ON THE RELATION BETWEEN LACUNARITY AND FRACTAL DIMENSION*

Przemyslaw Borys

Department of Physical Chemistry and Technology of Polymers
Section of Physical Chemistry and Biophysics, Silesian University of Technology
Ks. M. Strzody 9, 44-100 Gliwice, Poland
Przemyslaw.Borys@polsl.pl

(Received April 1, 2009)

Version corrected according to Erratum, *Acta Phys. Pol. B* **42**, 2045 (2011)

I discuss the relation between fractal dimension and lacunarity. Commenting the known results, I propose a method for estimation of the scaling constant in the power law dependency. Additionally, I provide a simple new derivation of a known experimental relation for lacunarity and fractal dimension.

PACS numbers: 05.45.Df, 64.60.al

1. Introduction

Lacunarity is a measure designed to assist fractal analysis in case of images with similar fractal dimension [1]. It was introduced by Mandelbrot when one could see qualitative differences in the mass distribution, *i.e.* differences in the amount of holes[2]. In such case, lacunarities were proposed to differ.

2. Gliding box method

Typical procedure to obtain a value of lacunarity is the gliding box method [3–5]. A gliding box, is a box of specified size that moves in the processing through the whole image, as depicted in Fig. 1. It shifts pixel by pixel from one position to another, centering at each pixel of the image (*i.e.* the decomposition contains more boxes than we could depict in Fig. 2).

As the box moves through the image, the software calculates the number of mass points (black pixels) within the box at each position. After collecting values from all boxes, a histogram is constructed, as shown in Fig. 2. One obtains a statistics of the number of black pixels X within the boxes.

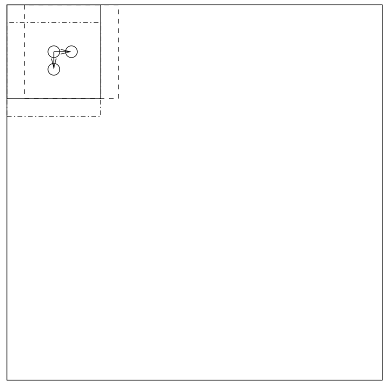

Fig. 1. The gliding box is initially placed in the left upper corner of the image, and then it glides pixel by pixel to subsequent positions (note, that there is no decomposition to adjacent boxes).

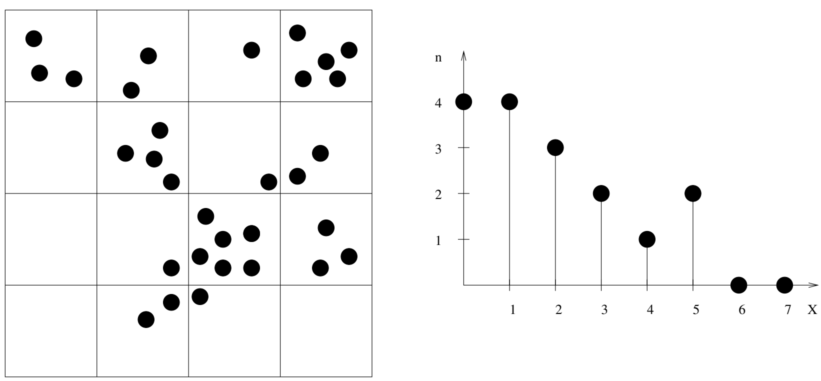

Fig. 2. The gliding box travels over the whole image, and being centered on each pixel (this is simplified in this picture), the software calculates the number of holes inside. These numbers increment the value of a histogram, that is depicted for the shown boxes on the right hand side of this picture.

Having the histogram, it is possible to characterize it by statistical moments. The lacunarity is then defined as a ratio of two expectation values:

$$\Lambda = \frac{E[X^2]}{EX^2}\,, \tag{1}$$

where X is the random variable describing the number of black pixels in the gliding box.

* Presented at the XXI Marian Smoluchowski Symposium on Statistical Physics Zakopane, Poland, September 13–18, 2008.

3. Lacunarity versus fractality

Because lacunarity is said to measure the holes' distribution, while the fractal dimension measures the mass distribution, some people say, that these two measures should be complementary, *i.e.* where one decreases, second may increase. In the work [6] we did experiments on this idea, and it appears, that such conclusion is met in many cases.

Some papers suggest even an equation for the relation between lacunarity and fractal dimension. In a paper of Pomonis [7], we can find a relation like

$$D_0 = 2.47 - 1.4\,\Lambda\,. \tag{2}$$

Similar finding was found for dendrite networks by Smith [8].

At first glance it seems odd to expect relations between fractal dimension and lacunarity since the later was introduced to distinguish images of the same fractal dimension. However, certain aspects of these measures are related, which will be shown later.

4. Scaling of lacunarity with box size

The scaling of lacunarity with the changes of the box size is quite natural and was noticed very early [5, 9]. I propose a similar but probably simpler reasoning here.

Expressing the expectations in (1) by their estimators, we can relate lacunarity to the generalized fractal dimension:

$$\Lambda = \frac{\frac{1}{N}\sum_{i=1}^{N} X_i^2}{\frac{1}{N^2}\left(\sum_{i=1}^{N} X_i\right)^2}\,. \tag{3}$$

To proceed we need a change in the estimators of the averages. Instead of a gliding box based average we will switch to a partition to adjacent boxes, like in the calculation of fractal dimension. This results in a worse sampling of the space, but converges to the same values

$$\Lambda = \frac{\frac{1}{M}\sum_{i=1}^{M} X_i^2}{\frac{1}{M^2}\left(\sum_{i=1}^{M} X_i\right)^2}\,, \tag{4}$$

where $M = 1/\varepsilon^2$ is the number of *adjacent* boxes with side length ε that fit in the image (side length of the image: equal 1).

Recall now the formula for generalized fractal dimension D_2 [10],

$$D_2 = -\lim_{\varepsilon \to 0} \frac{\log \sum_{i=1}^{M} X_i(\varepsilon)^2}{\log 1/\varepsilon}\,. \tag{5}$$

Knowing that it is possible to say how does the sum in the numerator of (4) behave, *i.e.*

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{M} X_i(\varepsilon)^2 = \varepsilon^{D_2}\,. \tag{6}$$

Substituting (6) to (4) and taking the value of denominator in (4) constant and equal to $1/k$ (it is just the average density squared), we obtain:

$$\Lambda = kM\varepsilon^{D_2}\,. \tag{7}$$

This is a known result. Expressing $M = 1/\varepsilon^2$, we have

$$\Lambda = k\varepsilon^{D_2-2}\,, \tag{8}$$

i.e. we see that as D_2 increases, Λ should also increase. This is in contrast to the results of [7, 8]. However, this does not discard these results as will be shown later.

5. Finding the constant for scaling relation

If we denote by Λ_0 the value of lacunarity at a scale ε_0, we can calculate it easily when ε_0 equals to one pixel (in such case only two values of X in histogram are possible):

$$\Lambda_0 = \frac{\frac{1}{M}\,M_{\mathrm{B}}}{\frac{1}{M^2}\,M_{\mathrm{B}}^2} = \frac{M}{M_{\mathrm{B}}}\,, \tag{9}$$

where M is the number of pixels in the image, M_{B} is the number of black pixels in the image. This is a new result that shows the dependency of lacunarity on the brightness of the image (or, using physical terms, on the mass found in the investigated structure).

We can combine (9) with (8) to obtain

$$\Lambda = \Lambda_0\left(\frac{\varepsilon}{\varepsilon_0}\right)^{D_2-2}\,. \tag{10}$$

This reasoning can be generalized to the gray scale images, as they can be converted to binary images by increasing resolution. If one has 256 gray scale levels in the image then we can divide each pixel to 256 sub-pixels (a box 16×16) and put in such box the amount of black pixels equal to the gray scale level. In this case the equation (10) transforms into

$$\Lambda = \Lambda_0\left(\frac{16\varepsilon}{\varepsilon_0}\right)^{D_2-2}\,, \tag{11}$$

$$\Lambda_0 = \frac{256M}{\sum_{i=0}^{M}(256-g_i)}\,, \tag{12}$$

where g_i is the gray level of the pixel i.

6. Finding the complementary relation between Λ and D_0

In the above scaling, one can see that the lacunarity increases as D_2 increases. In Sec. 3 we could see the contrary: when the fractal dimension increases, lacunarity lowers.

To have such behaviour we need special circumstances. This can happen when one has differences in the amount of mass in the two compared images. For example one can have a porous solid where the pores are distributed in similar frequency, but differ in shape (and, therefore, in mass per pore ratio).

Assume that the pores in two images have a diameter ε_α (larger than the pixel size ε_0). Then,

$$\frac{N_{B0}}{N_{B\alpha}} = \left(\frac{1/\varepsilon_0}{1/\varepsilon_\alpha}\right)^{D_0} \tag{13}$$

if we take $\varepsilon_\alpha/\varepsilon_0 = \alpha$, we have $N_{B0} = N_\alpha \alpha^{D_0}$. Then expanding α^{-D_0} into a series, we obtain:

$$\Lambda_0 = \frac{N}{\alpha^{D_0} N_{B\alpha}} \approx \frac{N}{N_{B\alpha}} (1 - \log \alpha D_0) \,. \tag{14}$$

This is the desired result, but we can see that it does not come from the scaling relation (8) but from the differences in the amount of mass in the image. This corresponds with the findings of [7, 8], where the lacunarities were evaluated at a particular length scale, where the scaling (8) plays a minor role.

7. Concluding remarks

In this paper I have investigated the lacunarity measure. I have proposed a new simple derivation of the relation for the scaling with a box length. What is more important, I have shown how to find the proportionality constant in this relation.

The paper also presents a way for obtaining a complementary relation between the values of lacunarity and fractal dimension that were reported from experiments.

REFERENCES

[1] B.B. Mandelbrot, *The Fractal Geometry of Nature*, Ed. W.H. Freeman, New York 1983.

[2] B.H. Kaye, *A Random Walk Through Fractal Dimensions*, VCH Publisher, New York 1989.

[3] R.E. Plotnick, R.H. Gardner, R.V. O'Neil, *Landscape Ecology* **3**, 201 (1993).

[4] R.E. Plotnick, R.H. Gardner, W.W. Hargrove, K. Prestegaard, M. Perlmutter, *Phys. Rev.* **E77**, 5461 (1996).

[5] C. Allain, M. Cloitre, *Phys. Rev.* **A44**, 3552 (1991).

[6] M. Krasowska, P. Borys, Z.J. Grzywna, *Acta Phys. Pol. B* **35**, 1519 (2004).

[7] G.S. Armatas, K.M. Kolonia, P.J. Pomonis, *Langmuir* **18**, 10421 (2002).

[8] T.G. Smith Jr., G.D. Lange, W.B. Marks, *J. Neurosci. Methods* **69**, 123 (1996).

[9] Q. Cheng, *Math. Geology* **29**, 919 (1997).

[10] E. Ott, *Chaos in Dynamical Systems*, Cambridge University Press, 1994.