N^A} - \sum_{i=2}^{\infty} \frac{A^i}{i(i-1)} + N\sum_{i=2}^{\infty} \frac{(A/N)^i}{i(i-2)} \\
&= \ln\frac{1-A}{N^A} - A\left(\sum_{i=1}^{\infty} \frac{A^i}{i(i+1)} - \sum_{i=1}^{\infty} \frac{(A/N)^i}{i(i+1)}\right) \\
&= \ln\frac{1-A}{N^A} + \ln(1-A)^{A-1} + \ln(1-A/N)^{N-A} \\
&= \ln\frac{(1-A)^A(1-A/N)^{N-A}}{N^A}\,.
\end{aligned}
$$

Next, we introduce hypergeometric series and functions and present some identities that were used in the proofs of Section 2.

A hypergeometric series $\sum_{k\geq 0} t_k$ is one in which $t_0 = 1$ and the ratio of two consecutive terms is a rational function of the summation index k, *i.e.*, in which

$$
\frac{t_{k+1}}{t_k} = \frac{P(k)}{Q(k)}\,,
$$

where P and Q are polynomials in k.

In the ratio $P(k)/Q(k)$ of consecutive terms let us consider the polynomials P and Q that are completely factored

$$
\frac{t_{k+1}}{t_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}x\,,
$$

where x is a constant. For $t_0 = 1$, the hypergeometric series (function) whose terms are the $t_k's$,
ie, the series $\sum_{k\geq 0} t_k x^k$, is denoted by

$$
{}_pF_q\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{array};x\right].
$$

The a_i's and the b_i's are called, respectively, the *upper* and *lower* parameters of the series. The b_i's are not permitted to be nonpositive integers or the series will obviously not make sense.

Gauss´ ${}_2F_1$ identity. If b is a non-positive integer or $c-a-b$ has positive real part, then

$$ {}_2F_1\left[\begin{matrix} a \;\; b \\ c \end{matrix} ;1\right] = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\,, $$

where $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$.

We have also the following hypergeometric series identities

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} A \;\; A \\ 1+A \end{matrix} ;1\right] &= \Gamma(1-A)\Gamma(1+A) \\ &= \frac{\pi A}{\sin(\pi A)} \qquad \text{(A.2)} \\ {}_3F_2\left[\begin{matrix} -1+A \quad -1+A \quad 1+A/2 \\ A/2 \quad A \end{matrix} ;1\right] &= \frac{\Gamma(2-A)\Gamma(A)(2-A)}{A} \\ &= \frac{\pi(A-1)(A-2)}{A\sin\pi A}. \qquad \text{(A.3)} \end{aligned} $$

For more on hypergeometric series see [21] and [22].

There are several relations satisfied by the coefficients α_n^2 and ${\alpha'_n}^2$ that can be expressed in terms of hypergeometric series. Here we list those that are important for analysis of the discussed series representations. First note that

$$ \begin{aligned} \alpha_n^2 - {\alpha'_n}^2 &\geq \frac{\sigma^2}{3-2H}(-1)^{n+1}\binom{2H}{n+1}(n+1-H) \\ &\quad \times \left(\frac{n+1}{n+1-2H} - \frac{n+2-2H}{n}\right) \\ &= \frac{2\sigma^2(2H-1)}{3-2H}(-1)^{n+1}\binom{2H}{n+1}\frac{(n+1-H)^2}{n(n+1-2H)} \\ &> 0\,. \end{aligned} $$

The following recurrent formulas prove to be useful for simulations and analysis of approximations

$$ \begin{aligned} \alpha_1^2 &= \sigma^2\frac{H(2H-1)(2-H)}{(3-2H)(1-H)}\,, \\ \alpha_{n+1}^2 &= \alpha_n^2\frac{(n+1-2H)^2(n+2-H)}{(n+1)(n+1-H)(n+2-2H)}\,, \\ {\alpha'_1}^2 &= 2(1-H)^2(3-2H)\alpha_1^2\,, \end{aligned} $$

$$
\begin{aligned}
{\alpha'_{n+1}}^2 &= {\alpha'_n}^2 \, \frac{n(n+1-2H)(n+2-H)(n+3-2H)}{(n+1-H)(n+2-2H)(n+1)(n+2)}\,, \\
\gamma_1 &= \frac{2}{H} - 1\,, \qquad \gamma_{n+1} = \gamma_n + \frac{1}{H}\,, \\
\gamma'_1 &= 1 + \frac{1}{\bar{H}}\,, \qquad \gamma'_{n+1} = \gamma'_n + \frac{1}{\bar{H}}\,.
\end{aligned}
$$

Thus the following power series can be represented as hyperbolical series

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \alpha_n^2 x^n \\
&= \frac{2\sigma^2 H(1-H)}{(3-2H)(2H-1)} \left({}_3F_2 \left[\begin{matrix} 1-2H \;\; 1-2H \;\; 2-H \\ 1-H \;\; 2-2H \end{matrix} \; ; x \right] - 1 \right), \\
&\sum_{n=1}^{\infty} {\alpha'_n}^2 x^n \\
&= 2\sigma^2 H(1-H)(2-H)(2H-1) \cdot x \cdot {}_5F_4 \left[\begin{matrix} 1 \;\; 1 \;\; 2-2H \;\; 3-H \;\; 4-2H \\ 2 \;\; 3 \;\; 2-H \;\; 3-2H \end{matrix} \; ; x \right].
\end{aligned}
$$

In particular, by (A.3) we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} \alpha_n^2 &= \frac{2\sigma^2 H(1-H)}{(3-2H)(2H-1)} \left(\frac{H}{1-H} \Gamma(2H)\Gamma(2(1-H)) - 1 \right) \\
&= \frac{2\sigma^2 H(1-H)}{(3-2H)(2H-1)} \left(\frac{H(2H-1)}{(1-H)\sin(2\pi(1-H))} - 1 \right). \quad \text{(A.4)}
\end{aligned}
$$

We have not been able to find an explicit formula for $\sum_{n=1}^{\infty} {\alpha'_n}^2$, however let us note the following upper bounds for the reminder of this series

$$
\begin{aligned}
\sum_{n=N+1}^{\infty} {\alpha'_n}^2 \le \frac{2\sigma^2}{3-2H} &\left(\frac{H(1-H)}{2H-1} \left(\frac{H(2H-1)}{(1-H)\sin(2\pi(1-H))} - 1 \right) \right. \\
&\left. -(2H-1) \sum_{n=N+1}^{\infty} (-1)^{n+1} \binom{2H}{n+1} \frac{(n+1-H)^2}{n(n+1-2H)} \right) - \sum_{n=1}^{N} \alpha_n^2 .
\end{aligned}
$$

Let us note the following relations

$$
\begin{aligned}
\alpha_n^2 &= \frac{\sigma^2}{3-2H} (-1)^{n+1} \binom{2H}{n+1} \left(n+1+H+\frac{2H^2}{n+1-2H} \right), \\
{\alpha'_n}^2 &= \frac{2\sigma^2(1-H)}{3-2H} (-1)^{n+1} \binom{2H}{n+1} \left(n+3(1-H)+\frac{2(1-H)^2}{n} \right), \quad \text{(A.5)}
\end{aligned}
$$

$$\begin{aligned}\frac{\alpha_n^2}{\gamma_n} &= \frac{\sigma^2 H}{3-2H}(-1)^{n+1}\binom{2H}{n+1}\left(1+\frac{2H}{n+1-2H}\right),\\ \frac{{\alpha'_n}^2}{\gamma'_n} &= \frac{2\sigma^2(1-H)}{3-2H}(-1)^{n+1}\binom{2H}{n+1}\left(1+\frac{2-2H}{n}\right).\end{aligned} \tag{A.6}$$

By the Gauss identity we have

$$\sum_{n=N+1}^{\infty}(-1)^{n+1}\binom{2H}{n+1}(n+1) = (-1)^N\binom{2H}{N+2}$$
$$\times(N+2)\,{}_2F_1\left[\begin{matrix}1\ N+2-2H\\ N+2\end{matrix};1\right] = 2H(-1)^N\binom{2H-2}{N}, \tag{A.7}$$
$$\sum_{n=N+1}^{\infty}(-1)^{n+1}\binom{2H}{n+1} = (-1)^N\binom{2H}{N+2}{}_2F_1\left[\begin{matrix}1\ N+2-2H\\ N+3\end{matrix};1\right]$$
$$= (-1)^N\binom{2H-1}{N+1}. \tag{A.8}$$

Additionally,

$$\sum_{n=N+1}^{\infty}(-1)^{n+1}\binom{2H}{n+1}\frac{1}{n+1-2H} = (-1)^N\binom{2H}{N+2}\frac{1}{N+2-2H}$$
$$\times{}_3F_2\left[\begin{matrix}1\ N+2-2H\ N+2-2H\\ N+3\ N+3-2H\end{matrix};1\right], \tag{A.9}$$
$$\sum_{n=N+1}^{\infty}(-1)^{n+1}\binom{2H}{n+1}\frac{1}{n} = (-1)^N\binom{2H}{N+2}\frac{1}{N+1}$$
$$\times{}_3F_2\left[\begin{matrix}1\ N+1\ N+2-2H\\ N+2\ N+3\end{matrix};1\right]. \tag{A.10}$$

The above relations lead directly to the explicit formulas presented in Theorem 3.

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