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TILTED CYLINDRICALLY SYMMETRIC SELF-SIMILAR SOLUTIONS

M. Sharif†, Sajid Sultan‡

Department of Mathematics, University of the Punjab
Quaid-e-Azam Campus Lahore-54590, Pakistan

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This paper is devoted to explore tilted kinematic self-similar solutions of the the general cylindrical symmetric spacetimes. These solutions are of the first, zeroth, second and infinite kinds for the perfect fluid and dust cases. Three different equations of state are used to obtain these solutions. We obtain a total of five independent solutions. The correspondence of these solutions with those already available in the literature is also given.

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1. Introduction

Einstein's theory of General Relativity (GR) relates the geometry (curvature) to the physical content of spacetime (matter) through the Einstein field equations (EFEs) given by

$$R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}\,, \tag{1}$$

where R_{ab} is the Ricci tensor, R is the Ricci scalar, κ is the coupling constant and T_{ab} is the energy-momentum tensor. These equations are coupled, second order, non-linear, partial differential equations (PDEs) and have no general solution.

Self-similarity is a scale transformation under which the set of field equations remains invariant with the assumption of appropriate matter field. This leads to the existence of scale-invariant solutions of the field equations which are called self-similar solutions. The beauty of this restriction is that

† msharif@math.pu.edu.pk
‡ sajidsultan@gmail.com

it reduces the EFEs to a set of ordinary differential equations (ODEs) which makes them easier to solve.

Cahill and Taub [1] were the pioneers who introduced the concept of self-similarity in GR corresponding to Newtonian self-similarity of the homothetic class. They studied spherically symmetric self-similar solutions of the EFEs for a perfect fluid in the cosmological context. Carr and Coley [2] discussed different types of self-similarity in GR. They mainly focussed on spatially homogenous and spherically symmetric self-similar solutions. Carr *et al.* [3] considered kinematics self-similar (KSS) vector associated with the critical behavior observed in the gravitational collapse of spherically symmetric perfect fluid with equation of state $p = k\rho$. This work was extended [4] to discuss the physical aspects of the solutions.

Sintes *et al.* [5] investigated KSS solutions of the infinite kind in the cases of plane, spherically or hyperbolically symmetric spacetime. Bicknell and Henriksen [6] considered the self-similar growth of black holes for a class of equation of state $p = c_{\rm s}^2\rho$, where $c_{\rm s}$ is the constant sound speed. They concluded that this growth is possible in the sufficiently bounded regions surrounding the black hole. Mitsuda and Tomimatsu [7] investigated the stability of self-similar solutions of gravitational collapse from the perspective of their nature as an attractor. They studied the critical phenomena and stability of a naked singularity. Harada and Maeda [8] explored spherical collapse of a perfect fluid with equation of state $p = k\rho$ by full general relativistic numerical simulations. Ori and Piran [9] described a family of general relativistic solutions for self-similar spherical collapse of an adiabatic perfect fluid including naked singularities. A counterexample to the cosmic-censorship hypothesis was also provided. In another paper, they [10] examined the structure of general relativistic spherical collapse solution for a perfect fluid with a barotropic equation of state.

Sharif and Aziz [11, 12] discussed perfect fluid and dust solutions for the special and the most general plane symmetric spacetimes. They explored the first, second, zeroth and infinite kinds with different equations of state when the KSS vector is tilted, orthogonal and parallel to the fluid flow. The same authors [13–15] discussed the properties of the self-similar solutions of the first kind for spherically, cylindrically and plane symmetric spacetimes. They have also studied the self-similar solutions for a special cylindrically symmetric spacetime [16]. Recently, self-similar solutions of the general cylindrically symmetric spacetime have been investigated [17] for the parallel and orthogonal cases. In this paper, we take the most general cylindrically symmetric spacetimes and find the KSS solutions of the first, zeroth, second and infinite kinds when the KSS vector is tilted to the fluid flow both for perfect fluid and dust cases.

The paper is organized as follows. Section 2 is devoted in formulation of the KSS vector for different kinds of self-similarity. In Sections 3 and 4, we investigate self-similar solutions for the tilted perfect fluid and dust cases respectively. Finally, Section 5 is furnished with summary of the results obtained.

2. Cylindrically symmetric spacetimes and kinematic self-similarity

The line element for the general cylindrically symmetric spacetimes is given as [18]

$$ds^2 = e^{2\nu(t,r)}dt^2 - e^{2\phi(t,r)}dr^2 - e^{2\mu(t,r)}d\theta^2 - e^{2\lambda(t,r)}dz^2\,, \tag{2}$$

where ν, ϕ, μ and λ are functions of t and r only. Notice that the four structure functions are used since we are using a co-moving frame. The energy-momentum tensor for a perfect fluid is given by

$$T_{ab} = [\rho(t,r) + p(t,r)]u_a u_b - p(t,r)g_{ab}\,, \qquad (a,b = 0,1,2,3)\,, \tag{3}$$

where ρ and p are the density and pressure respectively and u_a is the 4-velocity of the fluid. In co-moving coordinate system, the 4-velocity can be written as $u_a = (e^{\nu}, 0, 0, 0)$. For the line element (2), the EFEs take the form

$$\begin{aligned}
8\pi G\rho &= e^{-2\nu}(\mu_t\phi_t + \lambda_t\phi_t + \lambda_t\mu_t) + e^{-2\phi}(-\mu_{rr} + \mu_r\phi_r \\
&\quad -\mu_r^2 - \lambda_{rr} + \lambda_r\phi_r - \lambda_r^2 - \mu_r\lambda_r)\,, &(4)\\
0 &= -\mu_{tr} - \lambda_{tr} + \mu_t\nu_r + \lambda_t\nu_r + \phi_t\mu_r + \lambda_r\phi_t \\
&\quad -\lambda_r\lambda_t - \mu_t\mu_r\,, &(5)\\
8\pi Gp &= e^{-2\nu}(-\mu_{tt} - \lambda_{tt} + \mu_t\nu_t + \lambda_t\nu_t - \lambda_t\mu_t - \mu_t^2 - \lambda_t^2) \\
&\quad +e^{-2\phi}(\mu_r\nu_r + \lambda_r\nu_r + \lambda_r\mu_r)\,, &(6)\\
8\pi Gp &= e^{-2\nu}(-\phi_{tt} - \lambda_{tt} + \nu_t\phi_t + \lambda_t\nu_t - \lambda_t\phi_t - \phi_t^2 - \lambda_t^2) \\
&\quad +e^{-2\phi}(\nu_{rr} + \nu_r^2 - \nu_r\phi_r + \lambda_r\nu_r + \lambda_{rr} - \lambda_r\phi_r + \lambda_r^2)\,, &(7)\\
8\pi Gp &= e^{-2\nu}(-\phi_{tt} - \mu_{tt} + \nu_t\phi_t + \mu_t\nu_t - \mu_t\phi_t - \phi_t^2 - \mu_t^2) \\
&\quad +e^{-2\phi}(\nu_{rr} + \nu_r^2 - \nu_r\phi_r + \mu_r\nu_r + \mu_{rr} - \mu_r\phi_r + \mu_r^2)\,. &(8)
\end{aligned}$$

The conservation of energy-momentum tensor, $T^{ab}{}_{;b} = 0$, yields the following equations

$$\phi_t = -\frac{\rho_t}{\rho + p} - \mu_t - \lambda_t\,, \qquad \nu_r = -\frac{p_r}{\rho + p}\,. \tag{9}$$

For a cylindrically symmetric spacetime, the vector field ξ can have the following form

$$\xi^a \frac{\partial}{\partial x^a} = h_1(t,r)\frac{\partial}{\partial t} + h_2(t,r)\frac{\partial}{\partial r}\,, \qquad (10)$$

where h_1 and h_2 are arbitrary functions of t and r. The tilted perfect fluid has both h_1 and h_2 non-zero while $h_1 = 0$ gives orthogonal case and $h_2 = 0$ the parallel case. This paper is devoted to investigate the KSS solutions for the tilted perfect fluid and dust cases.

A kinematic self-similar vector ξ is defined by

$$\pounds_\xi h_{ab} = 2\delta h_{ab}\,, \qquad \pounds_\xi u_a = \alpha u_a\,, \qquad (11)$$

where $h_{ab} = g_{ab} - u_a u_b$ is the projection tensor and α, δ are dimensionless constants. We can have different kinds of self-similarity according as $\delta \neq 0$ or $\delta = 0$ given below:

$(*)$ $\delta \neq 0$**:** Here the KSS vector for the tilted perfect fluid case takes the form

$$\xi^a \frac{\partial}{\partial x^a} = (\alpha t + \beta)\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}\,. \qquad (12)$$

The similarity index, $\frac{\alpha}{\delta}$, yields the following three possibilities

(*i*) $\alpha = 1$ (β can be taken to zero) — first kind,

(*ii*) $\alpha = 0$ (β can be taken to unity) — zeroth kind,

(*iii*) $\alpha \neq 0, 1$ (β can be taken to zero) — second kind,

where δ can be taken as unity. The self-similar variable for self-similarity of the first kind turns out to be $\xi = \frac{r}{t}$. In the zeroth kind, *i.e.*, $\alpha = 0$, the self-similar variable is $\xi = \frac{r}{e^t}$. For the second kind, the self similar variable becomes $\xi = r/(\alpha t)^{\frac{1}{\alpha}}$. For all these kinds, the metric functions are

$$\nu(t,r) = \nu(\xi), \quad \phi(t,r) = \phi(\xi)\,, \quad e^{\mu(t,r)} = re^{\mu(\xi)}\,, \quad e^{\lambda(t,r)} = re^{\lambda(\xi)}\,. \quad (13)$$

$(**)$ $\delta = 0$**:** Here the KSS vector can take the following form (when $\alpha \neq 0$)

$$\xi^a \frac{\partial}{\partial x^a} = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \qquad (14)$$

and the corresponding self-similar variable is $\xi = \frac{r}{t}$. The metric function will become

$$\nu(t,r) = \nu(\xi)\,, \quad \phi(t,r) = -\ln r + \phi(\xi)\,, \quad \mu(t,r) = \mu(\xi)\,, \quad \lambda(t,r) = \lambda(\xi)\,. \qquad (15)$$

The following equations of state (EOS) are used.

EOS(1): $p = k\rho^\gamma$, where k and γ are constants.

EOS(2): $p = kn^\gamma$, $\rho = m_b n + \frac{p}{\gamma - 1}$, where $k \neq 0$ and $\gamma \neq 0, 1$.

EOS(3): $p = k\rho$, $-1 \leq k \leq 1$, $k \neq 0$.

3. Tilted perfect fluid case

3.1. Self-similarity of the first kind

Using Eq. (13) in Eqs. (4) and (6)–(8), the mass density and pressure must take the following forms [19]

$$\kappa\rho(t,r) = \frac{1}{r^2}\rho(\xi)\,, \tag{16}$$

$$\kappa p(t,r) = \frac{1}{r^2}p(\xi)\,, \tag{17}$$

where the self-similar variable is $\xi = r/t$. If the EFEs and the equations of motion for the matter field are satisfied for $O[(r)^{-2}]$, we obtain a set of ODEs given by

$$\begin{aligned}
\dot\rho &= -(\dot\phi + \dot\mu + \dot\lambda)(\rho + p)\,, && (18)\\
2p - \dot p &= \dot\nu(\rho + p)\,, && (19)\\
0 &= \dot\mu\dot\phi + \dot\lambda\dot\phi + \dot\lambda\dot\mu\,, && (20)\\
\rho e^{2\phi} &= -\ddot\mu - \ddot\lambda - \dot\mu^2 - \ddot\lambda^2 - 2\dot\mu - 2\dot\lambda + 2\dot\phi \\
&\quad +\dot\mu\dot\phi + \dot\lambda\dot\phi - \dot\lambda\dot\mu - 1\,, && (21)\\
0 &= \ddot\mu + \ddot\lambda + \dot\mu^2 + \ddot\lambda^2 + \dot\mu + \dot\lambda - \dot\mu\dot\nu - \dot\lambda\dot\nu \\
&\quad -2\dot\phi - \dot\phi\dot\mu - \dot\phi\dot\lambda\,, && (22)\\
0 &= -\ddot\mu - \ddot\lambda - \dot\mu^2 - \ddot\lambda^2 - \dot\mu - \dot\lambda - \dot\lambda\dot\mu + \dot\mu\dot\nu + \dot\lambda\dot\nu\,, && (23)\\
pe^{2\phi} &= 1 + \dot\mu + \dot\lambda + 2\dot\nu + \dot\mu\dot\nu + \dot\lambda\dot\nu + \dot\lambda\dot\mu\,, && (24)\\
0 &= -\ddot\phi - \ddot\lambda - \dot\phi^2 - \ddot\lambda^2 - \dot\phi - \dot\lambda - \dot\phi\dot\lambda + \dot\phi\dot\nu + \dot\lambda\dot\nu\,, && (25)\\
pe^{2\phi} &= \ddot\nu + \ddot\lambda + \dot\nu^2 + \dot\lambda^2 + \dot\lambda + \dot\lambda\dot\nu - \dot\phi\dot\nu - \dot\phi\dot\lambda - \dot\phi\,, && (26)\\
0 &= -\ddot\phi - \ddot\mu - \dot\phi^2 - \ddot\mu^2 - \dot\phi - \dot\mu - \dot\phi\dot\mu + \dot\phi\dot\nu + \dot\mu\dot\nu\,, && (27)\\
pe^{2\phi} &= \ddot\nu + \ddot\mu + \dot\nu^2 + \dot\mu^2 + \dot\mu + \dot\mu\dot\nu - \dot\phi\dot\nu - \dot\phi\dot\mu - \dot\phi\,. && (28)
\end{aligned}$$

where dot represents derivative with respect to ξ. Since the first kind is not compatible with EOS(1) and EOS(2), hence no solution exists.

3.1.1. EOS(3)

If a perfect fluid satisfies EOS(3), then we have the following solution

$$\begin{aligned}
\nu &= \ln(c_0\xi^{1\pm\sqrt{2}})\,, \quad \phi = c_1\,, \quad \mu = c_2\,, \quad \lambda = c_3\,, \\
p &= \rho = \text{const.} && (29)
\end{aligned}$$

corresponding to the metric

$$ds^2 = (r)^{2\pm2\sqrt{2}}dt^2 - dr^2 - r^2(d\theta^2 + dz^2)\,. \tag{30}$$

3.2. Self-similarity of the zeroth kind

Here the quantities ρ and p take the form

$$\kappa\rho = \frac{1}{r^2}\{\rho_1(\xi) + r^2\rho_2(\xi)\}\,, \tag{31}$$
$$\kappa p = \frac{1}{r^2}\{p_1(\xi) + r^2 p_2(\xi)\}\,, \tag{32}$$

where the self-similar variable is $\xi = re^{-t}$ and the set of ODEs become

$$\dot{\rho_1} = -(\dot{\phi} + \dot{\mu} + \dot{\lambda})(\rho_1 + p_1)\,, \tag{33}$$
$$\dot{\rho_2} = -(\dot{\phi} + \dot{\mu} + \dot{\lambda})(\rho_2 + p_2)\,, \tag{34}$$
$$2p_1 - \dot{p_1} = \dot{\nu}(\rho_1 + p_1)\,, \tag{35}$$
$$-\dot{p_2} = \dot{\nu}(\rho_2 + p_2)\,, \tag{36}$$
$$\rho_1 e^{2\phi} = -\ddot{\mu} - \ddot{\lambda} - \dot{\mu}^2 - \ddot{\lambda}^2 - 2\dot{\mu} - 2\dot{\lambda} + 2\dot{\phi} + \dot{\mu}\dot{\phi} + \dot{\lambda}\dot{\phi} - \dot{\lambda}\dot{\mu} - 1\,, \tag{37}$$
$$\rho_2 e^{2\nu} = \dot{\mu}\dot{\phi} + \dot{\lambda}\dot{\phi} + \dot{\lambda}\dot{\mu}\,, \tag{38}$$
$$0 = \ddot{\mu} + \ddot{\lambda} + \dot{\mu}^2 + \ddot{\lambda}^2 + \dot{\mu} + \dot{\lambda} - \dot{\mu}\dot{\nu} - \dot{\lambda}\dot{\nu} - 2\dot{\phi} - \dot{\phi}\dot{\mu} - \dot{\phi}\dot{\lambda}\,, \tag{39}$$
$$p_1 e^{2\phi} = 1 + \dot{\mu} + \dot{\lambda} + 2\dot{\nu} + \dot{\mu}\dot{\nu} + \dot{\lambda}\dot{\nu} + \dot{\lambda}\dot{\mu}\,, \tag{40}$$
$$p_2 e^{2\nu} = -\ddot{\mu} - \ddot{\lambda} - \dot{\mu}^2 - \ddot{\lambda}^2 - \dot{\lambda}\dot{\mu} + \dot{\mu}\dot{\nu} + \dot{\lambda}\dot{\nu}\,, \tag{41}$$
$$p_1 e^{2\phi} = \ddot{\nu} + \ddot{\lambda} + \dot{\nu}^2 + \dot{\lambda}^2 + \dot{\lambda} + \dot{\lambda}\dot{\nu} - \dot{\phi}\dot{\nu} - \dot{\phi}\dot{\lambda} - \dot{\phi}\,, \tag{42}$$
$$p_2 e^{2\nu} = -\ddot{\phi} - \ddot{\lambda} - \dot{\phi}^2 - \ddot{\lambda}^2 - \dot{\phi}\dot{\lambda} + \dot{\phi}\dot{\nu} + \dot{\lambda}\dot{\nu}\,, \tag{43}$$
$$p_1 e^{2\phi} = \ddot{\nu} + \ddot{\mu} + \dot{\nu}^2 + \dot{\mu}^2 + \dot{\mu} + \dot{\mu}\dot{\nu} - \dot{\phi}\dot{\nu} - \dot{\phi}\dot{\mu} - \dot{\phi}\,, \tag{44}$$
$$p_2 e^{2\nu} = -\ddot{\phi} - \ddot{\mu} - \dot{\phi}^2 - \ddot{\mu}^2 - \dot{\phi}\dot{\mu} + \dot{\phi}\dot{\nu} + \dot{\mu}\dot{\nu}\,. \tag{45}$$

3.2.1. Equations of state

For EOS(1) with $k \neq 0$ and $\gamma \neq 0, 1$, Eqs. (31) and (32) become

$$\rho_1 = p_1 = 0\,, \quad p_2 = \frac{k}{\kappa^{\gamma-1}}\rho_2^{\gamma}\,. \quad \text{[Case I]} \tag{46}$$

For EOS(2) with $k \neq 0$ and $\gamma \neq 0, 1$, Eqs. (31) and (32) take the form

$$\rho_1 = 0 = p_1\,, \quad p_2 = \frac{k}{\kappa^{\gamma-1}m_b^{\gamma}}\left(\rho_2 - \frac{1}{\gamma-1}p_2\right)^{\gamma}\,. \quad \text{[Case II]} \tag{47}$$

EOS(3) yields two more cases for $k = -1$ [Case III] and for $k \neq -1$ [Case IV].

Case I: Solving the equations simultaneously, we obtain

$$\begin{aligned}\nu &= c_1\,,\ \phi = -\ln\xi + \ln(\xi^3 - c_3) + c_2\,,\ \mu = -\ln\xi + c_4\,,\ \lambda = -\ln\xi + c_5\,,\\ p_1 &= 0 = \rho_1\,, \quad p_2 = \text{const.}\,, \quad \rho_2 = \frac{-3(\xi^3 + c_2)}{e^{2c_1}(\xi^3 - c_3)}\,. \end{aligned} \tag{48}$$

The corresponding metric is

$$ds^2 = dt^2 - \left(\frac{r^3 - c_3 e^{3t}}{re^{2t}}\right)^2 dr^2 - e^{2t}(d\theta^2 + dz^2)\,. \tag{49}$$

The second solution is

$$\begin{aligned}\nu &= c_1\,, \quad \phi = \tfrac{1}{2}\ln\xi + c_2\,, \quad \mu = \tfrac{1}{2}\ln\xi + c_3\,, \quad \lambda = -\ln\xi + c_4\,,\\ \rho_1 &= 0 = p_1\,, \quad \rho_2 = p_2 = \text{const.}\end{aligned} \tag{50}$$

and the corresponding metric is

$$ds^2 = dt^2 - \frac{r}{e^t}(dr^2 + r^2 d\theta^2) - e^{2t}dz^2. \tag{51}$$

The third solution is

$$\begin{aligned}\nu &= c_1\,, \quad \phi = \tfrac{1}{2}\ln\xi + c_2\,, \quad \mu = -\ln\xi + c_3\,, \quad \lambda = \tfrac{1}{2}\ln\xi + c_4\,,\\ \rho_1 &= 0 = p_1\,, \quad \rho_2 = p_2 = \text{const.}\end{aligned} \tag{52}$$

and the metric takes the form

$$ds^2 = dt^2 - \frac{r}{e^t}dr^2 - e^{2t}d\theta^2 - \frac{r^3}{e^t}dz^2. \tag{53}$$

The Case II gives the same solutions as the Case I.
Case III: Here the solution becomes

$$\begin{aligned}\nu &= c_1\,, \quad \phi = -\ln\xi + c_2\,, \quad \mu = -\ln\xi + c_3\,, \quad \lambda = -\ln\xi + c_4\,,\\ \rho_1 &= 0 = p_1\,, \quad \rho_2 = p_2 = \text{const.}\end{aligned} \tag{54}$$

and the corresponding metric is

$$ds^2 = dt^2 - \frac{e^{2t}}{r^2}(dr^2 + r^2(d\theta^2 + dz^2)). \tag{55}$$

Case IV: This case gives the two solutions out of which the first is

$$\begin{aligned}\nu &= c_1\,, \quad \phi = 2\ln\xi + c_2\,, \quad \mu = -\ln\xi + c_3\,, \quad \lambda = -\ln\xi + c_4\,,\\ \rho_1 &= 0 = p_1\,, \quad \rho_2 = p_2 = \text{const.}\end{aligned} \tag{56}$$

and the corresponding metric is

$$ds^2 = dt^2 - \frac{r^4}{e^{4t}} dr^2 - e^{2t}(d\theta^2 + dz^2)\,. \tag{57}$$

The second solution is

$$\begin{aligned} \nu &= (1 \pm \sqrt{2}) \ln \xi + c_1\,, \quad \phi = c_2\,, \quad \mu = c_3\,, \quad \lambda = c_4\,, \\ \rho_1 &= p_1 = \text{const.}\,, \quad \rho_2 = 0 = p_2 \end{aligned} \tag{58}$$

and the corresponding spacetime is

$$ds^2 = \left(\frac{r}{e^t}\right)^{2\pm2\sqrt{2}} dt^2 - dr^2 - r^2\left(d\theta^2 + dz^2\right)\,. \tag{59}$$

3.3. Self-similarity of the second kind

Here the EFEs imply that

$$\kappa\rho = \frac{1}{r^2}\left\{\rho_1(\xi) + \frac{r^2}{t^2}\rho_2(\xi)\right\}, \tag{60}$$

$$\kappa p = \frac{1}{r^2}\left\{p_1(\xi) + \frac{r^2}{t^2}p_2(\xi)\right\}, \tag{61}$$

where the self-similar variable is $\xi = r/(\alpha t)^{1/\alpha}$. The set of ODEs become

$$\dot{\rho_1} = -(\dot{\phi} + \dot{\mu} + \dot{\lambda})(\rho_1 + p_1)\,, \tag{62}$$

$$\dot{\rho_2} + 2\alpha\rho_2 = -(\dot{\phi} + \dot{\mu} + \dot{\lambda})(\rho_2 + p_2)\,, \tag{63}$$

$$2p_1 - \dot{p_1} = \dot{\nu}(\rho_1 + p_1)\,, \tag{64}$$

$$-\dot{p_2} = \dot{\nu}(\rho_2 + p_2)\,, \tag{65}$$

$$\begin{aligned} \rho_1 e^{2\phi} &= -\ddot{\mu} - \ddot{\lambda} - \dot{\mu}^2 - \ddot{\lambda}^2 - 2\dot{\mu} - 2\dot{\lambda} + 2\dot{\phi} \\ &\quad + \dot{\mu}\dot{\phi} + \dot{\lambda}\dot{\phi} - \dot{\lambda}\dot{\mu} - 1\,, \end{aligned} \tag{66}$$

$$\alpha^2\rho_2 e^{2\nu} = \dot{\mu}\dot{\phi} + \dot{\lambda}\dot{\phi} + \dot{\lambda}\dot{\mu}\,, \tag{67}$$

$$\begin{aligned} 0 &= \ddot{\mu} + \ddot{\lambda} + \dot{\mu}^2 + \ddot{\lambda}^2 + \dot{\mu} + \dot{\lambda} - \dot{\mu}\dot{\nu} - \dot{\lambda}\dot{\nu} \\ &\quad - 2\dot{\phi} - \dot{\phi}\dot{\mu} - \dot{\phi}\dot{\lambda}\,, \end{aligned} \tag{68}$$

$$p_1 e^{2\phi} = 1 + \dot{\mu} + \dot{\lambda} + 2\dot{\nu} + \dot{\mu}\dot{\nu} + \dot{\lambda}\dot{\nu} + \dot{\lambda}\dot{\mu}\,, \tag{69}$$

$$\begin{aligned} \alpha^2 p_2 e^{2\nu} &= -\ddot{\mu} - \ddot{\lambda} - \dot{\mu}^2 - \ddot{\lambda}^2 - \alpha\dot{\mu} - \alpha\dot{\lambda} \\ &\quad - \dot{\lambda}\dot{\mu} + \dot{\mu}\dot{\nu} + \dot{\lambda}\dot{\nu}\,, \end{aligned} \tag{70}$$

$$p_1 e^{2\phi} = \ddot{\nu} + \ddot{\lambda} + \dot{\nu}^2 + \dot{\lambda}^2 + \dot{\lambda} + \dot{\lambda}\dot{\nu} - \dot{\phi}\dot{\nu} - \dot{\phi}\dot{\lambda} - \dot{\phi}\,, \tag{71}$$

$$\alpha^2 p_2 e^{2\nu} = -\ddot{\phi} - \ddot{\lambda} - \dot{\phi}^2 - \ddot{\lambda}^2 - \alpha\dot{\phi} - \alpha\dot{\lambda} - \dot{\phi}\dot{\lambda} + \dot{\phi}\dot{\nu} + \dot{\lambda}\dot{\nu}\,, \quad (72)$$

$$p_1 e^{2\phi} = \ddot{\nu} + \ddot{\mu} + \dot{\nu}^2 + \dot{\mu}^2 + \dot{\mu} + \dot{\mu}\dot{\nu} - \dot{\phi}\dot{\nu} - \dot{\phi}\dot{\mu} - \dot{\phi}\,, \quad (73)$$

$$\alpha^2 p_2 e^{2\nu} = -\ddot{\phi} - \ddot{\mu} - \dot{\phi}^2 - \ddot{\mu}^2 - \alpha\dot{\phi} - \alpha\dot{\mu} - \dot{\phi}\dot{\mu} + \dot{\phi}\dot{\nu} + \dot{\mu}\dot{\nu}\,. \quad (74)$$

3.3.1. Equations of State

For EOS(1) ($k \neq 0$ and $\gamma \neq 0, 1$), Eqs. (60) and (61) imply that

$$p_1 = 0 = \rho_2, \quad \alpha = \gamma, \quad p_2 = \frac{k}{\kappa^{\gamma-1}\gamma^2}\xi^{-2\gamma}\rho_1^\gamma, \quad \text{[Case I]} \quad (75)$$

$$p_2 = 0 = \rho_1, \quad \alpha = \frac{1}{\gamma}, \quad p_1 = \frac{k}{\kappa^{\gamma-1}\gamma^{2\gamma}}\xi^2\rho_2^\gamma. \quad \text{[Case II]} \quad (76)$$

When a perfect fluid satisfies EOS(2) for $k \neq 0$ and $\gamma \neq 0, 1$, it follows from Eqs. (60) and (61) that

$$p_2 = \frac{k}{m_b^\gamma \gamma^2 \kappa^{\gamma-1}}\xi^{-2\gamma}\rho_1^\gamma = (\gamma-1)\rho_2\,,$$

$$p_1 = 0, \quad \alpha = \gamma, \quad \text{[Case III]} \quad (77)$$

$$p_1 = \frac{k}{m_b^\gamma \gamma^{2\gamma} \kappa^{\gamma-1}}\xi^2\rho_2^\gamma = (\gamma-1)\rho_1\,,$$

$$p_2 = 0, \quad \alpha = \frac{1}{\gamma}. \quad \text{[Case IV]} \quad (78)$$

EOS(3) yields two more cases (Case V and Case VI) for $k = -1$ and $k \neq -1$ respectively.

Case I: Solving the ODEs simultaneously, we obtain the solutions as

$$\nu = c_1\,, \quad \phi = \tfrac{1}{2}\ln\xi + c_2\,, \quad \mu = -\ln\xi + c_3\,, \quad \lambda = -\ln\xi + c_4\,,$$

$$\rho_1 = \rho_2 = 0 = p_1 = p_2\,, \quad \alpha = \tfrac{3}{2} \quad (79)$$

and the corresponding metric is

$$ds^2 = dt^2 - \left(\frac{2}{3t}\right)^{2/3} r dr^2 - \left(\frac{3t}{2}\right)^{4/3}(d\theta^2 + dz^2)\,. \quad (80)$$

The second solution is

$$\nu = c_1\,, \quad \phi = 2\ln\xi + c_2\,, \quad \mu = -\ln\xi + c_3\,, \quad \lambda = 2\ln\xi + c_4\,,$$

$$\rho_1 = \rho_2 = 0 = p_1 = p_2\,, \quad \alpha = -3\,. \quad (81)$$

Its metric form is

$$ds^2 = dt^2 - r^4(-3t)^{4/3}dr^2 - \frac{1}{(-3t)^{2/3}}d\theta^2 - r^6(-3t)^{4/3}dz^2\,. \qquad (82)$$

The third solution is

$$\begin{aligned} \nu &= c_1\,, \quad \phi = 2\ln\xi + c_2\,, \quad \mu = 2\ln\xi + c_4\,, \quad \lambda = -\ln\xi + c_3\,, \\ \rho_1 &= \rho_2 = 0 = p_1 = p_2\,, \quad \alpha = -3 \end{aligned} \qquad (83)$$

and the corresponding metric is

$$ds^2 = dt^2 - r^4(-3t)^{4/3}(dr^2 + r^2d\theta^2) - \frac{1}{(-3t)^{2/3}}dz^2\,. \qquad (84)$$

It is mentioned here that the Cases II, III and IV have the same solutions as given by Eqs. (80), (82) and (84). It is also noted that the Case V has only one solution given by Eq. (80).

Case VI: For this case, we obtain the following solutions. The first solution is

$$\begin{aligned} \nu &= c_1, \quad \phi = -\ln\xi + c_2\,, \quad \mu = -\ln\xi + c_3\,, \quad \lambda = -\ln\xi + c_4\,, \\ \rho_1 &= 0 = p_1\,, \quad \rho_2 = p_2 = \text{const.}\,, \quad k = \frac{2\alpha - 3}{3} \end{aligned} \qquad (85)$$

and the corresponding metric is

$$ds^2 = dt^2 - \frac{(\alpha t)^{2/\alpha}}{r^2}dr^2 - (\alpha t)^{2/\alpha}(d\theta^2 + dz^2)\,. \qquad (86)$$

The second solution is

$$\begin{aligned} \nu &= c_1\,, \quad \phi = (2-\alpha)\ln\xi = c_2\,, \quad \mu = -\ln\xi + c_3\,, \quad \lambda = -\ln\xi + c_4\,, \\ \rho_1 &= 0 = p_1, \quad \rho_2 = p_2 = \text{const.}\,, \quad k = 1 \end{aligned} \qquad (87)$$

and the corresponding spacetime is

$$ds^2 = dt^2 - \frac{r^{2(2-\alpha)}}{(\alpha t)^{2(2-\alpha)/\alpha}}dr^2 - (\alpha t)^{2/\alpha}(d\theta^2 + dz^2)\,. \qquad (88)$$

The third solution is

$$\begin{aligned} \nu &= c_1\,, \quad \phi = -\ln\xi = c_2\,, \quad \mu = -\ln\xi + c_3\,, \quad \lambda = -\ln\xi + c_4\,, \\ \rho_1 &= 0 = p_1\,, \quad \rho_2 = p_2 = \text{const.}\,, \quad k = 1\,, \quad \alpha = 3 \end{aligned} \qquad (89)$$

and its metric is

$$ds^2 = dt^2 - \frac{(3t)^{2/3}}{r^2}dr^2 - (3t)^{2/3}(d\theta^2 + dz^2)\,. \tag{90}$$

The fourth solution is

$$\begin{aligned}
\nu &= (1 \pm \sqrt{2})\ln\xi + c_1\,, \quad \phi = c_2, \mu = c_3\,, \quad \lambda = c_4\,, \\
\rho_1 &= p_1 = \text{const.}\,, \quad \rho_2 = 0 = p_2\,, \quad k = -3 \pm 2\sqrt{2}
\end{aligned} \tag{91}$$

and the corresponding metric is

$$ds^2 = \left(\frac{r}{(\alpha t)^{1/\alpha}}\right)^{2\pm2\sqrt{2}} dt^2 - dr^2 - r^2(d\theta^2 + dz^2)\,. \tag{92}$$

3.4. Self-similarity of the infinite kind

Here the quantities ρ and p can be written in terms of ξ as

$$\kappa\rho = \rho_1(\xi) + \frac{1}{t^2}\rho_2(\xi)\,, \tag{93}$$

$$\kappa p = p_1(\xi) + \frac{1}{t^2}p_2(\xi)\,, \tag{94}$$

where the self-similar variable is $\xi = r/t$ and the set of ODEs become

$$\dot{\rho_1} = -(\dot{\phi} + \dot{\mu} + \dot{\lambda})(\rho_1 + p_1)\,, \tag{95}$$

$$\dot{\rho_2} + 2\rho_2 = -(\dot{\phi} + \dot{\mu} + \dot{\lambda})(\rho_2 + p_2)\,, \tag{96}$$

$$-\dot{p_1} = \dot{\nu}(\rho_1 + p_1)\,, \tag{97}$$

$$-\dot{p_2} = \dot{\nu}(\rho_2 + p_2)\,, \tag{98}$$

$$\rho_1 e^{2\phi} = -\ddot{\mu} - \ddot{\lambda} - \dot{\mu}^2 - \ddot{\lambda}^2 + \dot{\mu}\dot{\phi} + \dot{\lambda}\dot{\phi} - \dot{\lambda}\dot{\mu}\,, \tag{99}$$

$$\rho_2 e^{2\nu} = \dot{\mu}\dot{\phi} + \dot{\lambda}\dot{\phi} + \dot{\lambda}\dot{\mu}\,, \tag{100}$$

$$0 = \ddot{\mu} + \ddot{\lambda} + \dot{\mu}^2 + \ddot{\lambda}^2 - \dot{\mu}\dot{\nu} - \dot{\lambda}\dot{\nu} - \dot{\phi}\dot{\mu} - \dot{\phi}\dot{\lambda}\,, \tag{101}$$

$$p_1 e^{2\phi} = \dot{\mu}\dot{\nu} + \dot{\lambda}\dot{\nu} + \dot{\lambda}\dot{\mu}\,, \tag{102}$$

$$p_2 e^{2\nu} = -\ddot{\mu} - \ddot{\lambda} - \dot{\mu}^2 - \ddot{\lambda}^2 - \dot{\mu} - \dot{\lambda} - \dot{\lambda}\dot{\mu} + \dot{\mu}\dot{\nu} + \dot{\lambda}\dot{\nu}\,, \tag{103}$$

$$p_1 e^{2\phi} = \ddot{\nu} + \ddot{\lambda} + \dot{\nu}^2 + \dot{\lambda}^2 + \dot{\lambda}\dot{\nu} - \dot{\phi}\dot{\nu} - \dot{\phi}\dot{\lambda}\,, \tag{104}$$

$$p_2 e^{2\nu} = -\ddot{\phi} - \ddot{\lambda} - \dot{\phi}^2 - \ddot{\lambda}^2 - \dot{\phi} - \dot{\lambda} - \dot{\phi}\dot{\lambda} + \dot{\phi}\dot{\nu} + \dot{\lambda}\dot{\nu}\,, \tag{105}$$

$$p_1 e^{2\phi} = \ddot{\nu} + \ddot{\mu} + \dot{\nu}^2 + \dot{\mu}^2 + \dot{\mu}\dot{\nu} - \dot{\phi}\dot{\nu} - \dot{\phi}\dot{\mu}\,, \tag{106}$$

$$p_2 e^{2\nu} = -\ddot{\phi} - \ddot{\mu} - \dot{\phi}^2 - \ddot{\mu}^2 - \dot{\phi} - \dot{\mu} - \dot{\phi}\dot{\mu} + \dot{\phi}\dot{\nu} + \dot{\mu}\dot{\nu}\,. \tag{107}$$

3.4.1. Equations of state

For EOS(1) with $k \neq 0$ and $\gamma \neq 0, 1$, Eqs. (95) and (96) become

$$\rho_2 = 0 = p_2, \quad p_1 = \frac{k}{\kappa^{\gamma-1}}\rho_1^{\gamma}. \quad \text{[Case I]} \tag{108}$$

For EOS(2) with $k \neq 0$ and $\gamma \neq 0, 1$, Eqs. (95) and (96) take the form

$$\rho_2 = 0 = p_2, \quad p_1 = \frac{k}{m_b^{\gamma}\kappa^{\gamma-1}}\left(\rho_1 - \frac{p_1}{\gamma - 1}\right)^{\gamma}. \quad \text{[Case II]} \tag{109}$$

EOS(3) yields two more cases for $k = -1$ [Case III] and for $k \neq -1$ [Case IV].
Case I: Solving the ODEs simultaneously for this case, we have

$$\nu = c_1, \quad \phi = c_2\,, \quad \mu = c_3\,, \quad \lambda = c_4\,, \quad \rho_1 = \rho_2 = 0 = p_1 = p_2 \tag{110}$$

and the corresponding metric is

$$ds^2 = dt^2 - \frac{1}{r^2}dr^2 - d\theta^2 - dz^2 \tag{111}$$

which corresponds to Minkowski spacetime. The second solution is

$$\begin{aligned} \nu &= \ln(\ln\xi - \ln c_1) + c_2\,, \quad \phi = c_3\,, \quad \mu = c_4\,, \quad \lambda = c_5\,, \\ \rho_1 &= \rho_2 = 0 = p_1 = p_2\,. \end{aligned} \tag{112}$$

The corresponding metric is

$$ds^2 = \left[\ln\left(\frac{r}{c_1 t}\right)\right]^2 dt^2 - \frac{1}{r^2}dr^2 - (d\theta^2 + dz^2)\,. \tag{113}$$

The third solution becomes

$$\begin{aligned} \nu &= c_3\,, \qquad \phi = \ln\left(\frac{\xi - c_1}{\xi}\right) + c_2\,, \qquad \dot{\mu} = 0\,, \qquad \dot{\lambda} = 0\,, \\ \rho_1 &= \rho_2 = 0 = p_1 = p_2\,. \end{aligned} \tag{114}$$

The corresponding metric is

$$ds^2 = dt^2 - \frac{(r - c_1 t)^2}{r^4}dr^2 - (d\theta^2 + dz^2)\,. \tag{115}$$

It is mentioned here that the Cases II, III and IV have the same solutions as the Case I.

4. Tilted dust case

It is well-known that a perfect fluid is characterized by the pressure and it reduces to the dust fluid when $p = 0$.

4.1. Self-similarity of the first kind

When we substitute $p = 0$ in Eqs. (18)–(28), it follows from Eq. (19)

$$\dot{\nu}\rho = 0 \tag{116}$$

which gives the following two solutions. The first solution turns out to be

$$\nu = c_1\,, \quad \phi = c_2\,, \quad \mu = c_3\,, \quad \lambda = -\ln\xi + c_4\,, \quad \rho = 0 \tag{117}$$

with metric

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - t^2 dz^2\,. \tag{118}$$

The second solution is

$$\nu = c_1\,, \quad \phi = c_2\,, \quad \mu = -\ln\xi + c_3\,, \quad \lambda = c_4\,, \quad \rho = 0 \tag{119}$$

and the corresponding metric is

$$ds^2 = dt^2 - dr^2 - t^2 d\theta^2 - r^2 dz^2\,. \tag{120}$$

4.2. Self-similarity of the zeroth kind

When we take $p_1 = 0$ and $p_2 = 0$ in Eqs. (33)–(45), we obtain contradiction and consequently there is no self-similar solution of the zeroth kind.

4.3. Self-similarity of the second kind

When we substitute $p_1 = 0$ and $p_2 = 0$ in Eqs. (62)–(74), we obtain the following solution

$$\begin{aligned} \nu &= c_1\,, \quad \phi = \ln\left(\frac{2\xi^{3/2} - c_2}{\xi}\right), \quad \mu = -\ln\xi + c_3\,, \quad \lambda = -\ln\xi + c_4\,, \\ \rho_1 &= 0, \quad \rho_2 = \frac{4}{9e^{2c_1}}\left(\frac{3c_2}{c_2 - 2\xi^{3/2}}\right). \end{aligned} \tag{121}$$

The corresponding metric is

$$ds^2 = dt^2 - \left(\frac{3t}{2}\right)^{4/3}\left(\frac{4r^{3/2} - 3tc_2}{3rt}\right)^2 dr^2 - \left(\frac{3t}{2}\right)^{4/3}(d\theta^2 + dz^2)\,. \tag{122}$$

4.4. Self-similarity of the infinite kind

Taking $p_1 = 0 = p_2$ in Eqs. (97)–(109), it yields the same solutions as given by Eqs. (111) and (115).

5. Summary and discussion

Recently, Sharif and Aziz [16] have found the KSS solutions for a special cylindrically symmetric spacetimes. This work has been extended to investigate the KSS solutions of the general cylindrically symmetric spacetime when the fluid flow is parallel as well as orthogonal [17]. However, we have left the tilted case to deal it separately. In this paper, we have studied the case when KSS vector is tilted to the fluid flow. The solutions are found for the first, zeroth, second and the infinite kinds both for perfect fluid and dust cases. The summary of the independent solutions is given in the form of Table I.

TABLE I

Tilted KSS Solutions.

S. No.	Metric	Equation of state	Physical
I	$ds^2 = (r)^{2\pm 2\sqrt{2}}dt^2$ $-dr^2 - r^2(d\theta^2 + dz^2)$	$\rho = -\frac{1}{8\pi r^2} < 0$, $p = -(3 \pm 2\sqrt{2})\rho > 0$	Unphysical
II	$ds^2 = dt^2$ $-(r/e^t)(dr^2 + r^2 d\theta^2) - e^{2t}dz^2$	$\rho = p = -\frac{3}{32\pi} < 0$	Unphysical
III	$ds^2 = dT^2 - e^{2T}(dR^2 + R^2 d\Theta^2$ $+dZ^2)$, $R > 0, \ -\infty < T < +\infty$, $0 < \Theta < 2\pi, \ -\infty < Z < +\infty$	$\rho = -p = \frac{3}{8\pi} > 0$	Physical
IV	$ds^2 = dt^2 - r^4(3t)^{4/3}(dr^2$ $+r^2 d\theta^2) - \frac{1}{(3t)^{2/3}}dz^2$	$\rho = 0 = p$	Physical
V	$ds^2 = dT^2 - dR^2 - R^2 d\Theta^2$ $+dZ^2$, $R > 0, \ -\infty < T < +\infty$, $0 < \Theta < 2\pi, \ -\infty < Z < +\infty$	$\rho = 0 = p$	Physical

The results can be discussed as follows:
I. $\rho = -\frac{1}{8\pi r^2} < 0,\ p = -(3 \pm 2\sqrt{2})\rho > 0$. This solution is unphysical as energy density is negative.
II. For this solution, we have $\rho = p = -\frac{3}{32\pi} < 0$ which is unphysical.
III. This is the expanding de Sitter universe (empty expanding flat space with cosmological constant $\Lambda > 0$). In this case $p < 0$ can be re-interpreted and considered by most astrophysicists as physical.
IV. Here $\rho = 0 = p$. This is an empty but curved spacetime as $R = 0,\ R_{ab} = 0$, but $R_{abcd}R^{abcd} = \frac{64}{27t^4} \neq 0$ singular at $t = 0$ (extendible).
V. This gives $\rho = 0 = p,\ R_{abcd}R^{abcd} = 0$, *i.e.* empty flat, Minkowski spacetime in cylindrical coordinates.

We have seen that the number of independent solutions are very few. Also, most of these solutions coincide with those already available in the literature. Thus we can conclude that one may obtain already known spacetimes by specializing a map adapted to some symmetries. However, the line element may look more complicated in such a map. Moreover, such a map may cover only a fragment of the spacetime.

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