Vol. 40 (2009) *ACTA PHYSICA POLONICA B* No 7

TRAVELING WAVES AND IMPACT-PARAMETER CORRELATIONS IN HIGH ENERGY QCD*

S. Munier

Centre de Physique Théorique, École Polytechnique, CNRS, Palaiseau, France

(Received April 27, 2009)

We report on a numerical check of one of the main assumptions that underly the recent conjecture that high-energy scattering may be a reaction-diffusion like process in the universality class of the FKPP equation, namely the fact that the QCD evolution is local in impact-parameter space.

PACS numbers: 13.60.Hb, 12.38.–t, 02.50.Ey, 05.10.–a

1. Introduction

Over the last few years, it has been argued [1,2] that high-energy scattering in QCD at fixed impact parameter is a peculiar reaction-diffusion process (For a review, see Ref. [3]). The evolution equation of the scattering amplitudes with the energy of the reaction in this regime has been conjectured to be in the universality class of the stochastic Fisher–Kolmogorov–Petrovsky–Piscounov equation (sFKPP). The latter is a stochastic nonlinear partial differential equation with one "space" variable and one "time" variable, which can be mapped to the logarithm of the relevant transverse momentum (or transverse size) scale and the rapidity, respectively.

One of the important points on which the conjecture relies is that the QCD evolution is local in impact parameter space in such a way that evolution at each impact parameter may indeed be described by a one-dimensional equation like the sFKPP equation. There are general arguments to support this assumption (they will be summarized below). However, these arguments are rather crude, and in particular, the effect of fluctuations is completely neglected. So far, more precise insight has not been achieved analytically. In the work of Ref. [4], on which we shall report here, we have checked numerically that this assumption is justified in a toy model which we think possesses the main characteristics of QCD.

* Presented at the Cracow Epiphany Conference on Hadron Interactions at the Dawn of the LHC, Cracow, Poland, January 5–7, 2009.

We first give the analytical arguments for the decoupling of the different impact parameters, then we introduce the toy model, and finally, we comment on our numerical results.

2. General picture and arguments for the decoupling

Let us consider the scattering of two mesons each of them made of a quark–antiquark pair (This process is a gedanken experiment that could model for example γ^*–γ^* scattering or deep inelastic e–p scattering). One may view the energy evolution of this process in the following way. The center-of-mass energy is increased by boosting one of the mesons to a higher rapidity. A fast meson appears not as a bare $q\bar{q}$ pair, but as a collection of many partons (essentially gluons). The building up of the latter may be represented in the QCD color dipole model [5]. In this model, the increase of the rapidity of the initial $q\bar{q}$ pair, that forms a color dipole, results in a probability that a gluon be emitted by this system, which in the large number of color limit, is equivalent to the splitting of the initial dipole to two new dipoles. The splitting process goes on with the new dipoles as the rapidity is increased, each of them evolving independently, until the maximum rapidity is reached. The splitting rate of a dipole whose endpoints have transverse coordinates (x_0, x_1) into two dipoles (x_0, x_2) and (x_1, x_2) as the result of a gluon emission at position x_2 reads [5]

$$\frac{dP}{d(\bar{\alpha}y)}(x_{01} \to x_{02}, x_{12}) = \frac{|x_0 - x_1|^2}{|x_0 - x_2|^2 |x_1 - x_2|^2} \frac{d^2 x_2}{2\pi}\,. \tag{1}$$

When the density of dipoles at a given impact parameter becomes so high that the scattering amplitude $T(r)$ of a dipole of size r with this system, roughly proportional to the local number of dipoles of sizes of the order of r (the proportionality factor goes like α_s^2), reaches its unitarity limit, this splitting process slows down: this stage is called *saturation*, and its precise mechanism has not been fully clarified in QCD yet.

Thanks to the features of the solutions to the sFKPP equation, it was shown in Refs. [1,2] that the scattering amplitude has the form of a traveling wave which moves towards larger values of $\log(1/r^2)$ when the rapidity is increased (see Fig. 1). For each value of the rapidity y, the position of the traveling wave is characterized by the value $1/Q_s(y)$ of r for which the amplitude reaches some predefined number, say 0.5.

Let us start with a single dipole at rest, and bring it gradually to a higher rapidity. As was just explained, during this process, the dipole may be replaced by two new dipoles, which themselves may split, and so on, eventually producing a chain of dipoles. Figure 2 pictures one realization of such a chain.

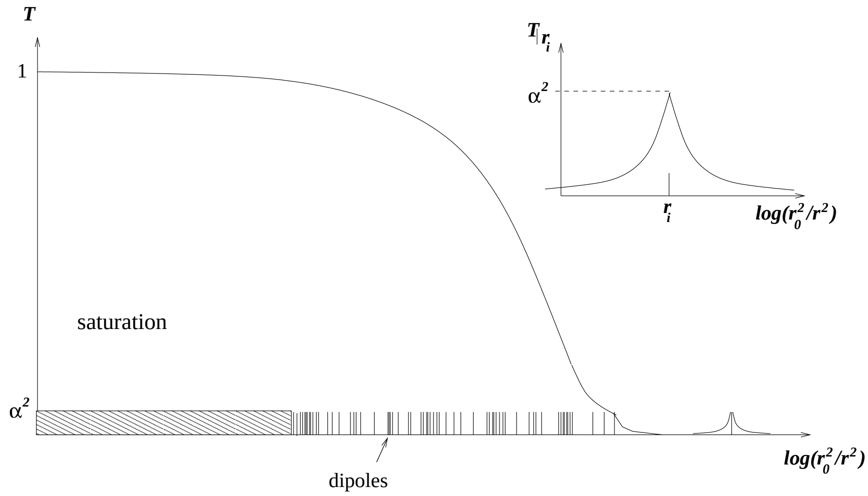

Fig. 1. [From Ref. [2]] Sketch of the scattering amplitude T of a dipole of size r off a frozen partonic configuration. The small lines on the axis denote the dipoles ordered by their logarithmic sizes. Up to fluctuations, T looks like a wave front. Inset: Scattering amplitude of a dipole of size r off a dipole of fixed size r_i as a function of $\log(1/r^2)$, for a central collision.

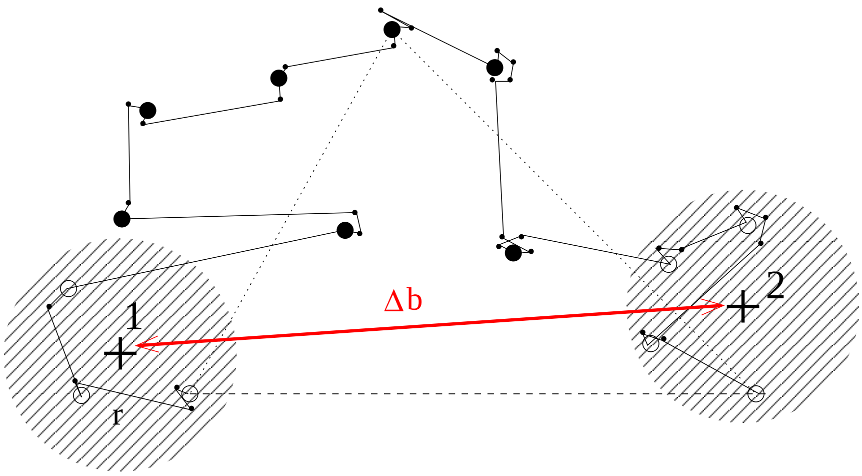

Fig. 2. [From Ref. [3]] One realization of the dipole chain obtained after QCD evolution over some rapidity interval. The initial quark and antiquark have positions x_0 and x_1, respectively. In the final chain, the points represent gluons and the straight lines that join them materialize the color dipoles. The regions 1 and 2 are supposed to decorrelate after the stage pictured in this figure.

According to Eq. (1), splittings to smaller-size dipoles are favored, and thus, one expects that the sizes of the dipoles get smaller on the average, and that in turn, the successive splittings become more local. The dipoles around region 1 and those around region 2 should have an independent evolution beyond the stage pictured in Fig. 2: further splittings will not mix in impact parameter space, and thus, the traveling waves around these regions should be uncorrelated. For a dipole in region 1 of size r to migrate to region 2, it should first split into a dipole whose size is of the order of the distance Δb between regions 1 and 2. (We assume in this discussion that the dipoles in region 2 relevant to the propagation of the local traveling waves, that is, those which are in the bulk of the wave front, also have sizes of the order of r.) Roughly speaking, the rate of such splittings may be estimated from the dipole splitting probability (1): it is of order $\bar{\alpha}(r^2/(\Delta b)^2)^2$, while the rate of splittings of the same dipole into a dipole of similar size in region 1 is of the order of $\bar{\alpha}$. Thus the first process is strongly suppressed as soon as regions 1 and 2 are more distant than a few units of r. Note that for $\Delta b \geq 1/Q_{\rm s}$, saturation may further reduce the emission of the first, large dipole, leading to an even stronger suppression of the estimated rate.

There is a second case that we should worry about. What could also happen is that some larger dipole has, by chance, one of its endpoints tuned to the vicinity of the coordinate one is looking at (at a distance which is at most $|\Delta r| \ll 1/Q_{\rm s}(y)$), and easily produces a large number of dipoles there. In this case, the position of the traveling wave at that impact parameter would suddenly jump. If such events were frequent enough, then they would modify the average wave velocity and thus the one-dimensional picture. We may give a rough estimate of the rate at which dipoles of size smaller than Δr are produced. Assuming local uniformity for the distribution of the emitting dipoles, the rate (per unit of $\bar{\alpha}y$) of such events can be written

$$\int\limits_{r_0>\Delta r} \frac{d^2 r_0}{r_0^2} \int\limits_{\varepsilon<\Delta r} d^2\varepsilon\, n(r_0) \left(\frac{\varepsilon}{r_0}\right)^2 \frac{1}{2\pi} \frac{r_0^2}{\varepsilon^2 (r_0-\varepsilon)^2}\,, \tag{2}$$

where we integrate over large dipoles of size $r_0 > \Delta r$ ($n(r_0)$ is their number density) emitting smaller dipoles (of size $\varepsilon < \Delta r$) with a probability $d^2\varepsilon\, r_0^2/(2\pi\varepsilon^2(r_0-\varepsilon)^2)$. The factor $(\varepsilon/r_0)^2$ accounts for the fact that one endpoint of the dipole of size r_0 has to be in a given region of size ε in order to emit the dipoles at the right impact parameter. To estimate this expression, we first use $n(r_0) = T(r_0)/\alpha_{\rm s}^2$ and use for T the simplified expression

$$T(r_0) = \Theta(r_0 - 1/Q_{\rm s}) \;+\; (r_0^2 Q_{\rm s}^2)^{\gamma_c}\, \Theta(1/Q_{\rm s} - r_0)$$

which splits the front into a saturated region ($r_0 > 1/Q_{\rm s}$) and a tail with geometric scaling ($r_0 < 1/Q_{\rm s}$). Using $r_0 - \varepsilon \approx r_0$ in the emission kernel, the integration is then easily performed and one finds a rate whose dominant term is

$$\frac{\pi}{2\alpha_{\rm s}^2}\frac{((\Delta r)^2 Q_{\rm s}^2)^{\gamma_c}}{1-\gamma_c}\,. \tag{3}$$

For $(\Delta r)^2 \ll (\alpha_{\rm s}^2)^{1/\gamma_c}/Q_{\rm s}^2$, that is, ahead of the bulk of the front, this term is parametrically less than 1 and is in fact of the order of the probability to find an object in this region that contributes to the normal evolution of the front [6]. Hence there is no extra contribution due to the fact that there are many dipoles around at different impact parameters.

The arguments given here are based on estimates of average numbers of dipoles, on typical configurations, and at this stage, we are not able to account analytically for the possible fluctuations. The latter often play an important role, so one should check more precisely locality of the evolution in impact parameter.

A numerical check was recently achieved in the case of a toy model that has an impact-parameter dependence in Ref. [4]. Let us briefly describe the model.

3. A model incorporating an impact-parameter dependence

3.1. Parton splittings

In order to arrive at a model that is tractable numerically, we only keep one transverse dimension instead of two in 3+1-dimensional QCD. However, we cannot consider genuine 2+1-dimensional QCD because we do not wish to give up the logarithmic collinear singularities at $x_2 = x_0$ and $x_2 = x_1$. A splitting rate which complies with our requirements is:

$$\frac{dP}{d(\bar{\alpha}y)} = \frac{1}{4}\frac{|x_{01}|}{|x_{02}||x_{12}|}dx_2\,. \tag{4}$$

We can further simplify this probability distribution by keeping only its collinear and infrared asymptotics. If $|x_{02}| \ll |x_{01}|$ (or the symmetrical case $|x_{12}| \ll |x_{01}|$), the probability reduces to $dx_2/|x_{02}|$ ($dx_2/|x_{12}|$ respectively). The result of the splitting is a small dipole (x_0, x_2) together with one close in size to the parent. So for simplicity we will just add the small dipole to the system and leave the parent unchanged. In the infrared region, a dipole of size $|x_{02}| \gg |x_{01}|$ is emitted with a rate given by the large-$|x_{02}|$ limit of the above probability. The probability laws (1), (4) imply that a second dipole of similar size should be produced while the parent dipole disappears. To retain a behaviour as close as possible to that in the collinear limit, we will instead just generate a single large dipole and maintain the parent. To do this consistently one must include a factor of two in the infrared splitting rate, so as not to modify the average rate of production of large dipoles.

In formulating our model precisely, let us focus first on the distribution of the sizes of the participating dipoles. (The simplifying assumptions made above enable one to choose the sizes and the impact parameters of the dipoles successively). We call r the modulus of the emitted dipole, r_0 the modulus of its parent and $Y = \bar{\alpha} y$. The splitting rate (4) reads in this simplified model

$$\frac{dP_{r_0 \to r}}{dY} = \theta(r - r_0)\frac{r_0 dr}{r^2} + \theta(r_0 - r)\frac{dr}{r}\,, \tag{5}$$

and the original parent dipole is kept. Logarithmic variables are the relevant ones here, so we introduce

$$\rho = \log_2 \frac{1}{r} \qquad \text{or} \qquad r = 2^{-\rho}\,. \tag{6}$$

We can thus rewrite the dipole creation rate as

$$\frac{dP_{\rho_0 \to \rho}}{dY} = \theta(\rho_0 - \rho)\, 2^{\rho - \rho_0}\, \log 2\, d\rho + \theta(\rho - \rho_0)\, \log 2\, d\rho\,. \tag{7}$$

To further simplify the model, we discretize the dipole sizes in such a way that ρ is now an integer. This amounts to restricting the dipole sizes to negative integer powers of 2. The probability that a dipole at lattice site i (*i.e.* a dipole of size 2^{-i}) creates a new dipole at lattice site j is

$$\frac{dP_{i \to j}}{dY} = \int_{\rho_j}^{\rho_{j+1}} \frac{dP_{\rho_i \to \rho}}{dY} = \begin{cases} \log 2 & j \geq i \\ 2^{j-i} & j < i \end{cases}. \tag{8}$$

The rates $dP_{i\pm}/dY$ for a dipole at lattice site i to split to any lattice site $j \geq i$ or $j < i$, respectively, are then given by

$$\begin{aligned} \frac{dP_{i+}}{dY} &= \sum_{j=i}^{i_{\max}-1} \frac{dP_{i \to j}}{dY} = \log 2 (i_{\max} - i)\,, \\ \frac{dP_{i-}}{dY} &= \sum_{j=0}^{i-1} \frac{dP_{i \to j}}{dY} = 1 - 2^{-i}\,, \end{aligned} \tag{9}$$

where we have restricted the lattice to $0 \leq i < i_{\max}$, for obvious reasons related to the numerical implementation.

Now we have to address the question of the impact parameter of the emitted dipole. In QCD, the collinear dipoles are produced near the endpoints of the parent dipoles. Let us take a parent of size r_0 at impact parameter b_0. We set the emitted dipole (size r) at the impact parameter b such that

$$b = b_0 \pm \frac{r_0 \pm r \times s}{2}\,, \tag{10}$$

where s has uniform probability between 0 and 1. It is introduced to obtain a continuous distribution of the impact parameter unaffected by the discretisation of r. This prescription is quite arbitrary in its details, but the latter do not influence significantly the physical observables. Each of the two signs that appear in the above expression is chosen to be either $+$ or $-$ with equal weights. We apply the same prescription when the emitted dipole is larger than its parent.

3.2. Scattering amplitude

We explained before that in QCD, the scattering amplitude of an elementary probe dipole of size $r_i = 2^{-i}$ with a dipole in an evolved Fock state is proportional to the number of objects which have a size of the same order of magnitude and which sit in an area of radius of order r_i around the impact point of the probe dipole. Since in our case, the sizes are discrete, the amplitude is just given, up to a factor, by the number of dipoles that are exactly in the same bin of size as the probe, namely

$$T(i,b_0) = \alpha_s^2 \times \#\{\text{dipoles of size } 2^{-i} \text{ at impact parameter } b \text{ satisfying } |b-b_0| < r_i/2\}. \quad (11)$$

3.3. Saturation

We now have to enforce unitarity, that is the condition

$$T(i,b) \leq 1 \quad (12)$$

for any i and b. This condition is expected to hold due to gluon saturation in QCD. However, saturation is not included in the original dipole model. The simplest choice is to veto splittings that would locally drive the amplitude to values larger than 1. In practice, for each splitting that gives birth to a new dipole of size i at impact parameter b, we compute $T(i,b)$ and $T(i,b \pm r_i/2)$, and throw away the produced dipole whenever one of these numbers gets larger than one.

Given the definition of the amplitude T, this saturation rule implies that there is a maximum number of objects in each bin of size and at each impact parameter, which is equal to $N_{\text{sat}} = 1/\alpha_s^2$.

4. Numerical results

We take as an initial condition a number N_{sat} of dipoles of size 1 ($i = 0$), uniformly distributed in impact parameter between $-r_0/2$ and $r_0/2$. The impact parameters b_j that are considered are, respectively, 0, 10^{-6}, 10^{-4}, 10^{-2} and 10^{-1}.

The number of events generated is typically 10^4, which allows one to measure the average and variance of the position of the traveling waves to a sufficient accuracy.

We have checked that at each impact parameter, we get traveling waves whose average position $\langle\rho_s\rangle$ grows linearly with rapidity at a velocity less than the expected mean-field velocity for this model (that is to say the velocity that would be found in the same model without fluctuations). $N_{\rm sat}$ was varied from 10 to 200.

Fig. 3 represents the correlations between the positions of the wave fronts at different impact parameters, defined as

$$\langle\rho_{\rm s}(Y,b_1)\rho_{\rm s}(Y,b_2)\rangle - \langle\rho_{\rm s}(Y,b_1)\rangle\langle\rho_{\rm s}(Y,b_2)\rangle\,. \tag{13}$$

We set $N_{\rm sat}$ to 25 in that figure, but we also repeated the analysis for different values of $N_{\rm sat}$ between 10 and 200.

In Fig. 3, we see very clearly the successive decouplings of the different impact parameters, from the most distant to the closest one, as rapidity increases. Indeed, the correlation functions flatten after some given rapidity

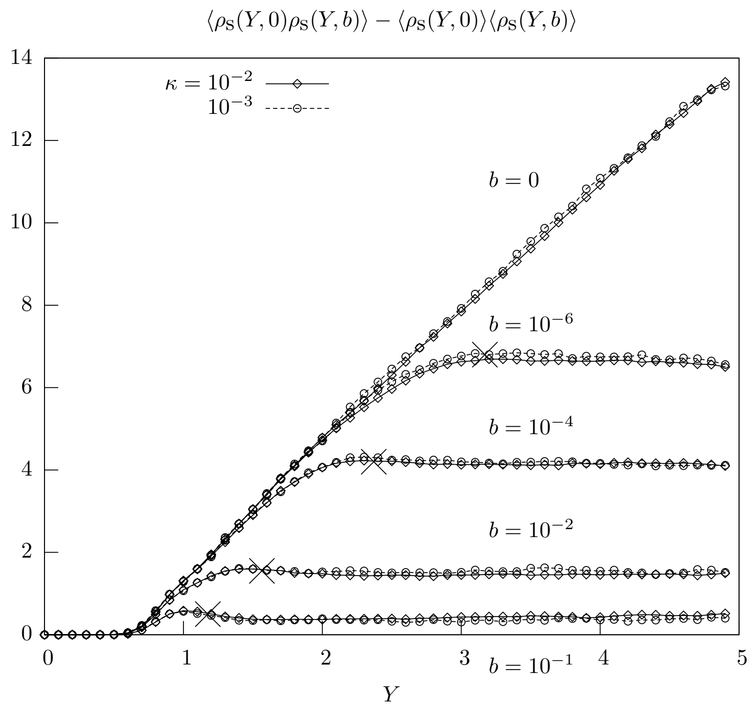

Fig. 3. [From Ref. [4]] Correlations of the positions of the traveling wave fronts.

depending on the difference in the probed impact parameters, which means that the evolutions decouple. This decoupling is expected as soon as the traveling wave front reaches dipole sizes which are smaller than the distance between the probed impact parameters, *i.e.* at Y such that $|b_2 - b_1| \approx 1/Q_{\rm s}(Y) = 2^{-\rho_{\rm s}(Y)}$. From the data for $\rho_{\rm s}(Y)$, we can estimate quantitatively the values of the rapidities at which the traveling waves decouple between the different impact parameters. (It is enough to invert the above formula for the relevant values of $b_2 - b_1$). These rapidities are denoted by a cross in Fig. 3 for the considered impact parameter differences. Our numerical results for the correlations are nicely consistent with this estimate, since the correlations start to saturate to a constant value precisely on the right of each such cross.

We conclude that the different impact parameters indeed decouple, as was expected from a naive analytical estimate. What is true for our toy model should go over to full QCD, since we have included the main features of QCD. When looking at the data more carefully however, it turns out that the model with impact parameter does not reduce exactly to a one-dimensional model of the sFKPP type. This is a point that would deserve more work. We refer the reader to Ref. [4] for all details of our numerical investigations.

REFERENCES

[1] S. Munier, R.B. Peschanski, *Phys. Rev. Lett.* **91**, 232001 (2003).

[2] E. Iancu, A.H. Mueller, S. Munier, *Phys. Lett.* **B606**, 342 (2005) [`arXiv:hep-ph/0410018`].

[3] S. Munier, *Phys. Rep.* **473**, 1 (2009).

[4] S. Munier, G.P. Salam, G. Soyez, *Phys. Rev.* **D78**, 054009 (2008).

[5] A.H. Mueller, *Nucl. Phys.* **B415**, 373 (1994).

[6] E. Brunet, B. Derrida, A.H. Mueller, S. Munier, *Phys. Rev.* **E73**, 056126 (2006).