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EXACT KINEMATICS IN GLUON CASCADE ON THE LIGHT-FRONT*

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In this talk we discuss the problem of kinematic effects in the development of the gluon cascade at high energies. The modification to the original dipole kernel for the dipole evolution at small x is proposed which accommodates important kinematical corrections. The techniques presented in this talk utilize the perturbation theory on the light-front. Using these techniques we construct recurrence relations for the wave-functions of gluons with arbitrary number of gluon components and with exact kinematics. In some special cases the recurrence relations can be solved exactly. By combining similar techniques for the fragmentation amplitudes one can derive the Parke–Taylor scattering amplitudes.

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1. Introduction

The limit of high energies is one of the most intriguing aspects of strong interactions. The hadron scattering amplitude in this regime is governed by the exchange of the Pomeron, an object with quantum numbers of the vacuum. In Quantum Chromodynamics this t-channel exchange can be computed in perturbation theory and is given by the BFKL singularity [1], which corresponds to a compound state of two reggeized gluons. The kernel of the resulting equation is currently known up to the next-to-leading logarithmic approximation [2, 3] and leads to the power growth of the scattering cross section with the center-of-mass energy $\sqrt{s}$.

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An alternative derivation of this equation was constructed from the s-channel point of view [4] in the light-front perturbation theory. In the latter approach an incoming particle, onium (a heavy quark–antiquark pair), develops a light-front wave-function which consists of many soft gluons. Here, soft means that the longitudinal components of the emitted gluons are much smaller than those of the parent gluons, while the transverse momenta are unrestricted. These emissions can be resummed by the equation analogous to the BFKL equation: the dipole evolution equation [4]. In the leading logarithmic approximation, the kernel eigenvalues of both equations are identical. The next-to-leading approximation was computed recently [5] and the kernel eigenvalue agrees with the t-channel computation [2, 3] up to a numerical constant.

Apart from the running coupling constant one major source of the next-to-leading order corrections are the kinematical effects. They can be effectively resummed by imposing kinematical constraints [6,7] onto the momenta of emitted gluons in the cascade.

In this paper we discuss the problem of the kinematical effects in the development of the onium and gluon wave-functions in the light-front perturbation theory [8, 9]. By including corrections in the energy denominators, the modified version of the dipole kernel is derived which is expressed in terms of the Bessel functions of the second kind. The diffusion pattern in the transverse coordinates is strongly suppressed due to the exponential nature of the modified kernel. The light-front formulation is also very convenient for the description of the wave-functions with exact kinematics. The recurrence relations for the wave-functions have a particularly simple form and can be explicitly solved when the incoming gluon is on-shell and for the case of the helicity-conserving splittings. These techniques can be also extended to the computation of the fragmentation amplitudes. By considering the scattering of the evolved wave function onto the gluon target, one can derive the MHV amplitudes on the light-front.

The outline of the paper is the following. In the next section we recall the basics of the light-front formulation. In Sec. 3 we consider the development of the onium wave function at high energy and show how it can be improved by including the kinematical effects coming from the energy denominators. In Sec. 4 we derive the recurrence relations for the wave-functions with exact kinematics and with arbitrary number of gluon components. The resummed expressions for the wave-functions and fragmentation functions are given there. In Sec. 5 we sketch the derivation of the Parke–Taylor amplitudes in the light-front formulation.

2. Light-front formulation

The front form for the relativistic dynamics of particles was first introduced by Dirac [10]. He pointed out that this form might be particularly useful because the Hamiltonian does not have the square root, and therefore the negative energies might be eliminated. Later on, the front-form appeared in the connection with the current algebra [11]. The characteristic feature of the light-front perturbation theory is the simplified structure of the vacuum [12]. The hadron viewed in the infinite momentum frame can be described as a nearly static collection of partons whose internal motions are slowed down due to the time dilation effect [13].

The infinite momentum frame can be most easily formulated as the change of the variables [8, 9] from the coordinates

$$(t, x, y, z)\,,$$

to

$$(\tau, x, y, \mathcal{Z})\,,$$

where

$$\tau = \frac{t+z}{\sqrt{2}} \qquad \text{and} \qquad \mathcal{Z} = \frac{t-z}{\sqrt{2}}\,.$$

The variable τ is the light-front time.

Of particular interest is the structure of the Poincare group in the new frame. One finds that

$$P^{\mu} = \left(\eta, P^1, P^2, H\right)\,,$$

with

$$\eta = \frac{P^0 + P^3}{\sqrt{2}}\,, \qquad H = \frac{P^0 - P^3}{\sqrt{2}}\,.$$

The above four generators together with the following combinations

$$B_1 = \frac{K_1 + J_2}{\sqrt{2}}\,, \qquad B_2 = \frac{K_2 - J_1}{\sqrt{2}}$$

and J_3 (where $K_i = M_{i0},\ M_{ij} = \varepsilon_{ijk} J_k$) have particular interesting commutation relations. One can show that the commutation relations are the same as the ones among the symmetry operators of the two-dimensional non-relativistic quantum mechanics. The identification is the following

$$\begin{aligned}
\eta &\rightarrow \text{mass,}\\
H &\rightarrow \text{Hamiltonian,}\\
P_{\mathrm{T}} = (P^1, P^2) &\rightarrow \text{momentum,}\\
J_3 &\rightarrow \text{angular momentum,}\\
B_1, B_2 &\rightarrow \text{Galilean boosts in } x \text{ and } y \text{ directions.}
\end{aligned}$$

The set of the generators $\eta, P_{\rm T}, J_3, B_1, B_2$ leaves the planes $\tau =$ const. invariant. Therefore, these operators are called kinematical generators. To illustrate the isomorphism with the Galilean group one can consider the free particle for which the mass shell condition holds

$$m^2 = P^\mu P_\mu = 2\eta H - P_{\rm T}^2 \,.$$

Solving for H one gets

$$H = \frac{P_{\rm T}^2 + m^2}{2\eta}$$

which is exactly of the non-relativistic form with identifications given above. The term $m^2/2\eta$ can be interpreted as the internal binding energy of the particle.

Much of the simplifications arising in the light-front formulation of quantum field theory is related to this isomorphism. The other remaining generators rotate the direction of the infinite momentum axis. They commute with the Hamiltonian and with each other and are called the dynamical generators.

2.1. QFT in the infinite momentum frame

Given definitions of the infinite momentum frame given above, one can formulate the quantum field theory in such frame [8, 9]. The basics of this formulation is to decompose any given covariant Feynman graph into a sum of non-covariant graphs which are ordered in the light-front time τ. A major simplification occurs in the infinite momentum frame, as some of the diagrams vanish. For example, the order of g^2 diagram for $2 \to 2$ scattering in the scalar ϕ^3 theory with an exchange in the s-channel can be decomposed into two diagrams shown in Fig. 1. It can be shown [12] that the right diagram vanishes. This happens essentially due to the fact that η (light-front 'mass') is conserved at each vertex and is positive definite.

The rules of the light-front perturbation theory can be summarized as follows:

1. Draw all possible Feynman graphs for the process in consideration.
2. Decompose each diagram into a set of τ-ordered graphs.
3. For each line assign the momentum k_μ. Variables $\eta, k_{\rm T}$ are conserved at each vertex. The light-front energy is given by $k^- = \frac{k_{\rm T}^2+m^2}{2\eta}$.
4. In the case of theory with spin, one needs to assign a spinor $u(k,\lambda)$ for each fermion, and a polarization vector ε_μ for the gluon (photon) line.

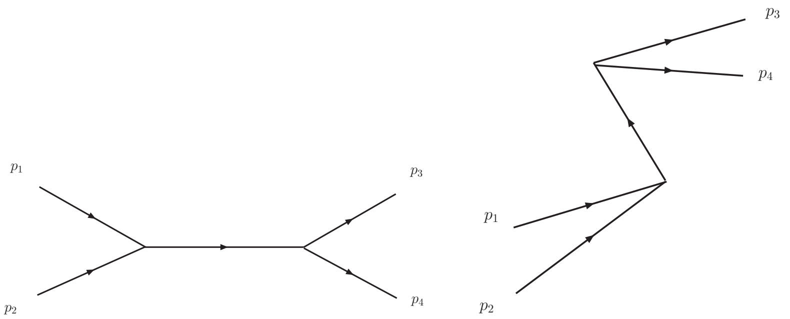

Fig. 1. The $2 \to 2$ process in the ϕ^3 theory, decomposed into two different time ordered diagrams. Left: diagram which is left in the infinite momentum frame. Right: vacuum graph which vanishes in this frame.

5. Include appropriate vertices see for example [8] in the case of QED or [14] in the case of QCD.

6. For each intermediate state include the energy denominator

$$\frac{1}{K^- - \sum_i k_i^-},$$

   where K^- is the total light-front energy for the initial state and $\sum_i k_i^-$ is the sum over the energies of the intermediate states.

7. Include the factor $\frac{\theta(\eta)}{\eta}$ for each intermediate line and finally integrate $\int d^2 k_{\mathrm{T}} \int d\eta$ for each independent momentum. Sum over helicities, time orderings and diagrams.

For the purpose of the subsequent discussion, let us focus on the energy denominator, point 6. It is easy to see that this denominator ties together the momenta of all the particles in a given intermediate state. In the case of a cascade with many gluons, this leads to rather complex expressions for the amplitudes. We will see in the next section that much of the simplification in the high energy limit comes from the approximations done to the energy denominators. To be precise, in the high energy limit one leaves the dominant energy in the denominator, which leads to the decoupling of this factor from the rest of the amplitude. As a consequence, this enables to write down closed evolution equation which iteratively generates subsequent splittings.

3. Dipole evolution kernel with kinematical effects

3.1. Dipole wave-function in the leading logarithmic approximation

We recall the original derivation of the dipole wave function at small x with soft gluons which was first developed by Mueller [4] using the techniques of the light-front perturbation theory. One introduces the heavy onium wave function which consists of the quark–antiquark pair

$$\Psi^{(0)}(z_1, \underline{k}_1)\,, \qquad \text{where} \qquad z_1 = k_1^+/P^+\,. \tag{1}$$

The initial momentum of the onium is P, the quark has four-momentum $k_1 = (k_1^+, k_1^-, \underline{k}_1)$ and the antiquark has momentum $P - k_1$. In the transverse coordinate space the onium wave-function reads

$$\Phi^{(0)}(z_1, \underline{x}_{01}) = \int \frac{d^2\underline{k}_1}{(2\pi)^2}\, e^{i\underline{k}_1\cdot\underline{x}_{01}}\, \Psi^{(0)}\left(z_1, \underline{k}_1\right)\,, \tag{2}$$

where $\underline{x}_{01}$ is the 2-dim. vector denoting the size of the dipole 01 in the transverse coordinate space. Such representation was shown to be very convenient for the purpose of studying the high energy limit [4]. This is because in this limit the longitudinal and transverse components of the momenta decouple.

The next step in this construction is to take into account the emission of a soft (small x) gluon with momentum k_2 from the onium.
The following assumptions were made:

- The emitted gluon is longitudinally soft: $k_2^+ \ll k_1^+$.
- The coupling of the gluon to the quark (antiquark) is eikonal.
- Since the gluon is longitudinally soft, the energy denominator for the graphs with two quarks and with one gluon emission was dominated by the gluon energy
$$D_1 \sim \frac{1}{k_2^-}\,.$$

Putting all these assumptions together, one arrives at the formula for the wave function with one emitted soft gluon

$$\Psi^{(1)}\left(z_1, \underline{k}_1; z_2, \underline{k}_2\right) = 2gt^a \frac{\underline{\varepsilon}_2\cdot\underline{k}_2}{\underline{k}_2^2}\left[\Psi^{(0)}\left(z_1, \underline{k}_1\right) - \Psi^{(0)}\left(z_1, \underline{k}_1 + \underline{k}_2\right)\right]\,, \tag{3}$$

where the gluon longitudinal momentum fraction is defined as $z_2 = k_2^+/P^+$, t^a is the color matrix in the fundamental representation, and $\underline{\varepsilon}_2$ is the polarization vector for the gluon. The Fourier transform to the transverse coordinate space is

$$\Phi^{(1)}\left(z_1, \underline{x}_{01}; z_2, \underline{x}_{02}\right) \;=\; -\frac{igt^a}{\pi}\left(\frac{\underline{x}_{20}}{x_{20}^2} - \frac{\underline{x}_{21}}{x_{21}^2}\right)\cdot\underline{\varepsilon}_2\, \Phi^{(0)}\left(z_1, \underline{x}_{01}\right)\,, \tag{4}$$

where $x_{ij}^2 \equiv \underline{x}_{ij}^2$. We see the advantage of using the coordinate space representation because the soft gluons factorize in (4). The modulus squared of the one gluon wave function has then explicitly the form

$$\left|\Phi^{(1)}\right|^2 (z_1, \underline{x}_{01}) = \int_{z_0}^{z_1} \frac{dz_2}{z_2} \int \frac{d^2\underline{x}_{02}}{2\pi} \frac{\underline{x}_{01}^2}{\underline{x}_{02}^2\, \underline{x}_{12}^2} \left|\Phi^{(0)} (z_1, \underline{x}_{01})\right|^2 . \tag{5}$$

In this form it is particularly transparent that in the soft gluon limit the transverse and longitudinal degrees of freedom decouple. The wave function with one soft gluon is just the wave function without any soft gluons times the branching probability which reads

$$\frac{\alpha_{\mathrm{s}} N_c}{\pi} \frac{d^2\underline{x}_{02}\, \underline{x}_{01}^2}{\underline{x}_{02}^2\, \underline{x}_{12}^2} . \tag{6}$$

This is the dipole splitting kernel in the leading logarithmic approximation in x which governs the dipole evolution equation [4].

3.2. Modified energy denominators in the dipole evolution

In deriving (3), and consequently (6), one uses crucial assumption about the strong ordering in the light-front energies of the partons

$$k_2^- \gg k_1^- . \tag{7}$$

If we consider more gluon emissions, then the above ordering is equivalent to the multi-Regge kinematics. This enables to make the approximations as described above and factorize the soft gluon contribution. However, for the consistency of the calculations we should keep the inequality (7) (where $k_1^- = \frac{\underline{k}_1^2}{2k_1^+}$, $k_2^- = \frac{\underline{k}_2^2}{2k_2^+}$) exact. To be precise we should take

$$\frac{\underline{k}_2^2}{k_2^+} > \frac{\underline{k}_1^2}{k_1^+} , \tag{8}$$

with $k_1^+ > k_2^+$.

If there are more gluon emissions we will have

$$\ldots \frac{\underline{k}_{i_4}^2}{k_{i_4}^+} > \frac{\underline{k}_{i_3}^2}{k_{i_3}^+} > \frac{\underline{k}_{i_2}^2}{k_{i_2}^+} > \frac{\underline{k}_{i_1}^2}{k_{i_1}^+} , \tag{9}$$

and

$$\ldots < k_{i_4}^+ < k_{i_3}^+ < k_{i_2}^+ < k_{i_1}^+ ,$$

where the indices $i_1, \ldots, i_4$ enumerate subsequent emissions along one branch of the gluon cascade. As a result of the ordering (8) the region in the transverse momenta $\underline{k}_2^2$ for the gluon emission is limited

$$\Theta\left(\underline{k}_2^2 - \underline{k}_1^2\right) + \Theta\left(\underline{k}_1^2 - \underline{k}_2^2\right) \Theta\left(\underline{k}_2^2 - \underline{k}_1^2 \frac{k_2^+}{k_1^+}\right) . \tag{10}$$

This means that there is a constraint which restricts the transverse momenta of the daughter gluon (labeled 2). It is given by the step function $\Theta(\underline{k}_2^2 - \underline{k}_1^2 \frac{k_2^+}{k_1^+})$. The constraint (8), (10) means that the momenta of the emitted gluons (those with k_2) are cutoff in the infrared.

It is evident that the kinematical constraint on the transverse momenta emerges in the light-front perturbation theory from more exact treatment of the energy denominators in the graphs. In the t-channel formulation of the BFKL Pomeron, the analogous consistency constraint arises when one takes into account the fact that virtualities of the exchanged gluons are dominated by the transverse parts. This leads to the constraint on the transverse momenta of the emitted gluons [6, 7].

One can include this constraint into the dipole evolution at small x. Let us take the more exact version of the energy denominator which includes the energy of the parent emitter

$$\frac{1}{k_1^- + k_2^-} .$$

With this modification the formula (3) for the dipole wave function in momentum space with one gluon becomes

$$\Psi^{(1)}(z_1, \underline{k}_1; z_2, \underline{k}_2) = 2gt^a \frac{\varepsilon_2 \cdot \underline{k}_2}{\underline{k}_2^2 + \underline{k}_1^2 \frac{k_2^+}{k_1^+}} \left[\Psi^{(0)}(z_1, \underline{k}_1) - \Psi^{(0)}(z_1, \underline{k}_1 + \underline{k}_2)\right] . \tag{11}$$

We still keep the vertex to be eikonal and the only modifications are in the energy denominator. Let us define the auxiliary scale

$$\overline{Q}^2 \equiv \underline{k}_1^2 \frac{k_2^+}{k_1^+} = \underline{k}_1^2\, z , \qquad z = \frac{z_2}{z_1} , \tag{12}$$

and perform the two-dimensional Fourier transform of (11) to the coordinate space

$$\begin{aligned} \Phi^{(1)}(z_1, \underline{x}_{01}; z_2, \underline{x}_{02}) &= 2gt^a \int \frac{d^2\underline{k}_1}{(2\pi)^2} \frac{d^2\underline{k}_2}{(2\pi)^2} e^{i\underline{k}_1 \cdot \underline{x}_{01} + i\underline{k}_2 \cdot \underline{x}_{02}} \\ &\quad \times \left[\Psi^{(0)}(z_1, \underline{k}_1) - \Psi^{(0)}(z_1, \underline{k}_1 + \underline{k}_2)\right] \frac{\varepsilon_2 \cdot \underline{k}_2}{\underline{k}_2^2 + \overline{Q}^2} . \end{aligned} \tag{13}$$

If one insists that the recoil of the parent emitter is very small one arrives at the following (approximated) result for the wave function with one gluon emission

$$2gt^a\,\Phi^{(0)}\left(z,\underline{x}_{01}\right)\frac{i}{2\pi}\overline{Q}_{01}K_1\left(\overline{Q}_{01}\,x_{02}\right)\frac{\underline{\varepsilon}_2\cdot\underline{x}_{02}}{x_{02}}\,, \tag{14}$$

where now

$$\overline{Q}_{01}^2\equiv\frac{1}{x_{01}^2}\frac{k_2^+}{k_1^+}\,.$$

Here K_1 is the Bessel function of the second kind. Expression (14) is an amplitude for one gluon emission in the coordinate space improved by taking into account the next, subleading term in the energy denominator. This expression obviously reduces to the original LL dipole formula, compare (4). One needs to expand the Bessel function K_1 for the small values of the argument

$$\overline{Q}_{01}K_1\left(\overline{Q}_{01}\,x_{02}\right)\;\simeq\;\frac{1}{x_{02}}\,,\qquad\text{for}\qquad\frac{x_{02}}{x_{01}}\sqrt{z}\to 0\,. \tag{15}$$

Expression (14) becomes in this limit

$$2gt^a\,\Phi^{(0)}\left(\underline{x}_{01},z\right)\,\frac{i}{2\pi}\frac{\underline{\varepsilon}_2\cdot\underline{x}_{02}}{x_{02}^2}\,, \tag{16}$$

which is the original LL formula [4] as expected. Therefore, Eq. (14) improves the original formula (16).

Finally, one can construct the improved dipole kernel by adding the contribution from both graphs and squaring them. This gives

$$d^2\underline{x}_2\left(\overline{Q}_{01}K_1\left(\overline{Q}_{01}x_{02}\right)\frac{\underline{\varepsilon}_2\cdot\underline{x}_{02}}{x_{02}}-\overline{Q}_{01}K_1\left(\overline{Q}_{01}x_{12}\right)\frac{\underline{\varepsilon}_2\cdot\underline{x}_{12}}{x_{12}}\right)^2\,. \tag{17}$$

We will refer to it as the *quasilocal* case because in this approximation we only keep terms in the energy denominators which refer to the daughter and the parent dipole, without any other dipoles in the cascade. It is straightforward to verify that expression (17) simplifies to expression (6) when $z\to 0$. The kernel (17) is very similar in form to the one with the massive gluon, which also can be expressed in terms of the K_1 functions. Here, however the argument of the Bessel functions depends on $\overline{Q}_{01}$, and consequently on the longitudinal momentum z. The transverse and longitudinal momenta are not separated any more, even though we can still use a single, closed integral equation for the evolution of the dipole amplitude in the rapidity. In the LL approximation, the evolution depended only on the previous step in rapidity, with the branching that was independent of the rapidity or x.

The modified kernel (17) contains branchings which depend explicitly on the longitudinal variable, and therefore on all the steps in the evolution in rapidity. This is a qualitative difference as this means that there is now a 'memory' in the evolution of the system of dipoles. The probability of the emission of next dipoles depends on the evolution variable ('time') z.

3.3. Diffusion in impact parameter space

As is clear from the form of the modified kernel (17), the corrections from the energy denominators imply large modifications of the diffusion properties in the impact parameter space. The modified Bessel functions exponentially suppress the production of large size dipoles above a z-dependent characteristic size

$$\overline{Q}_{01}^2 K_1^2\left(\overline{Q}_{01} x_{02}\right) \simeq \frac{\pi}{2}\frac{\overline{Q}_{01}}{x_{02}} e^{-2x_{02}\overline{Q}_{01}}\,, \qquad \frac{x_{02}}{x_{01}}\sqrt{z} \to \infty\,.$$

The effective cut-off size, proportional to $x_{01}/\sqrt{z}$, grows with decreasing z. This cutoff on the dipole size is analogous to the coherence effect in the cascade of the gluon emissions [15]. The maximal opening angle prevents the gluons from being emitted into a certain kinematic regime. Here, the effect is to prevent the emission of very large dipoles. As a result, diffusion in the impact parameter space is very much suppressed. We note however, that we do not expect the Froissart bound to be satisfied. This can only happen if there is a finite mass gap in the theory which will limit the range of the interactions. This can be phenomenologically achieved by the substitution

$$\overline{Q}_{01}^2 \to \overline{Q}_{01}^2 + m^2\,.$$

In this way one will get an amplitude which has exponential tails in impact parameter and this will lead to a behavior consistent with the Froissart bound [16] (modulo normalization).

4. Gluon cascades with exact kinematics

4.1. Light-front wave functions

The improved dipole kernel derived in the previous section contains only a part of the corrections due to the kinematics. One can show that the light-front perturbation techniques enable to compute the wave function of the gluon with arbitrary number of components while keeping both the denominators and the vertices exact. In the case when the gluon is on-shell one can resum the wave function and arrive at the closed expression.

Here, we consider a gluon with four momentum P, and the color index a, that develops a virtual fluctuation into states containing n gluons with momenta $(k_1, \dots k_n)$ and color indices $(a_1, a_2, \dots, a_n)$, correspondingly. For the initial gluon with virtuality $-Q^2$ and a vanishing transverse momentum one has in the light-front variables,

$$P^\mu = \left(P^+, -\frac{Q^2}{2P^+}, \underline{0} \right) ,$$

with $P^\pm = \frac{1}{\sqrt{2}}(P^0 \pm P^3)$. After $n-1$ splittings the wave function contains n gluons in the final state. These n gluons have the corresponding momenta $k_1, \dots, k_n$. Each of this momenta has components $k_i^\mu = (z_i P^+, k_i^-, \underline{k}_i)$, with z_i being the fraction of the initial P^+ momentum which is carried by the gluon labeled by i and $\underline{k}_i$ being the transverse component of the gluon momentum. The rules of the light-front perturbation theory [9, 12] require to evaluate the energy denominators for each of the intermediate states. For the last intermediate state with n gluons the corresponding energy denominator reads

$$\overline{D}_n = P^- - \sum_{i=1}^{n} k_i^- = -\frac{1}{2P^+} \left(Q^2 + \frac{\underline{k}_1^2}{z_1} + \frac{\underline{k}_2^2}{z_2} + \ldots + \frac{\underline{k}_n^2}{z_n} \right) = -\frac{1}{2P^+} D_n \, , \quad (18)$$

where we have used

$$k_i^- = \frac{\underline{k}_i^2}{2 z_i P^+} \, ,$$

and introduced the auxiliary notation for the (rescaled) denominator D_n.

In what follows, we shall focus on the color ordered multi-gluon amplitudes, that is the amplitudes decomposed in the basis of color tensors, $T^{a_1 a_2 \dots a_n} = \mathrm{tr}\,(t^{a_1} t^{a_2} \dots t^{a_n})$, where $a_1, a_2, \dots, a_n$ are the color indices of the gluons.

4.2. Recursion relations

Using the techniques of the light-front perturbation theory it is straightforward to derive the recurrence relations for the wave functions with different number of the gluon components. We focus on components of the wave function in which all the gluons have positive helicities. In the light-cone gauge, $\eta \cdot A = 0$, with vector $\eta = (0, 1, \underline{0})$, the polarization four-vector of the gluon with four-momentum k is

$$\varepsilon^{(\pm)} = \varepsilon_\perp^{(\pm)} + \frac{\underline{\varepsilon}^{(\pm)} \cdot \underline{k}}{\eta \cdot k} \, \eta \, , \quad (19)$$

where $\varepsilon_{\perp}^{(\pm)} = (0, 0, \underline{\varepsilon}^{(\pm)})$, and the transverse vector is defined by $\underline{\varepsilon}^{(\pm)} = \mp\frac{1}{\sqrt{2}}(1, \pm i)$. For the same-helicity cascade it will be sufficient to take into account only one projection of the three gluon vertex (the four gluon vertex does not contribute in this case).

When one helicity, say $\lambda_1 = -1$, is different than the others $\lambda_2 = \lambda_3 = +1$, the vertex takes the following form in the light-front variables,

$$\tilde{V}_{-++}^{a_1 a_2 a_3}(k_1, k_2, k_3) = g\, f^{a_1 a_2 a_3}\, z_1\, \underline{\varepsilon}^{(-)} \cdot \left(\frac{\underline{k}_2}{z_2} - \frac{\underline{k}_3}{z_3} \right) , \tag{20}$$

where the δ-functions related to the conservation of the transverse and longitudinal '+' components of the momentum are implicit. Here $f^{a_1 a_2 a_3}$ is the structure constant for the SU(N_c) color group. For the case of interest, of the $(+ \to ++)$ transition, the amplitude is described by Eq. (20) with z_1 being the fraction of the + component of the momentum of the incoming gluon.The dependence of the vertex on the transverse momenta of the daughter gluons is given by the variable

$$\underline{v}_{23} \equiv \left(\frac{\underline{k}_2}{z_2} - \frac{\underline{k}_3}{z_3} \right) . \tag{21}$$

This variable has a simple interpretation on the light-front: it is the relative transverse light-front velocity of the two gluons. Interestingly enough, the same variable is present when we consider the change of the energy denominator due to the splitting. In a general situation, when the gluon with momentum k_1 belongs to a virtual gluon cascade, the energy denominator before the splitting of gluon 1 can be written as

$$D_n = D_{n/1} + \frac{\underline{k}_1^2}{z_1} , \tag{22}$$

where $D_{n/1} = Q^2 + \sum_{i>1} \frac{\underline{k}_i^2}{z_i}$ does not contain the energy of gluon 1. After the gluon splits into two gluons with momenta k_2 and k_3 we have

$$D_{n+1} = D_{n/1} + \frac{\underline{k}_2^2}{z_2} + \frac{\underline{k}_3^2}{z_3} . \tag{23}$$

In the light-front perturbation theory the transverse and the + components of the longitudinal momenta are conserved in each vertex. Therefore, we have that $\underline{k}_1 = \underline{k}_2 + \underline{k}_3$ and $z_1 = z_2 + z_3$. Using this fact, one can express the change of the energy denominator as

$$D_{n+1} - D_n = \xi_{23}\, \underline{v}_{23}^2 , \tag{24}$$

where $\xi_{23} \equiv \frac{z_2 z_3}{z_2+z_3}$. We see here that the change in the denominator is in the form of (twice) the kinetic energy with ξ_{23} being the reduced light-front mass for particles 2 and 3.

Putting the vertex and the denominator together, we can formulate the recursion relation for the wave function containing n number of gluons

$$\Psi_n(k_1,\ldots,k_i,k_{i+1},\ldots,k_n) = g\,\mathcal{S}_{i,i+1}\,\Psi_{n-1}(k_1,\ldots,k_{i\,i+1},\ldots,k_n)\,, \quad (25)$$

where $\Psi_{n-1}(k_1,k_2,\ldots,k_{i\,i+1},\ldots,k_n)$ (with $k_{i\,i+1} \equiv k_i + k_{i+1}$) is the wave function with $(n-1)$ gluons and the splitting operator is defined to be

$$\mathcal{S}_{i,i+1} \;\equiv\; \frac{1}{\sqrt{\xi_{i\,i+1}}}\frac{\underline{\varepsilon}^{(-)}\underline{v}_{i\,i+1}}{D_{n-1}+\xi_{i\,i+1}\,\underline{v}^2_{i\,i+1}} \;=\; \frac{1}{\sqrt{\xi_{i\,i+1}}}\frac{\underline{\varepsilon}^{(-)}\underline{v}_{i\,i+1}}{D_n}\,.$$

Formula (25) is the recurrence prescription for obtaining the wave function with n virtual gluons from the wave function with $n-1$ gluons. To obtain the full recurrence formula one needs to sum over the different possibilities of the splittings which gives the following result

$$\Psi_n(k_1,k_2,\ldots,k_n) = \frac{g}{D_n}\sum_{i=1}^{n-1}\frac{\underline{\varepsilon}^{(-)}\underline{v}_{i\,i+1}}{\sqrt{\xi_{i\,i+1}}}\,\Psi_{n-1}(k_1,\ldots,k_{i\,i+1},\ldots,k_n)\,, \quad (26)$$

where D_n is the denominator for the last intermediate state with n gluons.

4.3. Multi-gluon wave function for the initial on-shell gluon

It is interesting to consider the multi-gluon wave function that originates from subsequent splittings of an on-shell incoming gluon with the + helicity. One can also alternatively think about it as the incoming particle with a large momentum P^+ such that $P^- = -\frac{Q^2}{2P^+}$ is very small, at least as compared with the particles in the wave function. Again we consider only the situation where the gluons have all + helicities. We will use the complex representation of the transverse vectors: $v_{ij} = \underline{\varepsilon}^{(+)}\cdot\underline{v}_{ij}$, $v^*_{ij} = \underline{\varepsilon}^{(-)}\cdot\underline{v}_{ij}$. It is also useful to introduce the following notation,

$$v_{(i_1 i_2 \ldots i_p)(j_1 j_2 \ldots j_q)} = \frac{k_{i_1}+k_{i_2}+\ldots+k_{i_p}}{z_{i_1}+z_{i_2}+\ldots+z_{i_p}} - \frac{k_{j_1}+k_{j_2}+\ldots+k_{j_q}}{z_{j_1}+z_{j_2}+\ldots+z_{j_q}}\,, \quad (27)$$

$$\xi_{(i_1 i_2 \ldots i_p)(j_1 j_2 \ldots j_q)} = \frac{(z_{i_1}+z_{i_2}+\ldots+z_{i_p})(z_{j_1}+z_{j_2}+\ldots+z_{j_q})}{z_{i_1}+z_{i_2}+\ldots+z_{i_p}+z_{j_1}+z_{j_2}+\ldots+z_{j_q}}\,, \quad (28)$$

with $k_i \equiv \underline{\varepsilon}^{(+)}\cdot\underline{k}_i$. The variable $v_{(i_1 i_2 \ldots i_p)(j_1 j_2 \ldots j_q)}$ has the interpretation of the relative velocity of two groups of particles with momenta $k_{(i_1\ldots i_p)}$ and

$k_{(j_1+1\dots j_q)}$. The variable $\xi_{(i_1 i_2 \dots i_p)(j_1 j_2 \dots j_q)}$ is the reduced light-front mass for this system. The incoming state has the wave function,

$$\Psi_1(1) = 1\,, \tag{29}$$

where the global momentum conservation δ-functions are assumed implicitly. In the following discussion we will consider color ordering in the amplitudes, therefore we will suppress color degrees of freedom. In general, for the color-ordered amplitude, the gluon splitting acts on the wave function as derived in Eq. (26) (for on-shell initial state)

$$\begin{aligned}
&-D_{n+1}\,\Psi_{n+1}(1,2,\dots,n+1)\\
&= g\frac{v_{12}^*}{\sqrt{\xi_{12}}}\Psi_n(12,3,\dots,n+1) + g\frac{v_{23}^*}{\sqrt{\xi_{23}}}\Psi_n(1,23,\dots,n+1) + \dots\\
&\dots + g\frac{v_{n\,n+1}^*}{\sqrt{\xi_{n\,n+1}}}\Psi_n(1,2,\dots,n\,n+1)\,,
\end{aligned} \tag{30}$$

with $D_{n+1} = \underline{k}_1^2/z_1 + \underline{k}_2^2/z_2 + \dots + \underline{k}_{n+1}^2/z_{n+1}$. We have introduced the notation $\Psi_n(1,\dots,i-1\,i,\dots,n+1)$ where $i-1\,i$ means that it is the gluon with the momentum $k_{i-1\,i} = k_{i-1} + k_i$.

It turns out that the recurrence relation can be solved exactly, and one obtains the following closed expression for the wave function for the same helicity configurations

$$\begin{aligned}
&\Psi_n(1,2,\dots,n)\\
&= (-1)^{n-1} g^{n-1}\,\frac{1}{\sqrt{z_1 z_2 \dots z_n}}\,\frac{1}{\xi_{(12\dots n-1)n}\,\xi_{(12\dots n-2)(n-1\,n)}\,\cdots\,\xi_{1(2\dots n)}}\\
&\times\,\frac{1}{v_{(12\dots n-1)n}\,v_{(12\dots n-2)(n-1\,n)}\,\cdots\,v_{1(2\dots n)}}\,.
\end{aligned} \tag{31}$$

The resummed form (31) looks relatively simple. Its dependence on transverse momenta of gluons has been factorized into reciprocals of velocities $v_{(12\dots p)(p\dots n)}$ with the splitting index p which takes all possible positions from $p = 2$ to $p = n-1$.

4.4. Light-front fragmentation functions

The same techniques can be used to compute the fragmentation amplitude for the gluon. The difference is that one needs to start with the off-shell particle in the initial state and for the final state which consists of the on-shell particles we will not have the energy denominator.

We consider here the fragmentation of a single, off-shell gluon (labeled by $(12\dots n)$) into the final state of n on-shell gluons, $1,2,\dots n$. The n final state gluons have transverse momenta $\underline{k}_1,\dots,\underline{k}_n$ and the longitudinal momentum fractions $z_1,\dots,z_n$. The initial gluon has transverse momentum $\underline{k}_{(1\dots n)}$ and the longitudinal fraction $z_{(1\dots n)}$ where we again used shortcut notation $\underline{k}_{(1\dots n)} = \sum_{j=1}^n \underline{k}_j$ and $z_{(12\dots n)} = \sum_{i=1}^n z_i$. We will denote the fragmentation part of the amplitude for 1 to n gluons as $T[(12\dots n) \to 1,2,\dots,n]$.

Using the analogous techniques as described previously one arrives at the following expression for the fragmentation of the single gluon

$$T[(12\dots n) \to 1,2,\dots,n] = g^{n-1}\left(\frac{z_{(12\dots n)}}{z_1 z_2\dots z_n}\right)^{3/2} \frac{1}{v_{12}v_{23}\dots v_{n-1\,n}}\,. \tag{32}$$

It is interesting to note that the form of the above amplitude for the fragmentation is dual to that of the gluon wave function with n-components. To be more precise, the two expressions can be matched to each other upon the exchange of v_{ij} with $v_{(1\dots i_p)(i_p\dots n)}$ (modulo factors which depend on the longitudinal momenta).

5. Scattering amplitudes

Using the exact forms for the wave-functions and fragmentation amplitudes it is possible to compute the multi-gluon scattering amplitude. Since we have chosen the helicity conserving processes both in the wave-function and in the fragmentation function, we are restricted to the specific helicity configuration of the amplitude. The expressions derived previously enable us to compute the $2 \to n$ gluon scattering amplitude where the two initial gluons are incoming with helicity $+$ and the n outgoing gluons are also with helicity $+$. By means of crossing this corresponds to the amplitude with all outgoing gluons with $(-,-,+,\dots,+)$ helicity configuration.

The exact tree level amplitudes with an arbitrary number of the external on-shell gluons are known. These are the Parke–Taylor amplitudes [17] (see [18] for a comprehensive review) and can be recast in the following form

$$\mathcal{M}_n = \sum_{\{1,\dots,n\}} \mathrm{Tr}(t^{a_1}t^{a_2}\dots t^{a_n})\, m(p_1,\varepsilon_1;p_2,\varepsilon_2;\dots;p_n,\varepsilon_n)\,, \tag{33}$$

where $a_1,a_2,\dots,a_n$, $p_1,p_2,\dots,p_n$ and $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n$ are the color indices, momenta and helicities of n external gluons, respectively. The sum in (33) is over the $(n-1)!$ non-cyclic permutations of the set $\{0,1,\dots,n\}$.

The kinematical parts of the amplitude, denoted by $m(1,2,\dots,n) \equiv m(p_1,\varepsilon_1;p_2,\varepsilon_2;\dots;p_n,\varepsilon_n)$, are color independent and gauge invariant. The non-vanishing amplitude for the configuration $(-,-,+,\dots,+)$ at the tree level is given by the formula

$$m(1^-,2^-,3^+,\dots,n^+) \;=\; ig^{n-2}\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n{-}2\;n{-}1\rangle\langle n{-}1\;n\rangle\langle n1\rangle}\,, \tag{34}$$

where $\langle jk\rangle$ are the spinor products defined below.

In order to compare the amplitudes obtained within the light-front perturbation theory formalism with the MHV amplitudes we need to consider the scattering process of the evolved wave function onto the target. We will simplify the problem by analyzing the case where the evolved projectile gluon scatters on a single target gluon which is separated from the virtual gluons in the projectile by a large rapidity interval. The exchange between the projectile and the target will be treated in the high energy limit. In this limit the interaction between the projectile and the target is mediated by an instantaneous part of the gluon propagator in the light-cone gauge. We restrict the kinematics of the exchange, but still, the internal structure of the projectile gluon field is accurately represented. In principle, the applied technique could also be used to evaluate the scattering amplitudes without any kinematical restrictions, but an analysis of the completely general case would be much more complicated.

For the derivation of the MHV amplitudes it is useful to note that the variables $\underline{v}_{jk}$ which we used to construct the wave functions are related to the variables used in the framework of helicity amplitudes, see [18]. Namely, for given pair of on-shell momenta k_i and k_j we have that

$$\langle ij\rangle = \sqrt{z_i z_j}\,\underline{\varepsilon}^{(+)}\cdot\left(\frac{\underline{k}_i}{z_i}-\frac{\underline{k}_j}{z_j}\right)\,, \qquad [ij] = \sqrt{z_i z_j}\,\underline{\varepsilon}^{(-)}\cdot\left(\frac{\underline{k}_i}{z_i}-\frac{\underline{k}_j}{z_j}\right)\,, \tag{35}$$

where the symbols $[ij], \langle ij\rangle$ are the spinor products defined by

$$\langle ij\rangle = \langle i-|j+\rangle\,, \qquad [ij] = \langle i+|j-\rangle\,, \tag{36}$$

with the chiral projections of the spinors for massless particles

$$|i\pm\rangle \;=\; \psi_\pm(k_i) \;=\; \tfrac{1}{2}(1\pm\gamma_5)\psi(k_i)\,, \qquad \langle\pm i| \;=\; \overline{\psi_\pm(k_i)}\,, \tag{37}$$

for a given momentum k_i. Combining (21) and (35) we obtain

$$\langle\, ij\,\rangle = \sqrt{z_i z_j}\,\underline{\varepsilon}^{(+)}\cdot\underline{v}_{ij}\,, \qquad [\,ij\,] = \sqrt{z_i z_j}\,\underline{\varepsilon}^{(-)}\cdot\underline{v}_{ij}\,, \tag{38}$$

and the dependence on the transverse momenta in the light-front wave function can be expressed by $\langle\, ij\,\rangle$ and $[\,ij\,]$.

In order to derive the MHV amplitude in the high energy limit we consider the process in which the incoming gluon, labeled by 0, develops into a cascade of gluons $1, \ldots, n$ and scatters on a target gluon $a \to b$. For the high energy case the dominant contributions are given by the instantaneous exchange of the Coulomb gluon. The exchanged gluon can be attached to any gluon in the cascade. The process in question is shown schematically in Fig. 2.

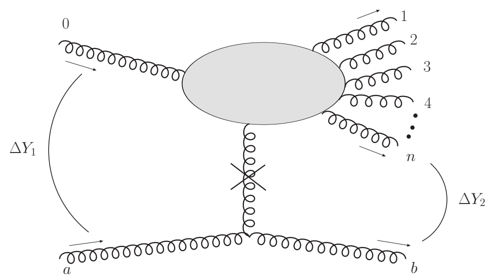

Fig. 2. The $2 \to n{+}1$ on-shell gluon amplitude in the high energy limit. The gluon 0 dissociates into the gluon cascade (indicated by a blob) which interacts via high energy gluon (with a cross) with the gluon $a \to b$. The large rapidity difference $\Delta Y_1 \sim \Delta Y_2$ between the light-front cascade and the lower gluon is taken. The arrows indicate the momentum flow: the gluons $0, a$ are incoming and $1, \ldots, n, b$ are outgoing. All the gluons have + helicity and it is conserved.

Therefore we include both initial and final state emissions. The gluons 0 and a are incoming with helicities +, and all gluons through the whole cascade down to the final state, carry the positive helicity. Also, we need to sum over all possible attachments of the exchanged gluon to the upper part of the diagram, shown in Fig. 2.

This summation can be represented by means of the general formula

$$\begin{aligned}
&\tilde{\Psi}_n(1,2,\ldots,n)\\
&=\sum_{m=1}^{n} \sum_{(1\leq n_1<n_2<\ldots<n_{m-1}\leq n)} \Psi_m((1\ldots n_1)(n_1{+}1\ldots n_2)\ldots(n_{m-1}{+}1\ldots n))\\
&\times T[(1\ldots n_1)\to 1,\ldots,n_1]\; T[(n_1+1\ldots n_2)\to n_1+1,\ldots,n_2]\;\ldots\\
&\ldots\; T[(n_{m-1}+1\ldots n)\to n_{m-1}+1,\ldots,n]\,. \qquad (39)
\end{aligned}$$

Here, $\tilde{\Psi}_n$ is the part of the amplitude which corresponds to the upper 'blob' in Fig. 2 with the momentum transfer given by $k_{(1\ldots n)}$. Putting in the exact expressions for the fragmentation amplitude T and the initial state wave function Ψ_n one obtains the explicit closed formula for the $\tilde{\Psi}_n$. The final result is the following general form for the $\tilde{\Psi}_n$ for an arbitrary number of emitted gluons

$$\tilde{\Psi}_n(1,2,\dots n) = g^{n-1}\,\frac{k_{(1\dots n)}}{k_1/z_1}\,\frac{1}{\sqrt{z_1 z_2 \dots z_n}}\,\frac{1}{z_1 z_2 \dots z_n}\,\frac{1}{v_{12}v_{23}\dots v_{n-1\,n}}\,. \quad (40)$$

Note that, $v_{01} = -\frac{k_1}{z_1}$ (as we have chosen the transverse momentum of particle 0 to vanish $\underline{k}_0 = 0$).

We use the relations between the light-front velocities and the spinor products defined above (38) to get

$$\tilde{\Psi}_n(1,2,\dots n) \;=\; g^{n-1}\;\frac{1}{\sqrt{z_1 z_2 \dots z_n}}\,\frac{1}{\sqrt{z_n}}\,\frac{k_{(1\dots n)}}{\langle 01\rangle\langle 12\rangle\langle 23\rangle\dots\langle n-1\,n\rangle}\,. \quad (41)$$

Putting the expression for the exchange of the Coulomb gluon in the high energy limit we recover in light-front perturbation theory the amplitude for $2\to n+1$ scattering

$$M(0;a\to 1,\dots,n;b) \;\simeq\; g^{n+1}\frac{\langle a0\rangle^4}{\langle a0\rangle\,\langle 01\rangle\,\langle 12\rangle\,\langle n-1\,n\rangle\,\langle nb\rangle\langle ba\rangle}\,, \quad (42)$$

which is equivalent to the MHV amplitude (34).

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