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GROUP ACTIONS AS STROBOSCOPIC MAPS OF ORDINARY DIFFERENTIAL EQUATIONS

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Discrete-time dynamical systems can be derived from group actions. In the present work possibility of application of this method to systems of ordinary differential equations is studied. Invertible group actions are considered as possible candidates for stroboscopic maps of ordinary differential equations. It is shown that flow of the Bloch equation is a unique suspension of an invertible map on the SU(2) group.

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1. Introduction

Discrete-time dynamical systems can be formulated in terms of group actions to exploit the group structure and get a better understanding of the corresponding dynamics. This approach was used to study non-invertible discrete-time dynamics on E(2) [1] and SU(2) [1–4]. The Shimizu–Leutbecher map [5, 6], a tool to study group structure, was solved for an arbitrary Lie group $\mathcal{G}$ [1]. Since the Shimizu–Leutbecher sequence generates the logistic map for $\mathcal{G} = \mathrm{SU}(2)$ [1, 3] a general solution of the logistic map was thus obtained. For $\mathcal{G} = \mathrm{E}(2)$ this approach led to a better understanding of the Harter-Heighway fractal curve [1]. A two-dimensional generalization of the logistic map was also introduced as a map on SU(2) and investigated [4].

On the other hand, structure of Kleinian groups is naturally studied in the setting of discrete-time dynamical systems, revealing in this way connections with fractals [7–9]. There were several attempts to study dynamics on Lie groups within continuous-time rather than discrete-time approach. Continuous-time dynamics on SU(2) and $\mathrm{SU}(2) \times \mathrm{SU}(2)$ was defined in the setting of ordinary differential equations (ODEs) [10, 11] while continuous iteration of maps was defined to find a correspondence between mappings and continuous-time evolution [12, 13].

The question now arises whether a general connection between discrete-time group actions and continuous-time dynamical systems can be established.

We investigate a possibility of relating group actions with stroboscopic maps of ODEs. Let us consider a continuous-time dynamical system given by a set of ODEs

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}\left(\boldsymbol{x}\left(t\right), t\right), \tag{1.1}$$

where $\boldsymbol{x} = [x_1,\ x_2,\ \ldots\ ,\ x_n]\,,\ \boldsymbol{f} = [f_1,\ f_2,\ \ldots\ ,\ f_n]$. Let $\boldsymbol{x}\left(t\right)$ be a solution to Eq. (1.1). Then the invertible map

$$S_T: \quad \boldsymbol{x}\left(t\right) \longmapsto \boldsymbol{x}\left(t+T\right), \tag{1.2}$$

is a stroboscopic map of strobe time T. Stroboscopic maps with several strobe times are standard tools to solve differential equations to mention only the Runge-Kutta methods. On the other hand, a discrete map can be also used to generate a continuous flow. Such a flow, non unique in general, is called a suspension of the map [14, 15].

In the present work we shall consider a class of invertible maps on a group and then try to find a flow, preferably unique, such that its trajectories connect continuously the iterates of the map.

The paper is organized as follows. In the next Section maps on a Lie group $\mathcal{G}$ are defined and method to deduce evolution of group parameters is described. Principal results are described in Sections 3 and 4. In Section 3 a class of invertible maps is considered. These maps are solved in Subsection 3.1. In Subsection 3.2 a simplified map is defined, solved and parameterized on the SU(2) group. In Subsection 3.3 symmetry and restrictions of the dynamics are determined. Results of the Subsection 3.2 are used in Section 4 to derive ODE for which the simplified map is a stroboscopic map. The map samples the flow of the ODE exactly and arbitrarily densely and it follows that the construction is unique. This ODE is the Bloch equation [16], also known as the Landau–Lifshitz equation. These results generalize our earlier findings [17]. In Section 5 computations for the simplified map on the SU(2) group are presented to elucidate dynamics of the Bloch equation. In the last Section the obtained results are summarized and perspectives of further research are outlined.

2. Group dynamical systems

Let us consider a dynamical system defined by the following map

$$G_{N+1} = \varphi\left(G_N,\ \ldots\right), \tag{2.1}$$

where $G_N \in \mathcal{G}$. Let $\mathcal{G}$ be a simple Lie group and $\mathfrak{g}$ its Lie algebra. Then $G_N \in \mathcal{G}$ can be written in exponential form

$$G_N = \exp\left(X_N\right) , \tag{2.2}$$

where $X_N \in \mathfrak{g}$ [18]. Any infinitesimal operator X_N of a n-parameter Lie group $\mathcal{G}$ is a linear combination of n generators I^k

$$X_N = \sum_{k=1}^{n} I^k c_N^k , \qquad I^k \in \mathfrak{g} , \tag{2.3}$$

where real parameters $c_N^1, \ldots, c_N^n$ are local coordinates of the Lie group element X_N.

Let $\mathfrak{g} = \mathfrak{su}\,(n)$. Substituting Eqs. (2.2), (2.3) into Eq. (2.1) and using completeness of the basis consisting of the generators I^k $\left(k = 1, \ldots, n^2 - 1\right)$ and the unit matrix $\mathbf{1}$ in the defining representation of $\mathfrak{su}\,(n)$ [19] we get a discrete-time dynamical system in parameter space [1, 20]

$$c_{N+1}^j = F^j\left(c_N^1,\ \ldots,\ c_N^{n^2-1}\right) , \qquad j = 1,\ \ldots,\ n^2 - 1 , \tag{2.4}$$

where F^j are continuous functions [21].

3. Discrete-time dynamics on a group

Let us consider an invertible discrete-time dynamical system on a Lie group $\mathcal{G}$

$$R_{N+1} = Q_N R_N Q_{N-1} R_{N-1} Q_{N-1}^{-1} R_N^{-1} Q_N^{-1} , \qquad N = 1, 2, \ldots , \tag{3.1}$$

i.e. $\varphi = Q_N R_N Q_{N-1} R_{N-1} Q_{N-1}^{-1} R_N^{-1} Q_N^{-1}$ in Eq. (2.1), where we assume knowledge of all group elements Q_M needed for the computations. Let us note here that apparently simpler non-invertible Shimizu–Leutbecher map $R_{N+1} = R_N Q R_N^{-1}$ has a complicated solution [1]. It turns out, however, that a relatively simple solution to (3.1) can be constructed upon introducing new variables S_N defined below.

3.1. Exact solution

We note that Eq. (3.1) can be reduced to two simpler equations. Indeed, introducing new quantity

$$S_N \stackrel{\mathrm{df}}{=} R_N Q_{N-1} R_{N-1} Q_{N-2} , \tag{3.2}$$

we can rewrite Eq. (3.1) as

$$R_{N+1} = S_{N+1} R_{N-1} S_{N+1}^{-1}, \quad N = 1, 2, \dots , \tag{3.3a}$$

$$S_{N+1} = Q_N S_N Q_{N-2}^{-1}, \quad N = 1, 2, \dots . \tag{3.3b}$$

To run dynamics defined by (3.1) or, alternatively, by (3.3), we have to impose initial conditions for Eq. (3.1), $R_0,\ R_1$, from which initial condition for Eq. (3.3b), $S_2 = R_2 Q_1 R_1 Q_0$, can be also computed. Equations (3.3) are easily solved

$$R_{2K} = S_{2K} S_{2K-1} \dots S_2 R_0 S_2^{-1} \dots S_{2K-1}^{-1} S_{2K}^{-1} , \tag{3.4a}$$

$$S_N = Q_{N-1} \dots Q_1 S_1 Q_{-1}^{-1} \dots Q_{N-3}^{-1} , \tag{3.4b}$$

where $K = 1, 2, \dots$, $N = 2, 3, \dots$ and similar equations can be written for R_{2K+1}.

3.2. Discrete-time dynamics on the sphere

In the case $\mathcal{G}$ = SU(2) the Hamilton's parameterization can be used. Let unit vectors $\boldsymbol{r}_N = \left(r_N^1 ,\ r_N^2,\ r_N^3\right)$, $\boldsymbol{q}_N = \left(q_N^1,\ q_N^2,\ q_N^3\right)$ and angles χ_N, α_N correspond to rotation axes and rotation angles, respectively. Then the rotation matrices $R_N,\ Q_N$ are defined as

$$R_N = \exp\left(i\tfrac{\chi_N}{2}\boldsymbol{\sigma}\cdot\boldsymbol{r}_N\right) , \qquad |\boldsymbol{r}_N| = 1 , \tag{3.5}$$

$$Q_N = \exp\left(i\tfrac{\alpha_N}{2}\boldsymbol{\sigma}\cdot\boldsymbol{q}_N\right) , \qquad |\boldsymbol{q}_N| = 1 , \tag{3.6}$$

where $i^2 = -1$, and $\boldsymbol{\sigma} = \left[\sigma^1,\ \sigma^2,\ \sigma^3\right]$ is the pseudo vector with the Pauli matrices as components [22]. With parameterization (3.5), (3.6) Eq. (3.1) induces dynamics of vectors $\boldsymbol{r}_N$ on unit sphere. We shall consider a special case $Q_N \equiv Q$

$$R_{N+1} = Q R_N Q R_{N-1} Q^{-1} R_N^{-1} Q^{-1} . \tag{3.7}$$

The solution of (3.7) is immediately obtained from (3.4):

$$R_{2K} = Q^{2K} P^K R_0 P^{-K} Q^{-2K} , \tag{3.8a}$$

$$\tilde{R}_{2K+1} = Q^{2K} P^K \tilde{R}_1 P^K Q^{-2K} , \tag{3.8b}$$

where

$$P \overset{\mathrm{df}}{=} Q^{-1} S_1 Q^{-1} = Q^{-1} R_1 Q R_0 ,$$

$$\tilde{R}_{2K+1} \overset{\mathrm{df}}{=} Q^{-1} R_{2K+1} Q,\ K = 0,\ 1,\ 2,\ \dots .$$

It follows that equations (3.8a), (3.8b) generate analogous dynamics. Matrices Q, P are parameterized as

$$Q = \exp\left(i\tfrac{\alpha}{2}\boldsymbol{\sigma}\cdot\boldsymbol{q}\right), \qquad |\boldsymbol{q}| = 1\,, \tag{3.9a}$$

$$P = \exp\left(i\tfrac{\beta}{2}\boldsymbol{\sigma}\cdot\boldsymbol{p}\right), \qquad |\boldsymbol{p}| = 1\,, \tag{3.9b}$$

where α, β are the corresponding angles of rotations while $\boldsymbol{p} = \left[p^1, p^2, p^3\right]$ and $\boldsymbol{q} = \left[q^1, q^2, q^3\right]$ are unit vectors. We still have to impose initial condition R_0 while R_1 is computed as $R_1 = QPR_0^{-1}Q^{-1}$.

A sufficient condition that points generated according to (3.8a) form a periodic trajectory (a finite set of points) is that for some integer K the following conditions hold:

$$K\beta = 2m\pi\,, \qquad 2K\alpha = 2n\pi\,, \tag{3.10}$$

for some integer m, n; in this case the parameter $\beta/(2\alpha) = m/n$ is rational.

Let us note that due to properties of the Pauli matrices the vector $\boldsymbol{x}(\gamma)$ defined by

$$\boldsymbol{\sigma}\cdot\boldsymbol{x}(\gamma) = S\ \boldsymbol{\sigma}\cdot\boldsymbol{x}\ S^{-1} = \exp\left(i\tfrac{\gamma}{2}\boldsymbol{\sigma}\cdot\boldsymbol{s}\right)\ \boldsymbol{\sigma}\cdot\boldsymbol{x}\ \exp\left(-i\tfrac{\gamma}{2}\boldsymbol{\sigma}\cdot\boldsymbol{s}\right), \tag{3.11}$$

where $\boldsymbol{x} = \left[x^1, x^2, x^3\right]$, $\boldsymbol{s} = \left[s^1, s^2, s^3\right]$ are unit vectors and γ is a corresponding angle of rotation, is easily computed as

$$\boldsymbol{x}(\gamma) = \cos(\gamma)\,\boldsymbol{x} + \sin(\gamma)\,\boldsymbol{x}\times\boldsymbol{s} + (1-\cos(\gamma))\,(\boldsymbol{s}\cdot\boldsymbol{x})\,\boldsymbol{s}\,. \tag{3.12}$$

Using this result it is possible to write the solution (3.8a) in a closed form. Indeed, equation (3.8a) can be explicitly written as

$$\begin{aligned}\boldsymbol{\sigma}\cdot\boldsymbol{r}_{2K} \;=\;& \exp(iK\alpha\boldsymbol{\sigma}\cdot\boldsymbol{q})\exp\left(iK\tfrac{\beta}{2}\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)\ \boldsymbol{\sigma}\cdot\boldsymbol{r}_0\\ &\times\exp\left(-iK\tfrac{\beta}{2}\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)\exp(-iK\alpha\boldsymbol{\sigma}\cdot\boldsymbol{q})\,.\end{aligned} \tag{3.13}$$

Applying Eq. (3.12) twice we find the closed form solution of Eq. (3.7):

$$\begin{aligned}\boldsymbol{r}(K\alpha) \;=\;& \cos(K\alpha)\,\boldsymbol{t}(K\alpha) + \sin(K\alpha)\,\boldsymbol{t}(K\alpha)\times\boldsymbol{q}\\ &+ (1-\cos(K\alpha))\,(\boldsymbol{q}\cdot\boldsymbol{t}(K\alpha))\,\boldsymbol{q}\,,\end{aligned} \tag{3.14a}$$

$$\begin{aligned}\boldsymbol{t}(K\alpha) \;=\;& \cos(\lambda K\alpha)\,\boldsymbol{r}_0 + \sin(\lambda K\alpha)\,\boldsymbol{r}_0\times\boldsymbol{p}\\ &+ (1-\cos(\lambda K\alpha))\,(\boldsymbol{p}\cdot\boldsymbol{r}_0)\,\boldsymbol{p}\,,\end{aligned} \tag{3.14b}$$

where $\boldsymbol{r}(K\alpha) \stackrel{\mathrm{df}}{=} \boldsymbol{r}_{2K}$, $\lambda \stackrel{\mathrm{df}}{=} \beta/(2\alpha)$ and $K = 0, 1, 2, \ldots$.

The sequence of vectors $\boldsymbol{r}_0,\ \boldsymbol{r}_2,\ \boldsymbol{r}_4, \ldots$, generated according to Eq. (3.14), samples a continuous curve $\mathcal{C}$. Indeed, it follows from Eq. (3.14) that for very small α and fixed λ vectors $\boldsymbol{r}\,(K\alpha)$ and $\boldsymbol{r}\,((K+1)\,\alpha)$ are very close on a unit sphere. Therefore for decreasing α, β and fixed $\lambda = \beta/\,(2\alpha)$ the sequence $\boldsymbol{r}_0,\ \boldsymbol{r}_2,\ \boldsymbol{r}_4, \ldots$ approximates the curve $\mathcal{C}$ more and more exactly. It is useful to introduce new variable $\theta \stackrel{\mathrm{df}}{=} K\alpha$ which can be treated as continuous since α is arbitrary. Using this we can rewrite Eq. (3.13) as:

$$\boldsymbol{\sigma}\cdot\boldsymbol{r}\,(\theta) = \exp\left(i\theta\boldsymbol{\sigma}\cdot\boldsymbol{q}\right)\exp\left(i\lambda\theta\ \boldsymbol{\sigma}\cdot\boldsymbol{p}\right)\ \boldsymbol{\sigma}\cdot\boldsymbol{r}_0\ \exp\left(-i\lambda\theta\boldsymbol{\sigma}\cdot\boldsymbol{p}\right)\exp\left(-i\theta\boldsymbol{\sigma}\cdot\boldsymbol{q}\right), \tag{3.15}$$

while Eq. (3.14) leads to explicit formula for the curve $\mathcal{C}$:

$$\boldsymbol{r}\,(\theta) = \cos(\theta)\,\boldsymbol{t}\,(\theta) + \sin(\theta)\,\boldsymbol{t}\,(\theta)\times\boldsymbol{q} + (1-\cos(\theta))\,(\boldsymbol{q}\cdot\boldsymbol{t}\,(\theta))\,\boldsymbol{q}\,, \tag{3.16a}$$

$$\boldsymbol{t}\,(\theta) = \cos(\lambda\theta)\,\boldsymbol{r}_0 + \sin(\lambda\theta)\,\boldsymbol{r}_0\times\boldsymbol{p} + (1-\cos(\lambda\theta))\,(\boldsymbol{p}\cdot\boldsymbol{r}_0)\,\boldsymbol{p}\,. \tag{3.16b}$$

3.3. Symmetry and restrictions of dynamics

Dynamical system (3.7) has continuous symmetry:

$$R_N \to Q^{\kappa} R_N Q^{-\kappa}, \qquad \forall\kappa\in\mathbb{R}. \tag{3.17}$$

This symmetry is equivalent to rotation of $\boldsymbol{r}_N$ around $\boldsymbol{q}$ about an arbitrary angle $\kappa\alpha$. It can be thus expected that dynamics of the quantity $\boldsymbol{r}_N\cdot\boldsymbol{q}$ should decouple from other degrees of freedom in (3.7) [2, 4]. Indeed, it follows from (3.14a) that

$$\boldsymbol{r}\,(\theta)\cdot\boldsymbol{q} = \boldsymbol{t}\,(\theta)\cdot\boldsymbol{q}\,. \tag{3.18}$$

Since $|\boldsymbol{t}\,(\theta)| = |\boldsymbol{q}| = 1$ it follows from the Schwartz inequality that $-1 \le \boldsymbol{t}\,(\theta)\cdot\boldsymbol{q} \le 1$. Now, for given $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}_0$ we have

$$A_1 \le \boldsymbol{t}\,(\theta)\cdot\boldsymbol{q} \le A_2\,, \qquad -1 \le A_1\,, \quad A_2 \le 1\,. \tag{3.19}$$

The constants $A_{1,2}$ depending on the parameters $\boldsymbol{p}$, $\boldsymbol{q}$ and the initial condition $\boldsymbol{r}_0$ can be computed from (3.14b) by elementary means

$$A_{1,2} = c \mp \sqrt{b^2 + (a-c)^2}, \tag{3.20a}$$

where

$$a = \boldsymbol{r}_0\cdot\boldsymbol{q}\,, \quad b = (\boldsymbol{r}_0\times\ \boldsymbol{p})\cdot\boldsymbol{q}\,, \quad c = (\boldsymbol{p}\cdot\boldsymbol{r}_0)\,(\boldsymbol{p}\cdot\boldsymbol{q})\,. \tag{3.20b}$$

It thus follows that the motion on the sphere $|\boldsymbol{r}\,(\theta)| = 1$ is bounded by two parallels: $A_1 \le \boldsymbol{r}\,(\theta)\cdot\boldsymbol{q} \le A_2$.

4. Differential equation

The map (3.7), as follows from the solution shown in Eq. (3.14), is the stroboscopic map with strobe time α of a differential equation which will be deduced from the solution (3.15) where θ is a continuous variable (it can be derived from the form (3.16) as well). Differentiating Eq. (3.15) with respect to θ and using (3.8) we get (see Exercise 41.3 in Ref. [22] for similar computations)

$$\frac{d\boldsymbol{\sigma}\cdot\boldsymbol{r}\left(\theta\right)}{d\theta}=i\left[\boldsymbol{\sigma}\cdot\boldsymbol{u}\left(\theta\right),\quad \boldsymbol{\sigma}\cdot\boldsymbol{r}\left(\theta\right)\right], \tag{4.21}$$

where $[A,\ B]\stackrel{\mathrm{df}}{=}AB-BA$ and

$$\boldsymbol{u}\left(\theta\right)=\boldsymbol{q}+\lambda\boldsymbol{p}\left(\theta\right),\quad \lambda=\tfrac{\beta}{2\alpha}, \tag{4.22a}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\left(\theta\right)=\exp\left(i\theta\boldsymbol{\sigma}\cdot\boldsymbol{q}\right)\,\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\exp\left(-i\theta\boldsymbol{\sigma}\cdot\boldsymbol{q}\right). \tag{4.22b}$$

It follows that the sequence $\boldsymbol{r}_0,\boldsymbol{r}_2,\boldsymbol{r}_4,\dots$, generated from Eq. (3.14), samples the flow of Eq. (4.21) exactly. As it was remarked in Subsection 3.2 the vectors $\boldsymbol{r}_{2K}$ and $\boldsymbol{r}_{2(K+1)}$ are very close on a unit sphere for very small α and fixed λ, *cf.* Eq. (3.14). Therefore for decreasing α, β and fixed $\lambda=\beta/\left(2\alpha\right)$ the sequence of points on the unit sphere given by $\boldsymbol{r}_0,\ \boldsymbol{r}_2,\ \boldsymbol{r}_4,\dots$ can be arbitrarily dense.

Equations (4.21), (4.22) can be written in explicit form. Using Eqs. (3.11), (3.12) we get

$$\boldsymbol{p}\left(\theta\right)=\cos\left(2\theta\right)\boldsymbol{p}+\sin\left(2\theta\right)\boldsymbol{p}\times\boldsymbol{q}+\left(1-\cos\left(2\theta\right)\right)\left(\boldsymbol{q}\cdot\boldsymbol{p}\right)\boldsymbol{q}, \tag{4.23}$$

and hence

$$\frac{d\boldsymbol{r}\left(\theta\right)}{d\theta}=2\boldsymbol{r}\left(\theta\right)\times\boldsymbol{u}\left(\theta\right), \tag{4.24}$$

with $\boldsymbol{u}\left(\theta\right)$ given by Eqs. (4.22a), (4.23). Obviously the length of the vector $\boldsymbol{r}$ is a conserved quantity, and we shall put $\left|\boldsymbol{r}\left(\theta\right)\right|=1$. The angle θ can be treated as increasing with angular velocity $\omega=$ const., $\dfrac{d\theta}{dt}=\omega$ [22], and time variable can be introduced to obtain finally

$$\frac{d\boldsymbol{r}\left(t\right)}{dt}=2\boldsymbol{r}\left(t\right)\times\boldsymbol{u}\left(t\right), \tag{4.25a}$$

$$\boldsymbol{u}\left(t\right)=\omega\left\{\left[1+\lambda-\lambda\cos\left(2\omega t\right)\left(\boldsymbol{q}\cdot\boldsymbol{p}\right)\right]\boldsymbol{q}+\lambda\cos\left(2\omega t\right)\boldsymbol{p}+\lambda\sin\left(2\omega t\right)\boldsymbol{p}\times\boldsymbol{q}\right\}. \tag{4.25b}$$

Let us note that Eq. (4.25) is the Bloch equation [16]. Equation (4.25) has two invariants: $\left|\boldsymbol{r}\left(t\right)\right|=$ const., $\boldsymbol{u}\left(t\right)\cdot\frac{d\boldsymbol{r}(t)}{dt}=0$ which follow from the structure of (4.25a).

5. Computational results

We have performed several computations for the Bloch equation (4.25) and the discrete-time dynamical system (3.7), parameterized as described in Section 3.2, to show dynamics of the Bloch equation and to demonstrate how the map (3.7) samples the flow of Eq. (4.25). Exact solutions of the map (3.7) as well as of the Bloch equation (4.25) are given by (3.14) and (3.16), respectively.

Since ω determines time scale only we put $\omega = 1$. In all computations described below vectors $\boldsymbol{p}$, $\boldsymbol{q}$ are orthogonal, $\boldsymbol{p} = [1,\ 0,\ 0]$, $\boldsymbol{q} = [0,\ 0,\ 1]$ and the initial condition is $\boldsymbol{r}_0 = [0.6,\ 0,\ 0.8]$. Motion on the sphere is bounded by parallels $A_{1,2} = \mp \max |\boldsymbol{r}(\theta) \cdot \boldsymbol{q}|$ given by (3.20). For the present choice of $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}_0$ we have $A_{1,2} = \mp 0.8$. In all figures below the vector $\boldsymbol{q}$, parallels $A_{1,2}$ and the equator are plotted, where thin dashed lines indicate points invisible for the observer.

The solution (3.14), of discrete-time dynamical system (3.7) with Q, P given by (3.9) has been plotted in Fig. 1 for $\alpha = 0.01$, $\beta = 0.06$ where angles α, β have been given in degrees and $\lambda = \beta/(2\alpha) = 3$. The value of α is so small that points $\boldsymbol{r}_0,\ \boldsymbol{r}_2,\ \boldsymbol{r}_4,\ \ldots$ lie so close one to another that a seemingly continuous curve, sampling the flow of the Bloch equation (4.25) very densely, has been obtained. The dynamics has been also generated directly from (3.7) with $R_1 = QPR_0^{-1}Q^{-1}$ (the angle $\chi_0 \neq 0$ and arbitrary otherwise) to the same effect, the unit vectors $\boldsymbol{r}_N$ have been renormalized after each iteration to avoid numerical instabilities. The whole trajectory has three-fold symmetry with respect to the $\boldsymbol{q}$ axis and is bounded by two parallels $A_{1,2} = \mp 0.8$.

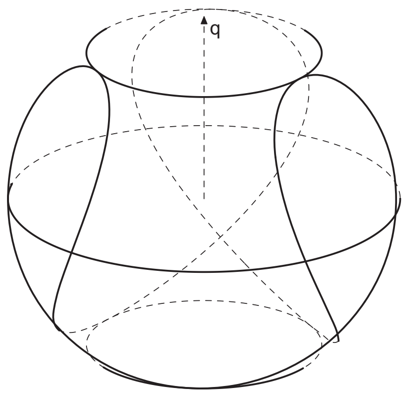

Fig. 1. The Bloch equation (4.25) and discrete-time dynamical system (3.7), $\alpha = 0.01$, $\beta = 0.06$, $\lambda = 3$.

In Fig. 2 dynamics of vectors $\boldsymbol{r}_N$ obtained from equation (3.14) has been plotted for $\alpha = 2$, $\beta = 8$ ($\lambda = 2$). We thus obtain forty five points marked with dots. The solution (3.14) has been also plotted for $\alpha = 0.01$, $\beta = 0.04$ ($\lambda = 2$). This set of points samples the Bloch equation densely. The seemingly continuous curve so obtained has two-fold symmetry with respect to the $\boldsymbol{q}$ axis.

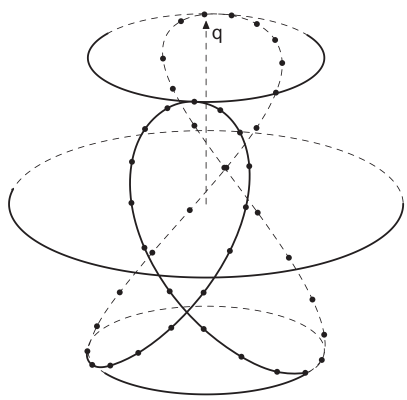

Fig. 2. The Bloch equation (4.25) and discrete-time dynamical system (3.7), $\alpha = 2$, $\beta = 8$ (dots) and $\alpha = 0.01$, $\beta = 0.04$ (solid and dashed lines), $\lambda = 2$.

In Fig. 3 initial stage of dynamics of vectors $\boldsymbol{r}_N$ (3.14) has been plotted for $\alpha = 0.01$, $\beta = 0.0205$ ($\lambda = 1.025$), dot on the upper parallel marks the initial vector $\boldsymbol{r}_0$. Had the value of λ be equal to one exactly the trajectory

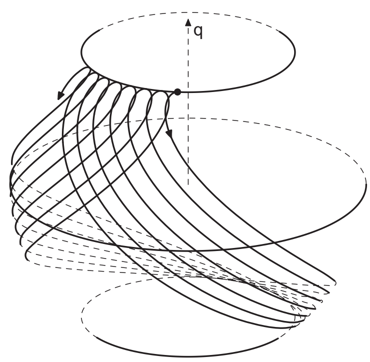

Fig. 3. The Bloch equation (4.25) and discrete-time dynamical system (3.7), $\alpha = 0.01$, $\beta = 0.0205$, $\lambda = 1.025$.

would consist of one closed loop. Since $\lambda = 1.025 = 41/40$ the whole closed curve consists of forty one loops (in Fig. 3 seven such loops have been shown) and samples densely the Bloch equation. In the case of close but irrational value of λ, *e.g.* $\lambda = \sqrt{2} - 0.389 = 1.0252\ldots$, the trajectory is ergodic on the whole spherical sector bounded by two parallels $A_{1,2} = \mp 0.8$.

6. Summary and discussion

We have introduced in Section 3 a class of discrete-time invertible maps (3.1) on an arbitrary group $\mathcal{G}$ and the exact solution (3.4) of this map has been found. Maps of form (3.1), parameterized on a Lie group, generate points in the parameter space which sample a trajectory in this space. This curve can be generated forward as well as backward from a given initial condition. This suggests that the group action (3.1) may correspond to a flow of a differential equation. The results described above generalize significantly our earlier results [17].

In Section 3.2 a simplified dynamics (3.7) has been parameterized on the unit sphere, *i.e.* for $\mathcal{G} = \mathrm{SU}(2)$, and it has been shown directly by constructing the solution (3.8a), (3.14) and (3.16) that the maps samples the flow of the Bloch equation (4.25) exactly and arbitrarily densely. It can be thus stated that the Bloch equation (4.25) is a unique suspension of the map (3.7).

In the special case $\boldsymbol{p}\cdot\boldsymbol{q} = \boldsymbol{0}$ we recover the solution obtained by H.K. Kim and S.P. Kim [23], see also Kobayashi papers where several methods to solve the Bloch equation were described earlier [24–26]. It seems that the present formulation leads to some progress in understanding the Bloch equation. The role of the parameter λ has been elucidated in Section 3.2. More exactly, it follows from the condition (3.10) that for fixed rational value of parameter $\lambda = \beta/(2\alpha)$ the flow of Eq. (4.25) generates closed trajectories consisting of a finite set of points. For $\alpha, \beta \to 0$ and fixed value of λ points obtained according to (3.14) sample trajectories of (4.25) exactly and arbitrarily densely. Moreover, it has been shown in Section 3.3 that dynamical system (3.7) has rotational symmetry with respect to the $\boldsymbol{q}$ axis from which dynamical restrictions for the quantity $\boldsymbol{r}(\theta)\cdot\boldsymbol{q}$ have been derived, see (3.20). It follows from (3.16) that for $\lambda = m/n$, with m, n relatively prime, the curve $\boldsymbol{r}(\theta)$ has m — fold symmetry with respect to the $\boldsymbol{q}$ axis. All these results have been visualized in Section 5 where computational results have been presented. The figures have been produced by code written in the MetaPost picture-drawing language [27].

The results described in the present paper can be generalized in several directions. First of all it should be determined when the general group dynamical system (3.1) samples a continuous curve. Whenever this is the case it should be possible to construct from a solution (3.4), computed for some Q_N's, an ODE for which the group action is a stroboscopic map. Of course, uniqueness of such construction should be investigated. A very simple choice of Q_N's is $Q_{2K} = S$, $Q_{2K+1} = T$, $K = 0, 1, \dots$. We found in our early computations that for $ST \neq TS$ dynamics of Eq. (3.1) was very complicated [28], yet there is a closed form solution (3.4). Finally, group actions on other groups can be considered.

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