Vol. 40 (2009) ACTA PHYSICA POLONICA B No 6

EVOLUTION EQUATIONS OF THE TRUNCATED MOMENTS OF THE PARTON DENSITIES. A POSSIBLE APPLICATION

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(Received March 20, 2009)

A possible application of the evolution equation for the truncated Mellin moments to determination of the parton distributions in the nucleon is presented. We find that the reconstruction of the initial parton densities at scale Q_0^2 from their truncated moments at a given scale Q^2 is exact and unique for small number of free parameters (≤ 3), even for the limited x-region of experimental data. For larger number of adjustable parameters the obtained fits are not unique and one needs an additional knowledge of the small-x behaviour of the parton densities to make the reconstruction procedure reliable. We apply successfully our method to HERMES and COMPASS spin-dependent valence quark data.

PACS numbers: 12.38.Bx

1. Introduction

Determination of unpolarised as well polarised parton distribution functions (PDFs) is nowadays a topic of intensive theoretical and experimental investigations. At the dawn of the LHC, where high center-of-mass energy $\sqrt{(s)} = 14\,\text{TeV}$ is available and all large Q^2 reactions are parton collisions, the precise knowledge of the PDFs is needed. Despite recent progress in experimental measurements and theoretical perturbative QCD analyses, our present knowledge of the partonic spin structure of the nucleon is still incomplete. According to the conservation law of the nucleon angular momentum, the nucleon spin is distributed among quarks and gluons. The spin of the partonic constituents as well as their orbital angular momentum contribute

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to the total nucleon spin of 1/2. Recent experiments on polarised deep inelastic lepton–nucleon scattering imply that quarks and anti-quarks carry only a small part of the proton's spin (about 30%) — less than half the prediction of relativistic quark model (about 75%). This result has stimulated theoretical activity to understand the proton spin structure. A possible explanation for the discrepancy may lie in a large gluon contribution, specially from the small-Bjorken x-region. The present RHIC data cannot distinguish between different (positive, negative and sign-changing) forms of the gluon distributions $\Delta G(x)$ and therefore one cannot definitely determine the quark and gluon contribution to the nucleon spin. Hence a primary goal of the spin program is to determine the gluon polarisation ΔG.

A knowledge of the low-x behaviour of the unpolarised or polarised nucleon structure functions enables the estimation of their Mellin moments and hence the sum rules. This is particularly important in view of the insufficient small-x experimental data. Theoretical perturbative QCD analysis is usually based on the evolution equations satisfied by the parton densities. Interactions between quarks and gluons violate the Bjorken scaling [1] and the parton distribution functions change with Q^2 according to the well-known Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equation [2–5]. The DGLAP equations can be solved with use of either the Mellin transform or the polynomial expansion in the x-space. This approach requires a knowledge of the initial parton densities at low-Q^2 scale for the wide range of x-values. Input parametrisations are fitted to the available experimental data. Standard DGLAP approach operates on the parton densities q. Hence their moments, which are *e.g.* the contributions to the proton's spin can be obtained by the integration of q over x. In this paper we propose an alternative approach, in which the main role is played by truncated moments of the quark and gluon distribution functions. The evolution equation for the nth truncated at x_0 moment $\int_{x_0}^{1} dx\, x^{n-1}\, q(x,Q^2)$ has the same form as that for the parton density itself with the modified splitting function $P'_{ij}(n,x) = x^n P_{ij}(x)$ [6]. The truncated moments approach refers directly to the physical values-moments (rather than to the parton densities), what enables one to use a wide range of deep-inelastic scattering data in terms of smaller number of parameters. In this way, no assumptions on the shape of parton distributions are needed. Using the evolution equations for the truncated moments one can also avoid uncertainties from the unmeasurable very small $x \to 0$ region. This approach allows for direct study of the behaviour of the truncated moments and can also be applied for determination of the parton densities. The latter is a topic of our paper. Determination of the quark and gluon distributions from their truncated moments can be done very simply via differentiation of the moment with respect to the point of truncation x_0. For practical purposes one can also reconstruct the parameters in PDF parametrisation using Marquardt's procedure. We describe this problem in detail and successfully apply our method to recent HERMES and COMPASS spin-dependent valence quark data.

The content of this paper is as follows. In the next section we recall the evolution equations for the truncated moments. Next, we present a generalisation of the obtained equations for the double truncated moments $\int_{x_{\rm min}}^{x_{\rm max}} dx\, x^{n-1}\, q(x,Q^2)$. Section 3 contains details for determination of the parton densities from their truncated moments. Our approach is presented with help of a general example for different input parametrisations of PDFs. In Section 4 we reconstruct the valence quark densities from recent HERMES and COMPASS data for their truncated first moments. In this way we can compare our predictions for the reconstructed parton densities with the PDFs fit and test the accuracy of our method. A summary and conclusions are given in Section 5.

2. Evolution equations for truncated moments

Standard PQCD approach is based on DGLAP evolution equations for parton densities [2–5]. The DGLAP equations can be solved with use of either the Mellin transform or the polynomial expansion in the x-space. The differentio-integral Volterra-like evolution equations change after the Mellin transform into simple differential and diagonal ones in the moment space and can be solved analytically. Then one can again obtain the x-space solutions via the inverse Mellin transform. The only problem is knowledge of the initial moments — integrals over the whole region $0 \leq x \leq 1$. The lowest limit $x \to 0$, which implies that the invariant energy $W^2 = Q^2(1/x-1)$ of the inelastic lepton–hadron scattering becomes infinite, will never be attained in experiments. Therefore, it is very useful to study the parton distributions only in a limited range of the Bjorken variable and hence their moments truncated at low-x_0.

In [6] we have derived the evolution equations for the truncated Mellin moments of the parton densities. We have found that the truncated moments obey the DGLAP-like evolution equation

$$\frac{d\bar{q}_n(x_0,Q^2)}{d\ln Q^2} = \frac{\alpha_{\rm s}(Q^2)}{2\pi}\,(P'\otimes\bar{q}_n)(x_0,Q^2)\,, \tag{2.1}$$

where

$$t = \ln\left(\frac{Q^2}{\Lambda^2}\right)\,, \tag{2.2}$$

$q(x,Q^2)$ is the parton distribution function and $\bar{q}_n(x_0,Q^2)$ denotes its nth moment truncated at x_0:

$$\bar{q}_n(x_0,Q^2)=\int\limits_{x_0}^{1} dx\, x^{n-1}\, q(x,Q^2)\,. \tag{2.3}$$

A role of the splitting function plays $P'(n,z)$:

$$P'(n,z)=z^n\,P(z) \tag{2.4}$$

and $\otimes$ abbreviates a Mellin convolution over x

$$\left(P\otimes f\right)(x,Q^2)=\int\limits_{x}^{1}\frac{dy}{y}\,P\left(\frac{x}{y}\right)\,f(y,Q^2)\,. \tag{2.5}$$

It can be shown, that the double truncated moments

$$\bar{q}_n(x_{\rm min},x_{\rm max},Q^2)=\int\limits_{x_{\rm min}}^{x_{\rm max}} dx\, x^{n-1}\, q(x,Q^2) \tag{2.6}$$

also fulfill the DGLAP-type evolution, namely

$$\frac{d\bar{q}_n(x_{\rm min},x_{\rm max},Q^2)}{d\ln Q^2} = \frac{\alpha_{\rm s}(Q^2)}{2\pi} \times\int\limits_{x_{\rm min}}^{1}\frac{dz}{z}P'(n,z)\bar{q}_n\left(\frac{x_{\rm min}}{z},\frac{x_{\rm max}}{z},Q^2\right). \tag{2.7}$$

We would like to emphasize that although the central role in the QCD studies play PDFs, dealing with their truncated moments can be also useful. This approach has the following major characteristics:
— refers directly to the physical values-moments (not to the parton densities), what enables one to use a wide range of experimental data in terms of smaller number of parameters. In this way, no assumptions on the shape of parton distributions are needed;
— allows one to study directly the evolution of moments and the scaling violation;
— one can avoid uncertainties from the unmeasurable very small $x\to 0$ and high $x\to 1$ region;
— the suitable evolution equations are exact and diagonal (there is no mixing between moments of different orders);
— can be used for different approximations: LO, NLO, NNLO *etc.* and in the polarised as well as unpolarised case.

Concluding, evolution equations for the truncated Mellin moments of the parton densities (2.1), (2.7) can be an additional useful tool in the QCD analysis of the nucleon structure functions. In the next section we examine one of the possible application, which is a reconstruction of the parton distributions.

3. The application of the evolution equation for truncated moments

The evolution equation (2.1) enables one to study the behaviour of the truncated moments (2.3) within different approximations (LO, NLO, NNLO *etc.*) and can be solved with use of standard methods of solving the DGLAP equations. In this way one can study the evolution of the truncated moments without making any assumption on the small-x behaviour of the parton densities themselves. One needs to know only the truncated moments of the parton distributions at the initial scale Q_0^2 (*e.g.* from the experimental data), what constrains a number of the input parameters. The solutions for truncated moments can be used in the determination of the parton distribution functions via differentiation

$$q(x,Q^2) = -x^{1-n}\,\frac{\partial\bar{q}_n(x,Q^2)}{\partial x}\,, \tag{3.1}$$

which results from (2.3). In order to reconstruct the parton densities from their truncated moments, we proceed the following steps:

1. Preparing available experimental data for moments $\bar{q}_n(x_0,Q_1^2)$ as a function of $x_{\min}\leq x_0\leq 1$ at the same scale Q_1^2.

2. Interpolation of the given data points into points which are Chebyshev nodes. This allows us to use the Chebyshev polynomials technique for solving the evolution equations.

3. Evolution of the truncated moments from Q_1^2 to Q_2^2 according to (2.1) for different $x_{\min}\leq x_0\leq 1$.

4. Reconstruction of the parton density $q(x,Q_2^2)$ from its truncated moment at the same scale Q_2^2 applying Marquardt procedure to fit free parameters.

Since numerical integration is more stable than numerical differentiation, we perform the final fitting (in step 4.) with respect to moments and not to their derivatives.

Let us test the above procedure on the nonsinglet function parametrised in a general form:

$$q(x, Q_0^2) = N(\alpha, \beta, \gamma)\, x^{\alpha}(1-x)^{\beta}(1+\gamma x)\,. \tag{3.2}$$

Now we create 'experimental data': for input parametrisation (3.2) we calculate truncated moments (2.3) at Q_0^2 and evolve them to Q^2. Obtained results $\bar{q}_n(x_0, Q^2)$ are our starting point in the above described 4-steps procedure. Finally we can compare the reconstructed parton densities with the assumed form (3.2).

In our test we construct simulated data sets, what may seem to be a "toy" proceeding. This approach has however a major advantage: knowing the result "in advance", we are able test the accuracy of the reconstruction of the parton densities. The presented evolution equations for the truncated moments will be really useful in such QCD analyses, where we know the moments (*e.g.* from direct measurements), while the PDFs are poorly known.

The results are presented in Fig. 1. We performed our test using three different input parametrisations $q(x, Q_0^2)$: one nonsingular ($\sim$ const.) and two singular $\sim x^{-0.4}$ and $\sim x^{-0.8}$ as $x \to 0$. One can see that our reconstruction is satisfactory independently on the input parametrisation. The agreement is very good even for the limited region of the data $x_0 \geq 0.01$.

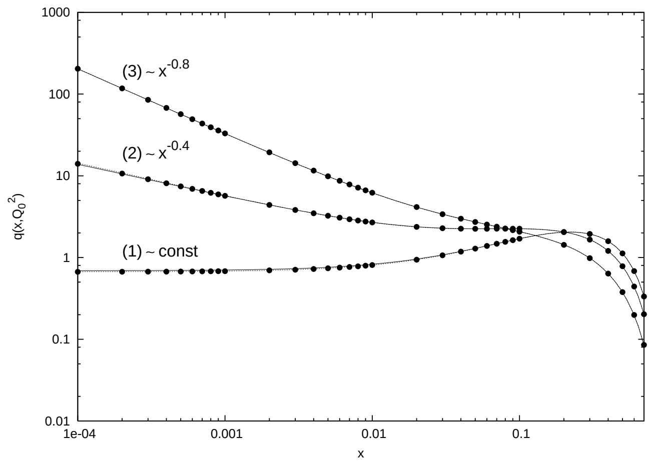

Fig. 1. Reconstruction of the initial parton density $q(x, Q_0^2)$ from its first truncated moment after back Q^2 evolution from $Q^2 = 10$ GeV^2 to $Q_0^2 = 1$ GeV^2 (solid). Comparison with the origin function $q(x, Q_0^2)$ (points). The dotted line (mostly overlapped by the solid one) represents the reconstruction from the limited x-region of available data $x_0 \geq 0.01$. Results are shown for 3 different small-x behaviour of $q(x, Q_0^2)$: (1)($\sim$ const.), (2)$\sim x^{-0.4}$ and (3)$\sim x^{-0.8}$.

It must be however emphasized that success of the determination of the parton densities from their truncated moments depends on the number of the fitted parameters and also on the available data. A lack of data for the very small-x is a constraint in the reliable determination of $q(x, Q^2)$ in this region. More free parameters need more data points and too many free parameters make a unique fit of the data not possible. In our test we have fitted three parameters (α, β and γ), obtaining almost the same values as those assumed in (3.2). In the next section we shall reconstruct the valence quark distributions from recent HERMES and COMPASS data. This needs to fit more parameters (minimum three for u-quarks and three for d-quarks) and will be another test of the accuracy of our method.

4. Determination of the spin-dependent valence quark densities from HERMES and COMPASS data for truncated first moments

In the previous section we have presented the general idea of the determination of the parton densities from their truncated moments. Now, we will try to reconstruct the polarised valence quark densities from recent HERMES [8] and COMPASS [9] data. These data (moments) have not been obtained in direct measurements but with use of a given PDF fit. Nevertheless, we can compare our predictions for reconstructed parton densities with a given fit (input parametrisation) and in this way test our method for a larger number of fitted parameters. The results are shown in Figs. 2–5.

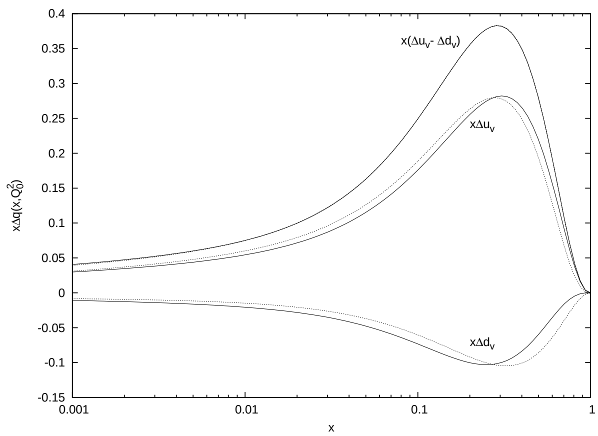

Fig. 2. Initial spin-dependent valence quark distributions $x(\Delta u_v - \Delta d_v)$, $x\Delta u_v$ and $x\Delta d_v$ at $Q_0^2 = 4$ GeV^2: dotted — reconstructed from HERMES data for the first truncated moment of the nonsinglet polarised function $g_1^{\rm NS}$ at $Q^2 = 5$ GeV^2 [8], solid — original BB fit [10]. Plots for $x(\Delta u_v - \Delta d_v)$ overlap each other.

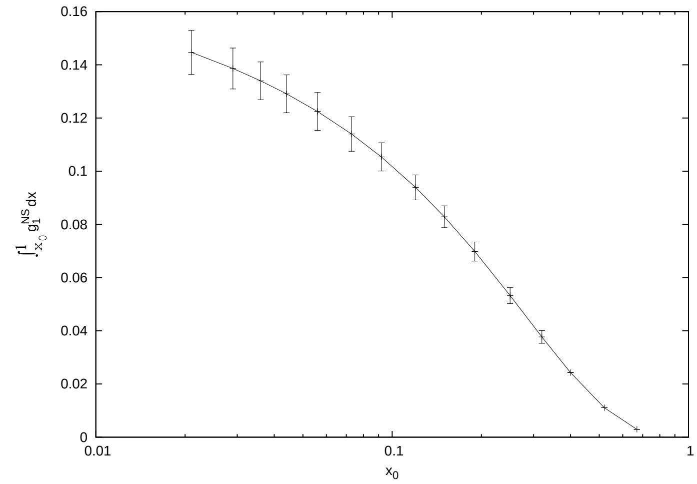

Fig. 3. The first truncated moment of the nonsinglet polarised structure function g_1^{NS} *versus* the truncation point x_0, calculated from the reconstructed fit (solid). $Q^2 = 5$ GeV^2. Comparison with HERMES data [8] based on BB fit [10] (points with error bars).

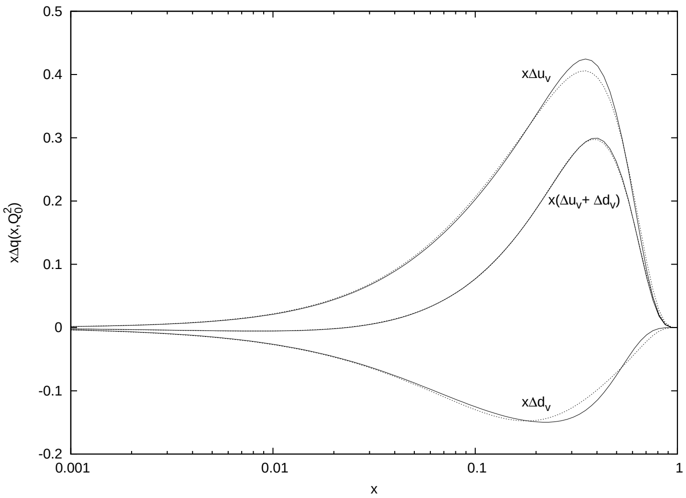

Fig. 4. Initial spin-dependent valence quark distributions $x(\Delta u_v + \Delta d_v)$, $x\Delta u_v$ and $x\Delta d_v$ at $Q_0^2 = 0.5$ GeV^2: dotted — reconstructed from COMPASS data for the first truncated moment of the function $\Delta u_v + \Delta d_v$ at $Q^2 = 10$ GeV^2 [9], solid — original DNS fit [11]. Plots for $x(\Delta u_v + \Delta d_v)$ overlap each other.

In Fig. 2 we plot the initial spin-dependent valence quark distributions $x\Delta u_v(x,Q_0^2)$ and $x\Delta d_v(x,Q_0^2)$, reconstructed from the HERMES data [8] for the first truncated moment of the nonsinglet polarised function

$$g_1^{\rm NS} = \tfrac{1}{6}(\Delta u_v - \Delta d_v)\,. \tag{4.1}$$

We compare our results to the original Blümlein–Böttcher (BB) fit [10]. Fig. 3 shows x_0 dependence of the first truncated moment

$$\bar{g}_1^{\rm NS}(x_0,Q^2) = \int\limits_{x_0}^{1} g_1^{\rm NS}(x,Q^2)dx \tag{4.2}$$

calculated from the reconstructed fit, compared with the experimental data. Figs. 4 and 5 contain the same analysis as Figs. 2 and 3 respectively, but for COMPASS $\Delta u_v + \Delta d_v$ data [9] with the original de Florian, Navarro, Sassot (DNS) fit [11].

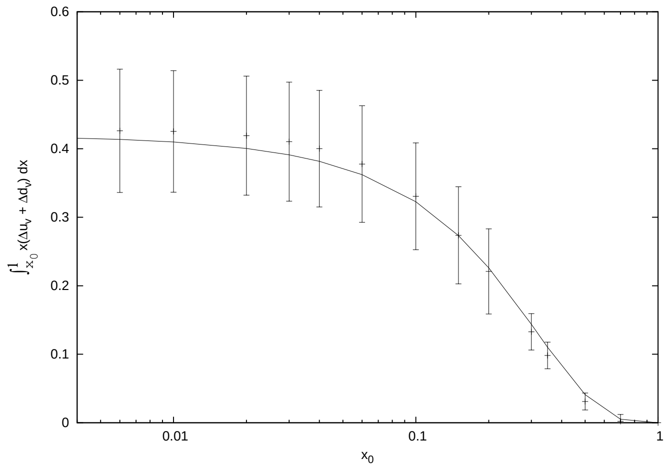

Fig. 5. The first truncated moment of the function $\Delta u_v + \Delta d_v$ *versus* the truncation point x_0, calculated from the reconstructed fit (solid). $Q^2 = 10$ GeV^2. Comparison with COMPASS data [9] based on DNS fit [11] (points with error bars).

One can see a satisfactory agreement between the reconstructed fits and experimental data. Reconstructed combined functions $x(\Delta u_v - \Delta d_v)$ and $x(\Delta u_v + \Delta d_v)$ overlap HERMES and COMPASS results, respectively. For the extracted valence quark densities alone the agreement is worse but still acceptable. We have found, however, that these fits are not unique and equally good agreement with the data can be obtained with use of the other (not only BB and DNS, respectively) sets of free parameters. When the number of adjustable parameters is large (> 3) and there are no experimental points from the low-x region $x < 0.001$, one cannot distinguish which fit is the best one. Only an additional constraint for small-x behaviour of the parton densities makes the fit procedure more reliable. In our test we found the best HERMES fit taking into account also the small-x ($x < 0.01$) $g_1^{\rm NS}$ data. Concluding, even for the large number of adjustable parameters (6 for HERMES and 8 for COMPASS data), the presented method of reconstruction can be a hopeful tool for determining parton densities from experimental results for their truncated moments.

5. Summary

We have shown how to determine the parton densities from their first truncated at x_0 Mellin moments. The method is based on the recently derived evolution equations for nth truncated moments $\bar{q}_n(x_0, Q^2)$. After evolution of the $\bar{q}_1$ results from a given scale Q^2 to the initial one Q_0^2, one can reconstruct the input quark and gluon distributions. In our analyses we have used different simulated data sets in order to test the accuracy of the reconstruction. In our first test we have fitted three parameters from the general form of the nonsinglet input function, obtaining a very good agreement with assumed original parametrisations. We have found, that for small number of the fitted parameters (≤ 3), the reconstruction is satisfactory independently on the small-x behaviour of the assumed input and even for the limited region of the data $x_0 \geq 0.01$. Next we have applied this technique to the experimental HERMES and COMPASS data for the polarised valence quarks. In this way, we have tried to reconstruct the original fits with larger number of free parameters (6 and 8, respectively). The results of the reconstruction have been again satisfactory. We have found however, that for larger number of adjustable parameters the obtained fits are not unique. We would like to emphasize that success of the determination of the parton densities from their truncated moments depends on the number of the fitted parameters and also on x-region of the available experimental data. An additional knowledge of the small-x behaviour of the parton densities, based either on the experimental data or the theoretical expectations, can make the fit procedure more reliable. Indeed, we have found the best HERMES fit taking into account also the small-x $g_1^{\rm NS}$ data.

Concluding, the presented evolution equations for the truncated moments allowing for a direct study of the Q^2-dependence of the moments, offer a new, promising tool towards improving our knowledge of the unpolarised and polarised partonic structure of the nucleon.

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