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EXPLICIT EXAMPLE OF LOCAL DIFFERENTIAL CALCULUS OVER FEDOSOV ALGEBRA

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In this paper example of local differential calculus over Fedosov algebra is constructed. The trivialization isomorphism for Fedosov $*$-algebra is used. The explicit formulas for deformed derivations are given up to power 2 of formal parameter. The resulting calculus can be considered as a building block for the theory of Seiberg–Witten map with Fedosov type of noncommutativity.

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1. Introduction

The aim of this paper is to give an explicit example of differential calculus over Fedosov algebra of formal power series with coefficients in functions on symplectic manifold. The term "differential calculus" refers here to an unital $\mathbb{N}$-graded algebra with nilpotent antiderivation *i.e.* to a structure analogous to the Cartan algebra of differential forms. The motivation for such investigation could be provided by the theory of Seiberg–Witten map [1]. The underlying noncommutativity of this theory is given by Moyal star product. One may ask about generalizations to other types of noncommutativity described by deformation quantization procedures *e.g.* to the Fedosov $*$-product on a symplectic manifold. When passing from Moyal to Fedosov product one encounters some difficulties. They generally originate in the fact that operators $\frac{\partial}{\partial x^i}$ which are derivations of both undeformed algebra of functions and Moyal algebra, are no longer derivations with respect to the Fedosov product. For this reason, the framework of the usual Cartan algebra cannot be used for developing consistent Seiberg–Witten map with Fedosov product. The construction presented in this paper may be regarded as a building block for such a theory. The approach presented here is similar to that developed in [2] and could be considered as its extension which enables explicit calculations (see concluding section for a discussion on relations between this paper and [2]). Fedosov-type deformations of algebra of differential forms were also analyzed in [3] with methods of geometry of supermanifolds. However, the resulting deformation does not preserve $\mathbb{N}$-graded structure. The deformation of Cartan algebra with Moyal noncommutativity was considered in [4]. In [5] global and general scheme for deformation of bimodule of sections of arbitrary vector bundle is presented. Unfortunately when trying to adopt these methods for the purpose of the deformation of Cartan algebra, one faces severe problems with constructing compatible deformation of both tensor and wedge product. The paper is organized as follows. First, the general scheme of Fedosov construction of the $*$-product on arbitrary symplectic manifold is recalled. Next (Section 3) the trivialization procedure for Fedosov algebras (originally formulated in [6]) is analyzed. The explicit formulas (up to h^2) for trivialization isomorphism with arbitrary underlying homotopy of symplectic connections are given. The concept of trivialization turns out to be crucial for construction of our example of differential calculus. This is described in the fourth section. Finally, some concluding comments are given.

2. Fedosov construction

This section is given mainly for the purpose of fixing the notations. Thus, the proofs are omitted, and the numbers of theorems in original formulation in [6] are given. For detailed insight into geometrical ideas behind Fedosov construction one may refer to [7]. Some further properties and examples can be found in [8]. In the first step, one constructs a bundle on the base manifold $\mathcal{M}$, called formal Weyl algebras bundle W, with fibres being algebras W_x consisting of formal power series

$$a(h, y) = \sum_{k,\, p \geq 0} h^k a_{i_1 \dots i_p} y^{i_1} \dots y^{i_p}\,, \tag{1}$$

where $y \in T_x M$, $a_{i_1 \dots i_p}$ are components of some symmetric covariant tensors in local Darboux coordinates and h is a formal parameter. One prescribes degrees to monomials in formal sum (1) according to the rule

$$\deg(h^k a_{i_1 \dots i_p} y^{i_1} \dots y^{i_p}) = 2k + p\,.$$

For nonhomogeneous a its degree is given by the lowest degree of nonzero monomials in formal sum (1). The fibrewise $\circ$-product is defined by the Moyal formula

$$a \circ b = \sum_{m=0}^{\infty} \left(-\frac{ih}{2}\right)^m \frac{1}{m!} \frac{\partial^m a}{\partial y^{i_1} \dots \partial y^{i_m}} \omega^{i_1 j_1} \dots \omega^{i_m j_m} \frac{\partial^m b}{\partial y^{j_1} \dots \partial y^{j_m}}\,.$$

This definition is invariant under linear symplectomorphisms *i.e.* under transformations of y^i generated by transitions between local Darboux coordinates on $\mathcal{M}$. We also consider bundle $W \otimes \Lambda$. Sections of this bundle can be locally written as

$$a = \sum h^k a_{i_1 \ldots i_p j_1 \ldots j_q}(x) y^{i_1} \ldots y^{i_p} dx^{j_1} \wedge \cdots \wedge dx^{j_q} \,.$$

The $\circ$-product in $W \otimes \Lambda$ is defined by the rule $(a \otimes \eta) \circ (b \otimes \xi) = (a \circ b) \otimes (\eta \wedge \xi)$. The commutator of $a \in W \otimes \Lambda^r$ and $b \in W \otimes \Lambda^s$ is given by $[a, b] = a \circ b - (-1)^{rs} b \circ a$. One introduces an operator δ acting on elements of $W \otimes \Lambda$ as follows

$$\delta a = dx^k \wedge \frac{\partial a}{\partial y^k} = -\frac{i}{h} \left[\omega_{ij} y^i dx^j, a \right] \,.$$

Similarly, δ^{-1} acting on monomial a_{km} with k-fold y and m-fold dx yields

$$\delta^{-1} a_{km} = \frac{1}{k+m} y^s \iota \left(\frac{\partial}{\partial x^s} \right) a_{km}$$

for $k + m > 0$ and $\delta^{-1} a_{00} = 0$. Both δ and δ^{-1} are nilpotent and for $a \in W \otimes \Lambda^k$ the Leibniz rule $\delta(a \circ b) = (\delta a) \circ b + (-1)^k a \circ \delta b$ holds. An arbitrary $a \in W \otimes \Lambda$ can be decomposed into

$$a = a_{00} + \delta \delta^{-1} a + \delta^{-1} \delta a \,.$$

Let a be a section of W. Symplectic connection ∂ can be extended to the Weyl bundle by the formula $\partial a = dx^i \wedge \partial_i a$, where $\partial_i a$ denotes covariant derivation of tensor fields in (1) with respect to $\frac{\partial}{\partial x^i}$. Using Darboux coordinates and connection coefficients $\Gamma^l{}_{jl}$ one may write ∂a in the form

$$\partial a = da + \frac{i}{h} \left[\frac{1}{2} \Gamma_{ijk} y^i y^j dx^k, a \right] \,,$$

with $\Gamma_{ijk} = \omega_{il} \Gamma^l{}_{jl}$ (any further raising or lowering of indices is also performed by means of ω). When dealing with sections of $W \otimes \Lambda$, we can compute ∂ using the rule

$$\partial(\eta \circ a) = d\eta \circ a + (-1)^k \eta \circ \partial a \,,$$

where η is a scalar k-form. The $\circ$-Leibniz rule holds for ∂ and one could be interested in other connections with this property, namely in the connections of the form

$$\nabla = \partial + \frac{i}{h} [\gamma, \cdot\,] \,,$$

with $\gamma \in C^\infty(W \otimes \Lambda^1)$. One can calculate that $\nabla^2 = \frac{i}{h}[\Omega, \cdot]$ with the curvature 2-form $\Omega = R + \partial\gamma + \frac{i}{h}\gamma \circ \gamma$, where $R = \frac{1}{4}R_{ijkl}y^i y^j dx^k \wedge dx^l$ and $R^i{}_{jkl}$ denotes the curvature tensor of symplectic connection on $\mathcal{M}$. The connection D is called Abelian if it is flat ($D^2 = 0$) *i.e.* if its curvature is a scalar form. Fedosov proves ([6] theorem 5.2.2) that for arbitrary symplectic connection ∂ there exists unique Abelian connection

$$D = -\delta + \partial + \frac{i}{h}[r, \cdot]\,,$$

with curvature form $\Omega = -1/2\omega_{ij}dx^i \wedge dx^j$ and r satisfying $\delta^{-1}r = 0$, $\deg r \geq 3$. The 1-form r is the unique solution of the equation

$$r = \delta^{-1}R + \delta^{-1}\left(\partial r + \frac{i}{h} r \circ r\right).$$

Section $a \in C^\infty(W)$ is called flat if $Da = 0$. Flat sections form subalgebra of the algebra of all sections of W. We denote this subalgebra by W_D. If the underlying symplectic connection is flat and we work in Darboux coordinates for which $\Gamma_{ijk} \equiv 0$, then Abelian connection reads $D = d - \delta$. The corresponding subalgebra of flat sections is called *trivial algebra* in this case. For $a \in C^\infty(W \otimes \Lambda)$ define $Q(a)$ as a solution of the equation

$$b = a + \delta^{-1}(D + \delta)b$$

with respect to b. One can prove that this solution is unique, and that Q is linear bijection. Clearly $Q^{-1}a = a - \delta^{-1}(D + \delta)a$. It turns out ([6] theorem 5.2.4) that Q establishes bijection between $C^\infty(\mathcal{M})[[h]]$ and W_D. For $f, g \in C^\infty(\mathcal{M})[[h]]$ the $*$-product is defined according to the rule

$$f * g = Q^{-1}(Q(f) \circ Q(g))\,.$$

In proofs of theorems related to Fedosov construction the *iteration method* is frequently used. Given an equation of the form

$$a = b + K(a) \tag{2}$$

one may try to solve it iteratively with respect to a, by putting $a^{(0)} = b$ and $a^{(n)} = b + K(a^{(n-1)})$. If K is linear and *raises degrees* (*i.e.* $\deg a < \deg K(a)$ or $K(a) = 0$) then it can be easily deduced that the unique solution of (2) is given up to degree n by $a^{(n)}$. We need one more theorem for further purposes.

Theorem 2.1 (Fedosov 5.2.6)**.** *Equation $Da = b$ (for some given $b \in C^\infty(W \otimes \Lambda^p)$, $p > 0$) has a solution if and only if $Db = 0$. The solution may be chosen in the form $a = -Q\delta^{-1}b$.*

3. Trivialization

In this section general methods developed by Fedosov are applied to the specific case of deformation quantization of symplectic manifold. The term *trivialization* refers to the procedure of establishing isomorphism between some given algebra W_D and the trivial algebra. The construction of this isomorphism is based on the following theorem.

Theorem 3.1 (Fedosov 5.4.3)**.** *Let* $D_t = d + \frac{i}{h}[\gamma(t), \cdot]$ *be a family of Abelian connections parameterized by* $t \in [0,1]$*, and let* $H(t)$ *be* t*-dependent section of* W *(called Hamiltonian) satisfying the following conditions:*

1. $D_t H(t) - \dot{\gamma}(t)$ *is a scalar form,*
2. $\deg(H(t)) \geq 3$.

Then, equation

$$\frac{da}{dt} + \frac{i}{h}[H, a] = 0 \tag{3}$$

has the unique solution $a(t)$ *for any given* $a(0) \in W \otimes \Lambda$ *and the mapping* $a(0) \mapsto a(t)$ *is an isomorphism for arbitrary* $t \in [0,1]$*. Moreover,* $a(0) \in W_{D_0}$ *if and only if* $a(t) \in W_{D_t}$*.*

The proof can be performed by integrating equation (3) to

$$a(t) = a(0) - \frac{i}{h} \int_0^t [H(\tau), a(\tau)]\, d\tau\,. \tag{4}$$

and using iteration method. We are interested in constructing isomorphism between W_D and the trivial algebra. This requires establishing homotopy of Abelian connections and compatible Hamiltonian. From now W_{D_0} will denote the trivial algebra.

Theorem 3.2 (Fedosov 5.5.1)**.** *Any algebra* W_D *on Fedosov manifold* (M, ω, ∂) *is locally isomorphic to the trivial algebra* W_{D_0} *on* $\mathbb{R}^n$*.*

Proof. Let $\mathcal{O}$ be a neighborhood of some point $x_0 \in M$, for which Darboux coordinates x^i may be chosen. The symplectic connection on $\mathcal{M}$ generates unique Abelian connection

$$D = d + \frac{i}{h}\left[\omega_{ij} y^i dx^j + \frac{1}{2}\Gamma_{ijk} y^i y^j dx^k + r, \cdot\right]\,,$$

where $r = 1/8 R_{ijkl} y^i y^j y^k dx^l + \dots$ is 1-form obtained from iterational procedure (2). Consider a local homotopy of symplectic connections $\boldsymbol{\partial}^{(t)}$ such that for connection coefficients we have $\boldsymbol{\Gamma}_{ijk}(0) = 0$ and $\boldsymbol{\Gamma}_{ijk}(1) = \Gamma_{ijk}$. It generates homotopy of local Abelian connections

$$D_t = d + \frac{i}{h}\left[\omega_{ij}y^i dx^j + \frac{1}{2}\boldsymbol{\Gamma}_{ijk}(t)y^i y^j dx^k + \boldsymbol{r}(t), \cdot\right] = d + \frac{i}{h}[\boldsymbol{\gamma}(t), \cdot\,],$$

satisfying $D_1 = D$ and $D_0 = d - \delta$ (trivial Abelian connection). Notice, that these Abelian connections have constant curvature $\boldsymbol{\Omega}(t) = -1/2\omega_{ij}dx^i \wedge dx^j$. We look for a Hamiltonian being a solution of the equation $D_t H(t) = \dot{\boldsymbol{\gamma}}(t)$. According to the theorem 2.1 one have to check condition $D_t\dot{\boldsymbol{\gamma}}(t) = 0$. We get $D_t\dot{\boldsymbol{\gamma}}(t) = \dot{\boldsymbol{\Omega}}(t) = 0$ and hence, the Hamiltonian may be written as $H(t) = -Q_t\delta^{-1}\dot{\boldsymbol{\gamma}}(t)$. Since d/dt commutes with δ^{-1} and the standard normalizing condition for an Abelian connection is $\delta^{-1}r = 0$, one obtains

$$H(t) = -\frac{1}{6}Q_t\left(\dot{\boldsymbol{\Gamma}}_{ijk}(t)y^i y^j y^k\right),$$

with $\deg(H(t)) \geq 3$. Thus assumptions of theorem 3.1 are fulfilled. The mapping defined therein is the desired isomorphism between W_{D_0} and W_D. $\square$

To obtain its explicit form we need explicit form of $H(t)$. Using iteration method one can calculate $H(t)$ up to the fifth degree

$$\begin{aligned} H(t) \;=\; & -\frac{1}{6}\dot{\boldsymbol{\Gamma}}_{ijk}(t)y^i y^j y^k - \frac{1}{24}\boldsymbol{\partial}_i^{(t)}\dot{\boldsymbol{\Gamma}}_{jkl}(t)y^i y^j y^k y^l \\ & -\frac{1}{120}\boldsymbol{\partial}_i^{(t)}\boldsymbol{\partial}_j^{(t)}\dot{\boldsymbol{\Gamma}}_{klm}(t)y^i y^j y^k y^l y^m - \frac{1}{80}\boldsymbol{R}_{ijpk}(t)\dot{\boldsymbol{\Gamma}}^p{}_{lm}(t)y^i y^j y^k y^l y^m \\ & +\frac{1}{32}h^2\boldsymbol{R}_{ijkl}(t)\dot{\boldsymbol{\Gamma}}^{ijk}(t)y^l + \ldots\,. \end{aligned} \tag{5}$$

Let $T^{-1} : W_{D_0} \to W_D$ denote isomorphism mentioned in theorem 3.1. Its inverse $T : W_D \to W_{D_0}$ is called *local trivialization* of W_D. Using (5) when iterating equation (4) one can compute first terms of T^{-1}. They read

$$\begin{aligned} T^{-1}(Q_0(a_0)) \;=\; & Q(a_0) + h^2 Q\left(\frac{1}{24}\omega^{ls}\frac{\partial a_0}{\partial x^s}\int_0^1 \frac{\partial \boldsymbol{\Gamma}_{ijk}(\tau)}{\partial x^l}\dot{\boldsymbol{\Gamma}}^{ijk}(\tau)d\tau \right. \\ & \left. + \frac{1}{16}\omega^{ls}\frac{\partial^2 a_0}{\partial x^s \partial x^k}\Gamma^{ijk}\Gamma_{ijl} + \frac{1}{24}\frac{\partial^3 a_0}{\partial x^i \partial x^j \partial x^k}\Gamma^{ijk}\right) + \ldots\,. \end{aligned}$$

Isomorphism T^{-1} depends on the choice of homotopy $\boldsymbol{\Gamma}(t)$. However for homotopies of the form $\boldsymbol{\Gamma}_{ijk}(t) = f(t)\Gamma_{ijk}$ ($f : [0,1] \to \mathbb{R}$, $f(0) = 0$, $f(1) = 1$), the result is independent (at least up to h^2) of the choice of f. In this case T^{-1} reads

$$T^{-1}(Q_0(a_0)) = Q(a_0) + h^2 Q\left(\frac{1}{48}\omega^{ls}\frac{\partial a_0}{\partial x^s}\frac{\partial \Gamma_{ijk}}{\partial x^l}\Gamma^{ijk} + \frac{1}{16}\omega^{ls}\frac{\partial^2 a_0}{\partial x^s \partial x^k}\Gamma^{ijk}\Gamma_{ijl} + \frac{1}{24}\frac{\partial^3 a_0}{\partial x^i \partial x^j \partial x^k}\Gamma^{ijk}\right) + \dots, \quad (6)$$

and conversely

$$T(Q(a_0)) = Q_0(a_0) - h^2 Q_0\left(\frac{1}{48}\omega^{ls}\frac{\partial a_0}{\partial x^s}\frac{\partial \Gamma_{ijk}}{\partial x^l}\Gamma^{ijk} + \frac{1}{16}\omega^{ls}\frac{\partial^2 a_0}{\partial x^s \partial x^k}\Gamma^{ijk}\Gamma_{ijl} + \frac{1}{24}\frac{\partial^3 a_0}{\partial x^i \partial x^j \partial x^k}\Gamma^{ijk}\right) + \dots .$$

The above form of trivialization isomorphism will be used in the next section.

4. Differential calculus

In this section we are going to construct a differential calculus based on noncommutative Fedosov algebra of formal series. We initially make use of some ideas of Madore and collaborators [11, 12] and then follow standard approach to Cartan algebra presented in [13]. First, let us recall algebraical definition of differential calculus [11, 14].

Definition 4.1. *A complex, unital and associative algebra $\mathcal{K}$ with product $\wedge$ is called differential calculus over $\mathcal{K}^0$ if it is $\mathbb{N}$-graded*

1. $\mathcal{K} = \bigoplus_{n\geq 0} \mathcal{K}^n$,
2. $\mathcal{K}^k \wedge \mathcal{K}^l \subset \mathcal{K}^{k+l}$,

and it is equipped with compatible nilpotent antiderivation $d : \mathcal{K} \to \mathcal{K}$

3. $d\mathcal{K}^l \subset \mathcal{K}^{l+1}$,
4. $d(\eta \wedge \xi) = (d\eta) \wedge \xi + (-1)^l \eta \wedge d\xi$ *for arbitrary $\eta \in \mathcal{K}^l$ and $\xi \in \mathcal{K}$* ,
5. $d^2 = 0$.

Let $\mathcal{O}$ be a neighborhood of some point x_0 for which trivialization theorem holds. Let $\Lambda^0 = \mathcal{A}$ be usual algebra of functions on $\mathcal{O}$ and $\Lambda^0_* = \mathcal{A}_*$ — algebra of formal series obtained by Fedosov deformation quantization procedure. Following ideas of [11, 12], we are going to choose set

$\mathcal{X} = \{X_1, \ldots, X_n\}$ of n derivations $X_i \in Der(\mathcal{A}_*)$, which is analogous to the frame in the classical geometry. We will use derivations of the form $X_i = \frac{i}{h}[\lambda_i \overset{*}{,} \cdot]$. The λ_is may be taken as

$$\lambda_i := \omega_{ij} Q^{-1} T^{-1} Q_0 x^j \,.$$

Using (6) one finds that

$$\lambda_i = \omega_{ij} x^j - \frac{h^2}{48} \frac{\partial \Gamma_{jkl}}{\partial x^i} \Gamma^{jkl} + \ldots \,.$$

Derivation X_i acting on $f \in \mathcal{A}_*$ yields

$$\begin{aligned} X_i(f) &= \frac{i}{h}[\lambda_i \overset{*}{,} f] = Q^{-1} T^{-1} \frac{\partial}{\partial y^i} (TQf) = \frac{\partial f}{\partial x^i} \\ &-h^2 \left\{ \frac{1}{48} \omega^{ls} \frac{\partial f}{\partial x^s} \frac{\partial}{\partial x^i} \left(\frac{\partial \Gamma_{mjk}}{\partial x^l} \Gamma^{mjk} \right) + \frac{1}{16} \omega^{ls} \frac{\partial^2 f}{\partial x^s \partial x^k} \frac{\partial (\Gamma^{mjk} \Gamma_{mjl})}{\partial x^i} \right. \\ &\left. + \frac{1}{24} \frac{\partial^3 f}{\partial x^m \partial x^j \partial x^k} \frac{\partial \Gamma^{mjk}}{\partial x^i} \right\} + \ldots \,. \end{aligned} \tag{7}$$

The most important properties of the "frame" $\mathcal{X}$ are consequences of the following lemma.

Lemma 4.2. *The commutation relations for λ_i are given by*

$$\frac{i}{h} [\lambda_i \overset{*}{,} \lambda_j] = -\omega_{ij} \,. \tag{8}$$

Proof. The straightforward calculation yields

$$\begin{aligned} \frac{i}{h} [\lambda_i \overset{*}{,} \lambda_j] &= \frac{i}{h} \left[\omega_{ik} Q^{-1} T^{-1} Q_0 x^k \overset{*}{,} \omega_{jl} Q^{-1} T^{-1} Q_0 x^l \right] \\ &= \omega_{ik} \omega_{jl} \frac{i}{h} Q^{-1} T^{-1} \left[Q_0 x^k \overset{\circ}{,} Q_0 x^l \right] = \omega_{ik} \omega^{kl} \omega_{jl} = -\omega_{ij} \,. \end{aligned}$$

□

Corollary 4.3. $X_i X_j = X_j X_i$ *for each* $X_i, X_j \in \mathcal{X}$ *.*

Proof. Using lemma 4.2 and the Jacobi identity one obtains for $f \in \mathcal{A}_*$

$$\begin{aligned} X_i X_j f &= -\frac{1}{h^2} [\lambda_i \overset{*}{,} [\lambda_j \overset{*}{,} f]] = \frac{1}{h^2} [f \overset{*}{,} [\lambda_i \overset{*}{,} \lambda_j]] + \frac{1}{h^2} [\lambda_j \overset{*}{,} [f \overset{*}{,} \lambda_i]] \\ &= -\frac{1}{h^2} [\lambda_j \overset{*}{,} [\lambda_i \overset{*}{,} f]] = X_j X_i f \,. \end{aligned}$$

□

Let $\mathcal{T}_*^k(\mathcal{X})$ denote the vector space (over $\mathbb{C}$) of mappings from $\mathcal{X}^k$ (k-fold product $\mathcal{X} \times \mathcal{X} \times \cdots \times \mathcal{X}$) to $\mathcal{A}_*$. $\mathcal{T}_*^k(\mathcal{X})$ has a natural structure of $\mathcal{A}_*$-bimodule given by the relations

$$\begin{aligned}(f * \eta)\,(X_{i_1}, \ldots, X_{i_k}) &= f * \eta\,(X_{i_1}, \ldots, X_{i_k})\ ,\\ (\eta * f)\,(X_{i_1}, \ldots, X_{i_k}) &= \eta\,(X_{i_1}, \ldots, X_{i_k}) * f\,,\end{aligned}$$

for $f \in \mathcal{A}_*$, $\eta \in \mathcal{T}_*^k(\mathcal{X})$ and $X_{i_1}, \ldots, X_{i_k} \in \mathcal{X}$. For $T \in \mathcal{T}_*^k(\mathcal{X})$ and $S \in \mathcal{T}_*^l(\mathcal{X})$, the tensor product $T \otimes_* S \in \mathcal{T}_*^{k+l}(\mathcal{X})$ may be defined as

$$(T \otimes_* S)(X_{i_1}, \ldots, X_{i_{k+l}}) := T(X_{i_1}, \ldots, X_{i_k}) * S(X_{i_{k+1}}, \ldots, X_{i_{k+l}})\,.$$

Theorem 4.4. *The product $\otimes_*$ has the following properties*

$$\begin{aligned}(S_1 + S_2) \otimes_* T &= S_1 \otimes_* T + S_2 \otimes_* T\,,\\ T \otimes_* (S_1 + S_2) &= T \otimes_* S_1 + T \otimes_* S_2\,,\\ (f * S) \otimes_* T &= f * (S \otimes_* T)\,,\\ S \otimes_* (T * f) &= (S \otimes_* T) * f\,,\\ (S * f) \otimes_* T &= S \otimes_* (f * T)\,,\\ (S \otimes_* T) \otimes_* U &= S \otimes_* (T \otimes_* U)\,,\end{aligned}$$

for S_1, S_2, S, T, U belonging to some (not necessarily the same) $\mathcal{T}_*^k(\mathcal{X})$, and $f \in \mathcal{A}_*$.

Proof is a straightforward consequence of properties of $\mathcal{A}_*$. One may introduce [11,12] the exterior derivative of $f \in \mathcal{A}_*$ as a mapping $d_* f \in \mathcal{T}_*^1(\mathcal{X})$ defined by

$$d_* f(X_i) := X_i(f)\,.$$

It can be easily observed that d_* fulfills the Leibniz rule

$$d_*(f * g) = (d_* f) * g + f * d_* g\,.$$

Our choice of $\mathcal{X}$ enables us to introduce "coframe" $\Theta = \{\theta^1, \ldots, \theta^n\}$ consisting of $\theta^j \in \mathcal{T}_*^1(\mathcal{X})$ defined by

$$\theta^j := d_* \left(\omega^{jk} \lambda_k\right) = d_* \left(Q^{-1} T^{-1} Q_0 x^j\right)\,.$$

By lemma 4.2 we calculate

$$\theta^j(X_i) = X_i \left(\omega^{jk} \lambda_k\right) = -\omega^{jk} \omega_{ik} = \delta^j{}_i\,. \tag{9}$$

(Concept of Θ dual to $\mathcal{X}$ is derived from [11, 12]). As a consequence one infers that each θ^j commutes with an arbitrary $f \in \mathcal{A}_*$, *i.e.*

$$f * \theta^j = \theta^j * f\,.$$

Define $\mathcal{B}_k$ as a set of all k-fold products $\theta^{i_1} \otimes_* \cdots \otimes_* \theta^{i_k}$ ($\mathcal{B}_1 = \Theta$).

Theorem 4.5. *$\mathcal{B}_k$ freely generates $\mathcal{A}_*$-bimodule $\mathcal{T}^k_*(\mathcal{X})$.*

Proof. For arbitrary $T \in \mathcal{T}^k_*(\mathcal{X})$ one has

$$T = T\left(X_{i_1}, \ldots, X_{i_k}\right) * \theta^{i_1} \otimes_* \cdots \otimes_* \theta^{i_k}\,, \tag{10}$$

and equation $r_{i_1 \ldots i_k} * \theta^{i_1} \otimes_* \cdots \otimes_* \theta^{i_k} = 0$ evaluated on $(X_{j_1}, \ldots, X_{j_k})$ yields $r_{j_1 \ldots j_k} = 0$. □

One concludes that $\mathcal{B}_k$ is $\mathcal{A}_*$-basis of $\mathcal{T}^k_*(\mathcal{X})$. We put $\Lambda^1_* = \mathcal{T}^1_*(\mathcal{X})$. Properties of $\mathcal{X}$ provide that construction of Λ^k_* for $k > 1$ may follow usual construction of Λ. The approach presented here is based on the classical textbook [13]. The omitted proofs are just identical to those in [13]. We call $\eta \in \mathcal{T}^k_*(\mathcal{X})$ alternating if

$$\eta\left(\ldots, X_{i_p}, \ldots, X_{i_q}, \ldots\right) = -\eta\left(\ldots, X_{i_q}, \ldots, X_{i_p}, \ldots\right)$$

for arbitrary $1 \leq p < q \leq k$. The subset of $\mathcal{T}^k_*(\mathcal{X})$ consisting of all alternating $\eta \in \mathcal{T}^k_*(\mathcal{X})$ is a $\mathcal{A}_*$-subbimodule of $\mathcal{T}^k_*(\mathcal{X})$. We put this submodule to be Λ^k_*. If $i_q = i_p$ for some $q \neq p$ then $\eta(X_{i_1}, \ldots, X_{i_k}) = 0$. Hence Λ^k_* vanish for $k > n$. The projection form $\mathcal{T}^k_*(\mathcal{X})$ to Λ^k_* can be chosen in the standard way. For $T \in \mathcal{T}^k_*(\mathcal{X})$ let

$$\mathrm{Alt}(T)\left(X_{i_1}, \ldots, X_{i_k}\right) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) T\left(X_{i_{\sigma(1)}}, \ldots, X_{i_{\sigma(k)}}\right)\,,$$

where S_k denotes the group of permutations of $\{1, \ldots, k\}$, $\mathrm{sgn}(\sigma) = 1$ for even and $\mathrm{sgn}(\sigma) = -1$ for odd permutations.

Theorem 4.6. *The Alt operation has the following properties*

$$\begin{aligned} \mathrm{Alt}(T) &\in \Lambda^k_*\,, \\ \mathrm{Alt}(f * T + S * g) &= f * \mathrm{Alt}(T) + \mathrm{Alt}(S) * g\,, \\ \mathrm{Alt}(\eta) &= \eta\,, \\ \mathrm{Alt}(\mathrm{Alt}(T)) &= \mathrm{Alt}(T)\,, \end{aligned} \tag{11}$$

for $T, S \in \mathcal{T}^k_*(\mathcal{X})$, $\eta \in \Lambda^k_*$ and $f, g \in \mathcal{A}_*$.

The second relation can be easily obtained from definition of Alt. The others are proven in [13]. For $\eta \in \Lambda^k_*$ and $\xi \in \Lambda^l_*$ the exterior product $\eta \wedge_* \xi \in \Lambda^{k+l}_*$ is defined as

$$\eta \wedge_* \xi := \frac{(k+l)!}{k!\,l!}\,\mathrm{Alt}(\eta \otimes_* \xi)\,.$$

Theorem 4.7. *The $\wedge_*$ product has the following properties*

$$\begin{aligned}
(\xi_1 + \xi_2) \wedge_* \eta &= \xi_1 \wedge_* \eta + \xi_2 \wedge_* \eta \,, \\
\eta \wedge_* (\xi_1 + \xi_2) &= \eta \wedge_* \xi_1 + \eta \wedge_* \xi_1 \,, \\
(f * \eta) \wedge_* \xi &= f * (\eta \wedge_* \xi) \,, \\
\eta \wedge_* (\xi * f) &= (\eta \wedge_* \xi) * f \,, \\
(\eta * f) \wedge_* \xi &= \eta \wedge_* (f * \xi) \,, \\
(\eta \wedge_* \xi) \wedge_* \zeta &= \eta \wedge_* (\xi \wedge_* \zeta) \,,
\end{aligned}$$

for $\eta \in \Lambda^k_*$, $\xi, \xi_1, \xi_2 \in \Lambda^l_*$ and $f \in \mathcal{A}_*$.

All except for the last of these relations are simple consequences of theorems 4.4 and 4.6. For associativity the proof is more elaborated, but can be performed exactly in the same way as in [13]. Omitting its details let us notice, that the key step is to prove the formula

$$\begin{aligned}
\mathrm{Alt}(\mathrm{Alt}(\eta \otimes_* \xi) \otimes_* \zeta) &= \mathrm{Alt}(\eta \otimes_* \xi \otimes_* \zeta) \\
&= \mathrm{Alt}(\eta \otimes_* \mathrm{Alt}(\xi \otimes_* \zeta)) \,. \qquad (12)
\end{aligned}$$

Theorem 4.7 justifies extension of exterior product to 0-forms. For $f \in \Lambda^0_* = \mathcal{A}_*$ and $\eta \in \Lambda^k_*$ we put $f \wedge_* \eta := f * \eta$ and $\eta \wedge_* f := \eta * f$. Notice that in general one cannot obtain relation analogous to $\eta \wedge \xi = (-1)^{kl} \xi \wedge \eta$. Fortunately, due to (9), the following formula holds

$$\theta^i \wedge_* \theta^j = -\theta^j \wedge_* \theta^i \,, \qquad (13)$$

for arbitrary $\theta^i, \theta^j \in \Theta$, and in general

$$\theta^{i_1} \wedge_* \theta^{i_2} \wedge_* \cdots \wedge_* \theta^{i_k} = \mathrm{sgn}(\sigma) \theta^{i_{\sigma(1)}} \wedge_* \theta^{i_{\sigma(2)}} \wedge_* \cdots \wedge_* \theta^{i_{\sigma(k)}} \,, \qquad (14)$$

for $\sigma \in S_k$ and $\theta^{i_1}, \ldots, \theta^{i_k} \in \Theta$. Using lemma 4.5 and formulas (11), (12) one can represent arbitrary $\eta \in \Lambda^k_*$ as

$$\eta = \frac{1}{k!} \eta \left(X_{i_1}, \ldots, X_{i_k} \right) * \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k} \,.$$

Applying (14) one reduces the above relation to

$$\eta = \sum_{1 \le i_1 < \cdots < i_k \le n} \eta \left(X_{i_1}, \ldots, X_{i_k} \right) * \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k} \,.$$

The $\mathcal{A}_*$-linear independence of the set $\mathcal{C}_k := \{\theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$ can be proven. Hence, $\mathcal{C}_k$ is $\mathcal{A}_*$-basis of Λ^k_* and

$$\dim(\Lambda^k_*) = \binom{n}{k} \,.$$

Moreover (14) guarantees that

$$\frac{1}{k!}\eta_{i_1\dots i_k} * \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k} = \frac{1}{k!}\eta_{[i_1\dots i_k]} * \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k} .$$

One also infers that any $\eta \in \Lambda^k_*$ can be written as

$$\eta = \frac{1}{k!}\eta_{i_1\dots i_k} * \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k}$$

in the unique manner, provided that $\eta_{i_1\dots i_k}$ is totally antisymmetric. Notice, that given two forms $\eta = \frac{1}{k!}\eta_{i_1\dots i_k} * \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k}$ and $\xi = \frac{1}{l!}\xi_{j_1\dots j_l} * \theta^{j_1} \wedge_* \cdots \wedge_* \theta^{j_l}$ their exterior product may be written as

$$\eta \wedge_* \xi = \frac{1}{k!l!}\,\eta_{i_1\dots i_k} * \xi_{j_1\dots j_l} * \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k} \wedge_* \theta^{j_1} \wedge_* \cdots \wedge_* \theta^{j_l} .$$

We are ready to extend d_* to forms of higher degree. Define

$$d_*\left(\frac{1}{k!}\eta_{i_1\dots i_k} * \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k}\right) := \frac{1}{k!}X_j\left(\eta_{i_1\dots i_k}\right) * \theta^j \wedge_* \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k} . \quad (15)$$

Suppose that $\frac{1}{k!}\eta_{i_1\dots i_k} * \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k} = \frac{1}{k!}\tilde{\eta}_{i_1\dots i_k} * \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k}$. Then replacing $X_j(\eta_{i_1\dots i_k})$ by $X_{[j}(\eta_{i_1\dots i_k]})$ in (15) and using relation $\eta_{[i_1\dots i_k]} = \tilde{\eta}_{[i_1\dots i_k]}$ one can obtain

$$d_*\left(\frac{1}{k!}\eta_{i_1\dots i_k} * \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k}\right) = d_*\left(\frac{1}{k!}\tilde{\eta}_{i_1\dots i_k} * \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k}\right) ,$$

hence d_* is well defined. Notice, that above definition of d_* is compatible with definition of d_* for 0-forms since $d_*f = (d_*f)(X_j) * \theta^j = X_j(f) * \theta^j$ for $f \in \mathcal{A}_*$.

Theorem 4.8. *The d_* operator has the following properties*

1. $d_*d_* = 0$,
2. $d_*(\eta \wedge_* \xi) = (d_*\eta) \wedge_* \xi + (-1)^k \eta \wedge_* (d_*\xi)$ *for* $\eta \in \Lambda^k_*$ *and* $\xi \in \Lambda^l_*$.

Proof. Using corollary 4.3 and formula (13) one calculates

$$\begin{aligned} d_*d_*\eta &= \frac{1}{k!}X_k(X_j(\eta_{i_1\dots i_k})) * \theta^k \wedge_* \theta^j \wedge_* \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k} \\ &= -\frac{1}{k!}X_j(X_k(\eta_{i_1\dots i_k})) * \theta^j \wedge_* \theta^k \wedge_* \theta^{i_1} \wedge_* \cdots \wedge_* \theta^{i_k} = -d_*d_*\eta = 0 . \end{aligned}$$

The Leibniz rule can be obtained by direct calculation involving application of Leibniz rule for X_j and k-fold use of formula (13). □

One concludes that $\Lambda_* = \Lambda^0_* \oplus \cdots \oplus \Lambda^n_*$ together with d_* is a differential calculus over $\mathcal{A}_*$.

5. Final comments

The main result of this paper is the explicit construction of local differential calculus over Fedosov algebra of formal power series with coefficients in functions on symplectic manifold. The approach presented here is generally inspired by some standard procedures of noncommutative differential geometry [11,12]. Quite similar analysis can be found in [2]. The main difference is that in our approach we do not *postulate* existence of λ_is with commutation relations given by lemma 4.2, but we rather *construct* them using trivialization procedure. The explicit (up to h^2) form of trivialization isomorphism has been calculated and thus, we are able to give explicit formulas for deformed derivations (7). Since they commute (corollary 4.3) we may proceed with standard methods of usual Cartan algebra. Hence, we omit all consistency conditions [11, 12] relating noncommutativity of algebra, noncommutativity of X_i, exterior product and exterior derivative. Moreover, the resulting differential calculus can be regarded as the deformation of the usual one *i.e.* obtained corrections vanish either at $h = 0$ or at $\Gamma_{ijk} \equiv 0$. On the other hand, construction presented here is local (restricted to some open subset for which trivialization theorem holds). It should be also stressed, that our choice of "noncommutative frame" $\mathcal{X}$, although yielding some useful properties, cannot be considered as the distinguished one. The application of presented construction to Seiberg–Witten theory requires also some concepts on "noncommutative connections". Such ideas were considered in [15] and they seem to be applicable also in the case of Fedosov quantization. This is hoped to be covered in author's forthcoming paper.

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