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A MULTIPLICATIVE LAW OF NETWORK TRAFFIC AND ITS CONSEQUENCES*

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A basic law of Network traffic is derived. The appearance of long tails and self-similar processes is clarified. The theory is confirmed by empirical measurements.

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1. Introduction

The rapidly growing computer Network has been the object of intensive studies for over a decade. Among the most commonly studied characteristics are packets count, packets sizes, or arrival times. Because of a different nature and the high complexity of Network Traffic the early attempts to model traffic variables as in the traditional teletraffic failed. The process of packets arrivals is no longer Poissonian, the inter-arrival times are not exponentially distributed and the traditional queuing theory is also not applicable.

The studies of Network traffic are mostly empirically founded. The majority of the publications is devoted to statistical analysis of observed variables and their consistency with various ad hoc proposed models. In the modelling of network traffic the predominant role has been attributed to self-similar processes, heavy-tailed distributions and fractional noises. Beginning from the pioneering work of Leland *et al.* [1] (see also [2–7], and many others), it is claimed that among the most characteristic features of Network traffic are *self-similarity* and a *bursty* nature. Such claim is supported by observation of the behavior of some traffic variables, like the throughput traffic

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at a given point of a network. The plots of such traffic time series observed during different periods of time (hours, minutes, seconds) look similar (see *e.g.* [5–7]), which suggests that changes of the time scale do not affect the distributions of the observed process.

The heuristic arguments for self-similarity in Network traffic [1] seem to be credible but do not explain the basic mechanism leading to the proposed model. Moreover, mathematically simple forms of self-similarity are not evident in the observed data. Even if they are, the proposed models provide a very limited information about probabilistic characteristics of the considered processes and, consequently, they are not valuable for predictions.

In this paper we show that there is an accurate way to model the underlying structure of Network time series. We show that behind the observed self-similarity and "burstines" there is a simple multiplicative law which plays also a fundamental role in traffic modeling. We derive here, on a fundamental level, a simple but rigorous model of packets traffic which can be further elaborated. Although this model is based on simplifying assumptions concerning the network, it is in perfect agreement with traffic modeling and observations.

We model rigorously the packet traffic between two arbitrary sites of a network separated by a number of nodes (routers) and study its properties. The main object of the study is the inter-arrival time IAT (or inter-delay time) between packets which changes as the packets emitted at a source travel to a destination. We have shown that the inter-delay times obey a simple multiplicative law which under the assumption of independence of the transverse traffic leads to log-normal distributions. As an application of the multiplicative law the distribution of the transmission processes has been derived. This in turns allowed to model the throughput traffic which has been one of the most frequently measured and discussed traffic variables. Using our model we have explained and corrected some statements concerning the nature of the apparent self-similarity, long memory and log-normal distributions of throughput traffic. In particular we have given a qualitative explanation for the common appearance of long tails. The aforementioned theoretical model has been also validated by real network traffic measurements and with the use of a network traffic simulator.

2. Multiplicative law

We begin with a theoretical study of inter-arrival times (IAT), which are understood as the time intervals between two consecutive packets sent through the network. The IAT change as the emitted packets travel to the destination. Our goal is to describe the distribution of the final delay at the destination.

We assume that the considered traffic between the source and the destination takes place on the same path and that changes of the time interval between packets take place on the routers, and are caused by *transverse* traffic, *i.e.* additional packets that enter between the considered ones. Although two packets sent from the same source to the same destination can, in principle, travel over different routes this does not happen in practice. Eventual redirection of packets on particular routers would require changes in the routing tables. But such actions are not frequent and can be neglected.

Let us, therefore, consider a chain of routers $R_1, R_2, \ldots, R_N$, as depicted in Fig. 1, through which a sequence of packets is send. Between each two consecutive packets there is initially some fixed time of delay, called the *inter-arrival time* (IAT), which can change on further routers. These changes are caused by the transverse traffic on the routers. The IAT can both increase and decrease. The increase is caused by the necessity of additional service. The decrease of IAT can happen when some packets from the transverse traffic leave the chain and the observed packets are stored in the buffers on the next routers. Moreover, we will assume that all considered packets require the same amount of service-time. This assumption is not very restrictive since the size of individual packets does not vary significantly (the size of packets is usually between 40 and 1500 bytes).

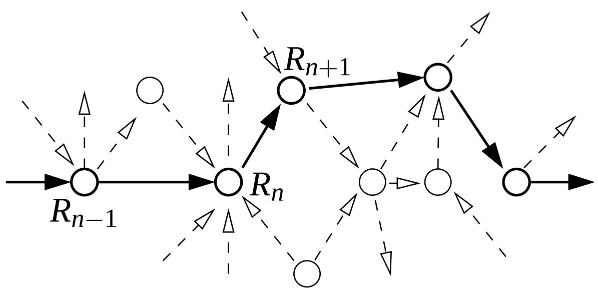

Fig. 1. A chain of routers.

The transverse traffic can influence packets delays in the following way. If an additional packet enters in between two observed packets then it increases the delay about the amount of time needed for its service. We assume that this amount of time is equal to σ. If an additional packet leaves the line and there is a queue on the next router then the gap can decrease by σ. The increase or decrease of the IAT is proportional to the size of the time gap.

The transverse traffic on each router will be expressed by the random variables ξ_k, $k = 1, 2, \ldots$. Let us derive first a traffic model under the assumption of non congested line, *i.e.* we assume that ξ_k are non negative integers. In such case ξ_k can be interpreted as the number of packets entering the unit gap on the k-th router.

Suppose that we observe two packets traveling from a source to a destination. By $\tau_1, \tau_2, \ldots$ we will denote the IAT between these packets after they pass through the routers $R_1, R_2, \ldots$, correspondingly. Let us assume, for simplicity, that $\tau_0 = 1 = \sigma$ and denote

$$\Delta\tau_k = \tau_k - \tau_{k-1}\,, \qquad k = 1, 2, \ldots\,. \tag{1}$$

Thus $\Delta\tau_k$ is the increase of the delay between two observed packets after they passed k-th router.

When the second packet leaves the first router the initial delay $\tau_0 = 1$ is augmented by the time needed for the service of the additionally arrived packets, *i.e.*

$$\Delta\tau_1 = \xi_1 = \xi_1\tau_0\,.$$

On the second router the increment of the delay is $\Delta\tau_2 = \xi_2\tau_1$ and, generally, we have

$$\Delta\tau_k = \xi_k\tau_{k-1}\,, \qquad k = 1, 2, \ldots \tag{2}$$

Using (1) and (2) we have

$$\tau_k = (1 + \xi_k)\tau_{k-1}\,, \tag{3}$$

which gives

$$\tau_k = \prod_{i=1}^{k}(1 + \xi_i)\,. \tag{4}$$

The above derived dependence (3), or equivalently (4), between the IAT variables τ_k and the transverse traffic represented by ξ_k will be called the *multiplicative law*.

In general, it can be assumed that ξ_k appearing in (3) are real valued random variables, bounded from below by -1. $\xi_k > 0$ means an increase of IAT caused by the transverse traffic, on the k-th router. $\xi_k < 0$ means a decrease of the IAT caused by packets leaving a unit gap due to the aforementioned cogestion and buffering.

Assuming that ξ_k are independent and representing $\ln\tau_k$ as the sum

$$\ln\tau_k = \sum_{i=1}^{k}\ln(1 + \xi_i)$$

we can apply a central limit theorem to the random variables $\ln(1+\xi_i)$. For example, if ξ_i are identically distributed then we have the following theorem:

Theorem 1 *Suppose that the random variables ξ_i, $i = 1, 2, \ldots$ are independent, identically distributed, and satisfy*

$$\xi_i > -1\,, \qquad \textit{for each} \ \ i\,.$$

Moreover, let

$$\mathrm{E}\ \ln^2(1+\xi_i) < \infty\,.$$

Then the distribution of the IAT τ_k are asymptotically log-normal $\Lambda(k\mu_\xi, k\sigma_\xi^2)$, with $\mu_\xi = \mathrm{E}\ \ln(1+\xi_i)$ and $\sigma_\xi^2 = \ln(1+\xi_i)$.

Recall that a random variable X has the log-normal distribution $\Lambda(\mu, \sigma^2)$ if $X > 0$ *a.e.* and the random variable $\ln X$ is normally distributed with mean μ and variance σ^2. Then the density of X is of the form

$$\frac{1}{x\sigma\sqrt{2\pi}}\ \exp\left[-\frac{1}{2\sigma^2}(\ln x - \mu)^2\right]\,, \qquad \text{for} \ \ x > 0\,.$$

The assumption of identical distributions of ξ_k can be relaxed due to various forms of central limit theorems (see, *e.g.* [8]). For non-identically distributed random variables ξ_k the asymptotic expression $k\mu_\xi$ for $\mathrm{E}\ln\tau_k$ can be obtained by applying the law of large numbers to

$$\ln\sqrt[k]{\tau_k} = \frac{1}{k}\sum_{i=1}^{k}\ln(1+\xi_i)\,.$$

Note that we can also estimate the ratio $\Delta\tau_k/\xi_k$ of the increase of the delay on k-th router to the the delay caused by the transverse traffic. Indeed, putting (4) to (2) we obtain

$$\Delta\tau_k = \xi_k\prod_{i=1}^{k-1}(1+\xi_i)\,,$$

from which we get

$$\frac{\Delta\tau_k}{\xi_k} = \prod_{i=1}^{k-1}(1+\xi_i)\,. \tag{5}$$

Therefore,

$$\tau_k = \frac{\Delta\tau_{k+1}}{\xi_{k+1}}$$

and the conclusion is that τ_k and $\Delta\tau_k/\xi_k$ have the same asymptotic distribution.

Theorem 1 implies that the distributions of IAT have heavy tails in the sense that their tails decay slower that any exponential distribution (see Sec. 3). This feature has been already confirmed empirically and called the “burstines” of packets traffic.

An elaboration of the traffic model in which we do not assume independence of the transverse traffic can be found in Appendix A.

3. Transmission time

Using the above derived model we can now study the probabilistic properties of transmission processes, natures of congestions, or the throughput traffic. We begin with the derivation of the distribution of transmission time, which is defined as the time needed for sending a number of packets from a source to a destination. This time consist of two components — the sum of time distances between consecutive packets and the sum of delays during the transmission. We neglect here the service-time of our packets.

Let us assume that the number of packets is equal to $n+1$ and that the initial time distance between packets is δ. Denote by $\tau^{(1)},\ldots,\tau^{(n)}$ the delays of consecutive packets when they reach the destination and by $T=T_n=\tau^{(1)}+\ldots+\tau^{(n)}$ the total delay. Then the transmission time is

$$n\delta+T\,. \tag{6}$$

The first component of (6) is deterministic thus our main concern is the distribution of the random variable T. Retaining the assumptions from Sec. 2 we obtain that IAT are log-normally distributed. However, if $n>1$ then the distribution of the sum of the total delay

$$\tau^{(1)}+\ldots+\tau^{(n)}\,. \tag{7}$$

is no longer log-normal. Log-normality is not preserved when taking the sums of independent random log-normally distributed random variables. In principle the distribution of the sum $\tau^{(1)}+\ldots+\tau^{(n)}$ is asymptotically normal, because $\tau^{(i)}$ are assumed to be independent and identically distributed random variables with finite second moment. It should be noticed, however, that the accuracy of the approximation of the distribution of T by a normal distribution depends heavily on how large is n and on the variances (“broadness”) of $\tau^{(i)}$.

For $\tau^{(i)}$ with large variances the distribution of their sum is very flat and akin to a uniform distribution on a large interval. In consequence the dominating role in the distribution of sums of $\tau^{(i)}$ is played by the largest value of $\tau^{(1)},\ldots,\tau^{(n)}$. In fact, we have the following result

Theorem 2 *Assume that the of $\tau^{(i)}$ are independent and have identical log-normal distributions $\Lambda(\mu, \sigma^2)$. Then the total delay $T_n = \tau^{(1)} + \ldots + \tau^{(n)}$ has asymptotically the Gumbel distribution*

$$P(T_n < t) = G(c_n t + d_n)\,,$$

where $G(t) = e^{-e^{-t}}$, $c_n = \sigma d_n / \sqrt{2 \ln n}$
and $d_n = \exp\left\{\mu + \sigma\left(\sqrt{2\ln n} - \frac{\ln(4\pi) + \ln\ln n}{2\sqrt{2\ln n}}\right)\right\}$.

The proof of this theorem is based on properties of subexponential distributions (see [9]). Recall that a random variable with the distribution function $F(t)$ is called subexponentially distributed if it has the following property:

$$\lim_{t\to\infty}(1 - F(t))e^{\alpha t} = \infty\,, \qquad \text{for each} \quad \alpha > 0\,.$$

Because the IAT $\tau^{(n)}$ are log-normally distributed they are subexponential (see [9]). Moreover, since they are also independent the distribution of the sum $\tau^{(1)} + \ldots + \tau^{(n)}$ is asymptotically equal to the distribution of $\max\{\tau^{(1)}, \ldots, \tau^{(n)}\}$. Since log-normal distributions belong to the domain of attraction of the Gumbel distribution, there exist constants $c_n > 0$ and d_n such that the limit distribution of $c_n^{-1}(\max\{\tau^{(1)}, \ldots, \tau^{(n)}\} - d_n)$ is $e^{-e^{-x}}$ (see [9] for the above explicit form of c_n and d_n).

4. Throughput traffic

We would like to present now the model of throughput traffic, which has been one of the most frequently measured and discussed traffic variables. In fact the easiest way to study experimentally network traffic is to register at some point of the network the number of packets, or bytes, per a time unit. This kind of measurements is a special case of throughput traffic. In various publications the meaning of the throughput traffic differs slightly. Nevertheless, the basic idea behind this concept is to measure the volume of traffic in regular time intervals. It has been observed that for different length of time intervals Δt the distribution of traffic volume can vary significantly. It is usually undetermined for short time intervals, then gradually, the distribution assumes more defined shape when Δt increases. Different hypotheses concerning the observed distributions, which can vary with Δt, have been proposed. Using the obtained results we will derive now rigorously the throughput traffic distributions.

Suppose we are interested in the throughput traffic between two nodes of a network and that we cannot observe any of the intermediate nodes. In such case the throughput traffic can be measured sending a number of test packets is at the source in regular time intervals δ. If the transmitted packets arrive with the same frequency δ, then the eventual transverse traffic can be neglected and we can simply say that, in average, there is no other traffic on the line. Otherwise, additional transverse traffic will cause delays proportional to its intensity. Therefore, measuring the delays of inter-arrival times we can infer about the traffic on the line. Such average traffic on the line connecting the source with the destination is, what we call here, the *throughput traffic*. In this approach we do not need to register all traveling packets.

In rigorous terms, the (average) throughput traffic on the line during the time interval Δt is represented by

$$n\delta(R_{t+\Delta t} - R_t)\,,$$

where R_t is the total number of test packets received up to time t. The stochastic process $\{R_t\}$ will be called the transmission process.

We derive first the distribution of R_t. Let $\tau^{(1)}, \tau^{(2)}, \ldots$ denote the time intervals between consecutive test packets. We assume that $\tau^{(k)}$ are independent and have the same distribution function $F(t)$. Then we have

$$\begin{array}{lcl} R_t = 0 & \Leftrightarrow & t < \tau^{(1)}\,, \\ R_t = 1 & \Leftrightarrow & \tau^{(1)} \le t < \tau^{(1)} + \tau^{(2)}\,, \\ & \vdots & \\ R_t = k & \Leftrightarrow & \tau^{(1)} + \ldots + \tau^{(k)} \le t < \tau^{(1)} + \ldots + \tau^{(k+1)}\,. \\ & \vdots & \end{array}$$

Therefore

$$P(R_t \le x) = P(\tau^{(1)} + \ldots + \tau^{[x]} > t)\,, \tag{8}$$

where the symbol $[\,\cdot\,]$ stands for the entire value of a real number. Since the random variables $\tau^{(k)}$ are independent

$$P(R_t \le x) = 1 - P(\tau^{(1)} + \ldots + \tau^{[x]} \le t) = 1 - F^{*[x]}(t)\,.$$

Let us assume that the process $\{R_t\}$ is also homogenous and denote

$$\Delta R \stackrel{\mathrm{df}}{=} R_{t+\Delta t} - R_t\,,$$

where Δt is fixed. It follows from the above considerations that

$$\Delta R = \tau^{(i)} + \ldots + \tau^{(i+k)}\,, \tag{9}$$

for some i, k. The number of components k of (9) depends on the length of the interval Δt.

If the number of components of (9) is about 1 then the distribution of ΔR coincides with the log-normal distribution. This means that there exists such time interval (threshold) Δ_0 for which the throughput traffic is log-normal. Increasing time interval we see the gradual departure from the log-normal distribution. We obtain instead the distribution of aggregations of log-normal distribution which, in turn, belongs to the family of subexponential distributions.

The log-normality in throughput traffic have been already observed experimentally (see [10–12], for example). Particularly interesting is Ref. [10] devoted to the statistical analysis of the throughput of large packets transfers. Several different distributions have been proposed, as possible models of the observed throughput, and tested. It was the log-normal distribution which fitted the best the observed data.

Using our model we can now correct some statements concerning the nature of the distribution of throughput traffic. It has been claimed in [12] that beginning from some threshold value of the aggregation level the distribution of packet log-normal. This claim is partially consistent with our theoretical model. Namely, for the aggregation level in the vicinity of the average IAT. It cannot be true, however, for significantly larger aggregations. The reason is very simple, an aggregation on a large time interval is also an aggregation of the time series already aggregated with the threshold value. The process of aggregation means forming sums of independent random variables with log-normal distributions. These sums are, however, no longer log-normally distributed. The distribution of the sum of independent log-normal random variables heavily depends on the variance of their components.

5. Self-similarity of Network traffic

Finally we would like to comment on the problem of self-similarity which is postulated in a majority of the publications on Network traffic. First, however, let us clarify the meaning of a self-similar process. Intuitively, self-similarity of a stochastic process means independence of the time scale. A mathematically rigorous definition, which is in agreement with the intuitive meaning, is that a stochastic process $Y(t)$ is called H-self-similar if, for each $a > 0$, its finite dimensional distributions coincide with the distributions of the scaled process $a^{-H}Y(at)$. The number H, $0 < H \leq 1$ is also called the *Hurst exponent*. Brownian motion is an example of a self-similar process with $H = 1/2$. However, the above definition of self-similarity is not used in practice. It is assumed instead that the values of the observed time series $x_k, k = 1, 2, \ldots$ are the stationary increments of some self-similar process. The self-similarity of a time series x_k, called also the *exact self-similarity*, means that x_k is a fixed point of renormalization group, *i.e.* the aggregated series

$$x_k^{(m)} = \frac{1}{m} \sum_{i=(k-1)m+1}^{km} x_i \,. \tag{10}$$

has the same distribution as the series $1/(m^{1-H})\, x_k$, for different levels of aggregations m (see [3]). In practice, this is usually limited to checking whether $\log \mathrm{E}\, |x_k^{(m)}|^r$ is a linear function of $\log m$, for each r.

Applying the latter definition we can explain the observed self-similar-like behaviour of average throughput traffic. Indeed, since the increments ΔR are of the form (9), *i.e.* the sums of subexponential random variables, ΔR are also subexponential. We see, however, that the distributions may be distinct for different levels of aggregations although their tails are of similar type and consequently also their moments. Since in practice not the distributions are compared but only a first few moments, it is claimed that the observed traffic time series is a self-similar process. We can also explain a commonly encountered phenomenon that the estimated Hurst exponent is not invariant with respect to the level of aggregation. Such behaviour is in agreement with (9) because the level of aggregation is in one-to-one correspondence with the length of the observations Δt. When Δt increases the distributions of the throughput traffic evolve from the log normal to the Gumbel distribution. The speed of the deviation of the distributions of throughput traffic from the basic log normal depends on how heavy are the tails or, equivalently, the intensity of the transverse traffic. In Appendix B it is also shown the dependence between Hurst exponents and connections length.

6. Measurements and simulations of network traffic

In this section we demonstrate briefly the consistency of the derived multiplicative law of network traffic with measurements and simulations (see our earlier publications [13–16] for details and further results).

Measurements

There exist several tools as *Ping* and *Traceroute* that permit to analyze the network configuration between two computers. We remind that the *Ping* command provides the *Round Trip Time* (RTT) needed by a probe packet to firstly travel from the emission node N_{e} toward the reception node N_{r}, and then from N_{r} toward N_{e}. We can estimate the transverse traffic of the connection path by analyzing the RTT modifications of successive probe packets. As for the *Traceroute* command, it helps to assess the connection path length in hops between two network entities. We have implemented an automatic script: first, we randomly select a destination *Internet Protocol* (IP) address IP_{r}. Then we obtain the path between IP_{r} and our network router IPe in respect with the $Traceroute$ command. Finally we extract the IAT distribution of *Ping* packets generating between IP_{e} and IP_{r}. The *Ping* command periodically sends *Internet Control Message Protocol* (ICMP) packets towards the desired IP address. Therefore the original IAT of the sending process presents a Dirac delta distribution. The peak is given by the time period between the emission of two successive packets. During the round trip, the transverse traffic of the connection path strongly affects the IAT distribution.

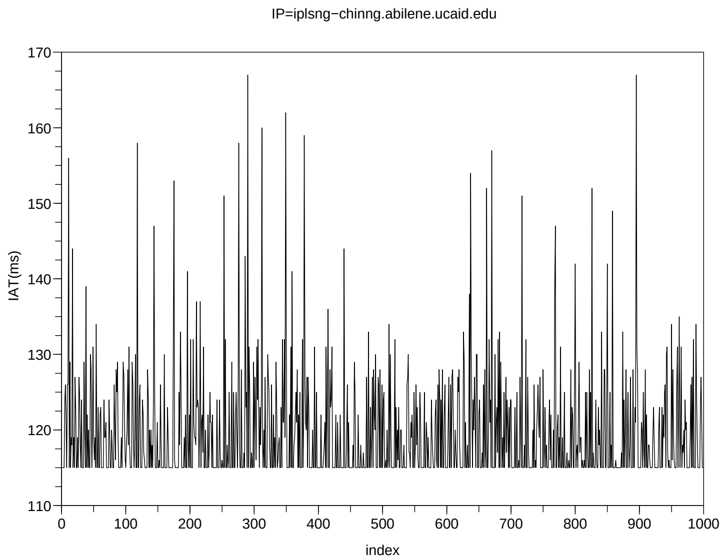

Fig. 2. *Ping* IAT process between a router located at the University of Luxembourg and the IP address 198.32.8.77 (`iplsng-chinng.abilene.ucaid.edu`) corresponding to a router based at the University of New York.

For instance in Fig. 2, we show the final IAT after a round trip between a probe router of the University of Luxembourg and a router located in the University of New York (United States). 1000 ICMP packets have been periodically sent every 0.1 second. We observe that the IAT process presents a high variability. Values are included in the range [115 ms–168 ms]. The distribution tail becomes long. Similar measurements have been done for other routers distributed all around the world. A typical distribution of IAT is log-normal with large σ, see *eg.* Fig. 3.

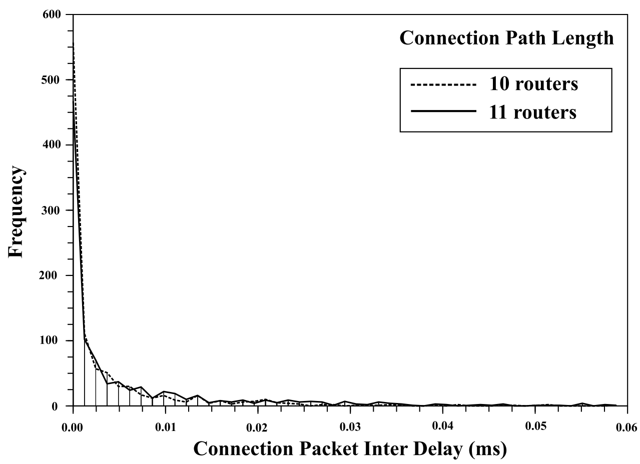

Fig. 3. Connection packet IAT distribution for the two last routers.

Simulations

We perceive the real impact of the connection path length in the final distribution shape. In order to evaluate the impact of the path length on the final IAT distribution, we have implemented a traffic simulator. In this way we achieve a better control of the interaction between the path length and the transverse traffic. We can examine the spread of the distributions δ_j as a function of j and observe the evolution of the IAT distributions recorded over several distinct connections in a well defined network as a function of the path length.

The inter-arrival time (IAT) modifications are connected to the network load. In practice the TCP protocol manages the source packets emission according to the network configuration. If the network load is heavy, the source is notified to reduce its sending rate. Therefore, TCP modulates the traffic source as the emission node adapts its flow in order to avoid congestions. Let n be the total number of packets sent from the emission node N_e towards the reception node N_{r} and the path length $\mathcal{L}$ be defined as the total quantity of nodes belonging to the path between N_{e} and N_{r}.

The data transmission reveals that the distributions $\delta_{\mathcal{L}}$ for the reception node N_{r} changes radically from their original shape δ_1. The δ_j distributions extend themselves after crossing each connection path node. The variances increase with j and the distribution tails become longer. In fact each node N_j has its own absolute transverse traffic which is coming from its neighbors. We observe that the transverse traffic of the node N_j modifies δ_j according to a multiplicative law (see Fig. 4). Therefore, the distributions $\delta_{\mathcal{L}}$ in N_{r} depend on the network load, especially on the transversal traffic, and on the connection length $\mathcal{L}$.

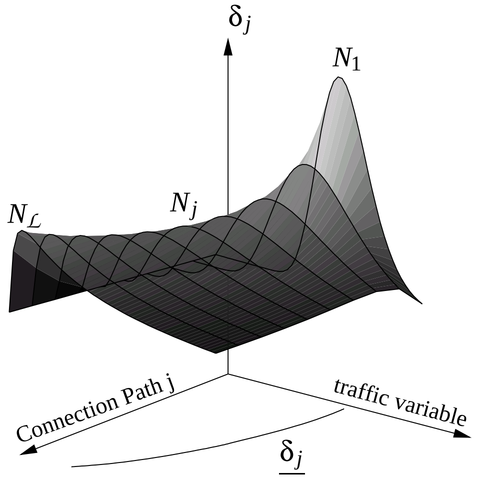

Fig. 4. Evolution of traffic variable distributions along the connection path $N_1 \mapsto N_{\mathcal{L}}$: the distribution tail becomes longer with the increasing path length.

7. Concluding remarks

In summary, in this work we have derived for a simple network a multiplicative law for packet delay which lead to log-normal distributions and explains self-similar-like behaviour of some traffic variables. In particular we have given a qualitative explanation for the common appearance of long tails. We believe that the above derived multiplicative low is a universal Network invariant and that our work will help to understand and predict Internet behaviour.

Appendix A

Martingale model of inter-arrival times

We begin with the following observation. Under the assumptions of Section 2, let us consider again the sequence $\tau_1, \tau_2, \ldots$ of IAT and their increments $\Delta\tau_n := \tau_n - \tau_{n-1}$, for $n = 1, 2, \ldots$ and note that equation (2) is equivalent to

$$\xi_n = \frac{\Delta\tau_n}{\tau_{n-1}}\,, \tag{A.1}$$

which means the relative change of the IAT after passing the n-th router. Denote by $\mathcal{A}_n$ the σ-algebra generated by independent random variables $\xi_1, \ldots, \xi_n$ which represents the available knowledge of transverse traffic. Thus $\mathcal{A}_n$ is identified with the accessible information after the packet passes n-th router.

Since $\tau_n = \prod_{i=1}^{n}(1+\xi_i)$ (see (4)) the random variables τ_n are $\mathcal{A}_n$-measurable. Moreover, if we assume, like in Section 2, that the random variables ξ_n, $n = 1, 2, \ldots$, are independent then, because of the independence of ξ_n and $\mathcal{A}_{n-1}$, we also have

$$\mathrm{E}\,(\tau_n|\mathcal{A}_{n-1}) = \tau_{n-1}\mathrm{E}\,(1+\xi_n)\,.$$

Putting $a_n := \mathrm{E}\,\xi_n$ we have

$$\mathrm{E}\,(\tau_n|\mathcal{A}_{n-1}) = (1+a_n)\tau_{n-1}\,. \tag{A.2}$$

Let us now consider the sequence

$$A_n := (1+a_1)(1+a_2)\ldots(1+a_n)A_0\,,$$

where A_0 is a constant. By (A.2)

$$\mathrm{E}\left(\frac{\tau_n}{A_n}\,\middle|\,\mathcal{A}_{n-1}\right) = \frac{\tau_{n-1}}{A_{n-1}}\,, \tag{A.3}$$

which means that the sequence $\left\{\frac{\tau_n}{A_n}\right\}$ is a martingale (see [17, 18]) with respect to the filtration $\{\mathcal{A}_n\}$. If in particular all a_n are equal to a, *e.g.* the case of identically distributed ξ_n, then $A_n = (1+a)^n A_0$.

The martingale property of the sequence of IAT times suggest the way in which our model can be generalized. Namely, generalizing the model for IAT we retain the basic structure of τ_n as described above but we relax the assumptions concerning the transverse traffic. We do not assume anything about the specific form form of the distribution of the transverse traffic but only that relation (A.3) holds, *i.e.* $\{\tau_n/A_n\}$ is a martingale with respect to $\{\mathcal{A}_n\}$ ($\mathcal{A}_n$ as above).

An immediate consequence of the above assumption is that the prediction of τ_{n+1} based on the knowledge of the traffic on the previous n routers amounts to the knowledge of τ_n and the average value of the transverse traffic at $n+1$. Indeed, the martingale property reduces now to relation (A.2).

For longer chains of routers limit theorems for martingales can be also applied. Namely, assuming that the random variables ξ_n have finite first moment and $\xi_n > -1$, $n = 1, 2, \ldots$, we obtain that $\{\tau_n/A_n\}$ is a positive martingale thus convergent *a.e.* to some limit M_∞. Moreover, we can also apply Kakutani's theorem (see [19]) to characterize this limit. Namely we have the following

Proposition 1 *Let* $\alpha_n = \mathrm{E}\sqrt{1+\xi_n}/\sqrt{1+a_n}$. *Then* $\mathrm{E}M_\infty = 1$ *if and only if* $\prod_n \alpha_n > 0$ *(equivalently* $\sum_n(1-\alpha_n) < \infty$*). If* $\prod_n \alpha_n = 0$ *(equivalently, if* $\sum_n(1-\alpha_n) = \infty$*) then* $M_\infty = 0$ *a.e.*

The first part, $\mathrm{E}\,M_\infty = 1$, of the above proposition is the non-trivial one from the point of view of application. It allows to estimate the mean value of τ_n for long chains. Namely assuming, for simplicity, that the constant A_0 equals 1 we have

$$\lim_n \mathrm{E}\,\tau_n = \lim_n A_n = \prod_n (1 + a_n) \; .$$

Particulary interesting is the case of the transverse traffic with the same distribution. Here we have $\mathrm{E}\,\xi_n = a$, for each n, an also $\alpha_n = \alpha$ (constant). It follows from the assumption on ξ_n that $\alpha \leq 1$. Thus we the following dichotomy. If $\alpha = 1$ then we have the nontrivial case of the convergent martingale or, if $\alpha \neq 1$, the limit M_∞ is 0, which means that the traffic eventually dies out. The latter case is irrelevant from the point of view of applications.

Appendix B

Traffic simulator and simulation results

We have implemented simulations on the top of the famous and commonly used *ns*-2. Our goal is to approximate scenarios closed to the Internet characteristics in order to extend our conclusions to real traffics. Here are some values commonly used to simulate the Internet [20]. The majority of the traffic on the Internet is associated to file transfers. The average transferred file borders 10 Kbytes. This means that an "average" file has no more than 10 TCP packets if we take the typical TCP packet size to be 1 Kbyte. So the majority of file transfers ends in the slow-start phase. These files are frequently called "mice". However, the most of the traffic in the Internet is transmitted by very long files called "elephants. A typical *Pareto* distribution describes the files size, with a shape parameter value between 1 and 2 (with an average of 10 Kbytes). The files size median is around 2.5 Kbytes. A Pareto distribution with a mean size of 10 Kbytes and a median size of 2.5 Kbytes corresponds to a shape parameter $\beta = 1.16$ and a minimum size of 1.37 Kbytes. The IAT distribution of new connections is frequently taken to be exponential. Let consider a network defined by the reduced topology defined in the Fig. 5.

We study a connection between two nodes N_{e} and N_{r} linked together by n routers $\{R_1\; R_2\; \ldots R_n\}$. In respect with [21], we fixed the maximal path length n to 22 hops for our simulations. Indeed the probability that a connection path met on the Internet would be composed by more than 22 nodes remains insignificant. According to our approach each virtual source emits data on the network in respect with the degree of its neighbor node belonging to the connection.

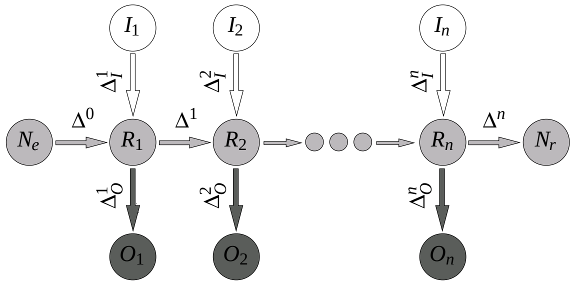

Fig. 5. Topology model: a data transaction between N_{e} and N_{r} is modulated by the transverse traffic crossing each connection node.

Here is a brief description of our simulator. Each router is connected to its own virtual source node and its own virtual reception node. We randomly select each link delay into the uniform set $[1 \mapsto 5\mathrm{ms}]$. Physically the delay Δ_{I}^{k} (respectively Δ_{O}^{k}) represents the average delay of all links towards (respectively from) R_k which send (respectively receive) packets to (respectively from) it. Finally all input flows of R_k are aggregated into one flow over a single link with the delay Δ_{I}^{k}. The same assumption is performed for the output flows. Each link is associated to a maximum rate of 10 Mbps. Even if that does not reflect the reality, we expect again that this assumption does not change radically the final traffic structure. Each virtual source node can create a connection towards any virtual reception node. Therefore, $n+1$ traffic sources (n virtual sources and N_{e}) share the connection path $N_{\mathrm{e}} \mapsto N_{\mathrm{r}}$. Each virtual source node can start up to 500 TCP sessions while the 20 s connection duration between N_{e} and N_{r}. A new FTP application is defined for each TCP agent. The new TCP connections arrive according to a *Poisson* process. Therefore, we generate their beginning using exponentially distributed random variables into the range $[0 \mapsto 7\mathrm{s}]$. We use the New Reno version with a maximum window size of 2000. The average time between the TCP sessions arrivals at each node rates 45 ms. Each node buffer is fixed to 100 packets. The sessions are generated with a random size according to a Pareto distribution with a 1.5 shape parameter and a 10 Kbytes mean. The network effect is measured on a connection between an emission node N_{e} and a destination node N_{r}. We are interested in the packet arrival process at the destination node N_{r}.

Results

We have simulated 20 000 random scenarios composed by 1 000 simulations for each connection path length varying from 3 to 22 hops (1 to 20 intermediate routers between N_{e} and N_{r}). We have recorded for each simulation the arrival time of all connection packets ($N_{\mathrm{e}} \mapsto N_{\mathrm{r}}$) at the destination node N_{r}. In fact, as mentioned above, one user is only interested by his data request. A simulation generates a IAT sequence for the connection packets. For instance we present in Fig. 6 the result for a connection composed by 12 routers.

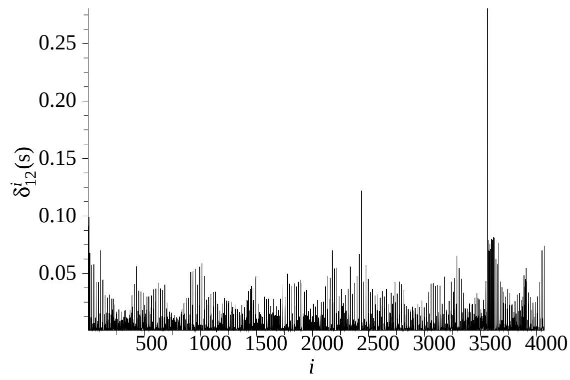

Fig. 6. Final IAT process of a 12 hops-length path TCP connection modulated by random transverse traffics.

For each simulation, we have calculated the *Hurst* exponent (H) and the geometric standard deviation of $\delta_{N_{\rm r}}$ which correspond to dispersion measures of the IAT distribution. The *Absolute Value Method* (AVM) often called the *Variance/Time Plot* offers the easiest computational way to perform H. This method is based on the following assumption: the variance of a n-length sequence X presents an order of n^{2H-2}. For each integer $m \in \left[2 \mapsto \frac{n}{2}\right]$, X is divided into non-overlapping blocks of length m. We compute the sample average $\overline{X^m_k}$ of each $k^{\rm th}$ block and its sample variance s^2_m. We perform a linear regression of the function $\log\left(s^2_m\right)$ *versus* $\log(m)$ which leads to a straight line with the slope β. For sufficiently large values of m, β estimates $2H - 2$. A parameter β in the range $[-1 \mapsto 0]$ indicates the self-similarity property. The geometric standard deviation σ is directly extracted from the IAT time series. The geometric standard deviation and the Hurst parameter of a connection increase exponentially with the connection path length n. In Fig. 7 the Hurst parameter always grows when the connection length increases even if we observe an anomaly for the $\{7, 8, 9\}$ length connections (respectively $j \in \{5, 6, 7\}$).

For a fixed connection length, we have calculated the median H for 1000 random simulations. In Fig. 8 we plot the value of σ for each simulation.

We show its median value for each j. We conclude that the function which links the geometric standard deviation to the connection path is strictly increasing (except the first point). The longer the connection path, the more dispersed the final distribution is (greater σ and H). This conclusion supports our modeling.

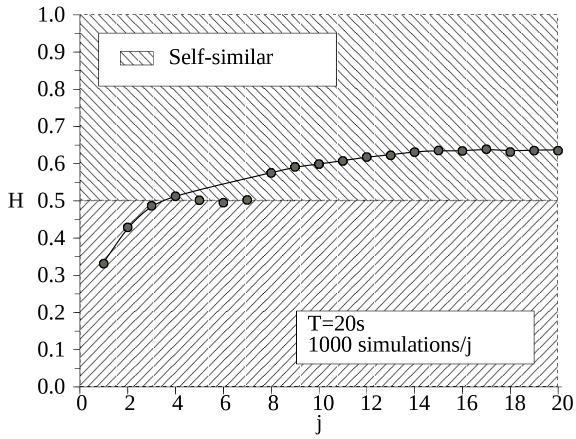

Fig. 7. H Evolution of δ_{N_r} in function of the connection intermediate routers quantity.

Concluding, we have shown in this section that a multiplicative law can indeed be observed in the Internet traffic behavior. Thus it becomes legitimate to use a log-normal modeling in order to describe the Internet traffic variability. The Internet traffic burstiness can be also explained with a simple modeling based on the IAT of connection packets.

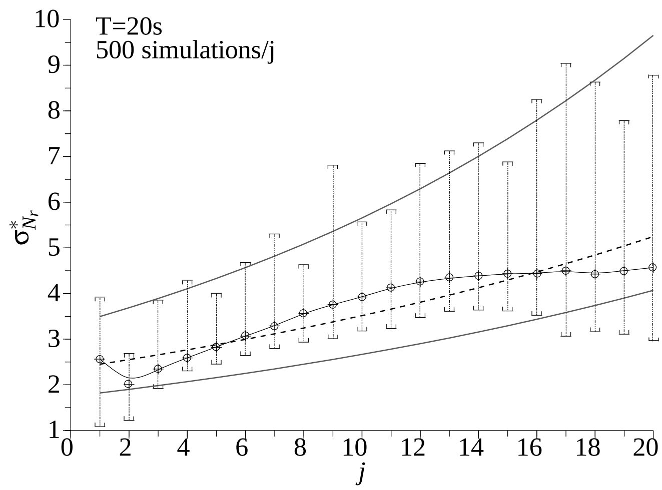

Fig. 8. Geometric standard deviation evolution of δ_{N_r} in function of the connection intermediate routers quantity.

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