NON-ABELIAN INFRA-RED CANCELLATIONS IN THE UNINTEGRATED NLO KERNEL* **

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We investigate the infrared singularity structure of Feynman diagrams entering the next-to-leading-order (NLO) DGLAP kernel (non-singlet). We examine cancellations between diagrams for two gluon emission contributing to NLO kernels. We observe the crucial role of colour coherence effects in cancellations of infra-red singularities. Numerical calculations are explained using analytical formulas for the singular contributions.

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1. Introduction

This study is part of the effort with the aim of constructing fully-exclusive (unintegrated) kernels for DGLAP [1] evolution in QCD at the complete NLO level. More details on this project can be found in Ref. [2]. Let us only mention that construction of the exclusive NLO DGLAP kernels in Ref. [2] is done following Curci–Furmanski–Petronzio (CFP) scheme [3] and we adopt this scheme also in our study. In short the CFP scheme uses axial gauge and dimensional regularisation ($\overline{\text{MS}}$) and generalises collinear factorization developed in Ref. [4]. The two particle-irreducible (2PI) evolution kernels K_0

$$M = C_0(1 + K_0 + K_0^2 + ...) = C_0\Gamma_0 \tag{1}$$

are contracted with the coefficient functions C_0, which are infra-red finite. All infra-red collinear divergences are encapsulated in Γ_0, which denotes the sum over kernels K_0. The DGLAP NLO kernel is then extracted according to the scheme [3] as a single pole in Γ_0. More details can be found in Ref. [3] and in Ref. [2] of these proceedings.

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As it is well known individual Feynman diagrams are not gauge-invariant and sizeable cancellations occur among them. In particular some graphs contributing to NLO DGLAP kernels may contain artificial infra-red singularities, which cancel in a bigger subset of diagrams and do not appear in the final results. Our aim is to analyse in detail such cancellations at the level of the exclusive distributions, before the phase space integration, for the diagrams contributing to NLO DGLAP kernels in the CFP scheme.

We shall analyse singular infra-red structure of the Feynman diagrams depicted in Fig. 1. They describe emission of two gluons off a quark and enter into calculation of the non-singlet NLO kernel. Diagrams of Fig. 1(a)–1(c) are bremsstrahlung type diagrams, including interference in Fig. 1(c). Their $C_{\rm F}^2$ part is the same as in the corresponding case of QED, hence we shall sometimes call them "abelian". The other diagrams of Fig. 1 include production of the gluon pair, see Fig. 1(f), and its interference with the previous bremsstrahlung diagrams, Figs. 1(d) and 1(e). Because of the presence of the triple-gluon vertex they are diagrams of the genuine non-abelian origin. Moreover, since the crossed-ladder diagram of Fig. 1(c) carries colour factor equal to $C_{\rm F}^2 - \frac{1}{2}C_{\rm F}C_{\rm A}$, it contributes to both "abelian" and "non-abelian" part of NLO kernel.

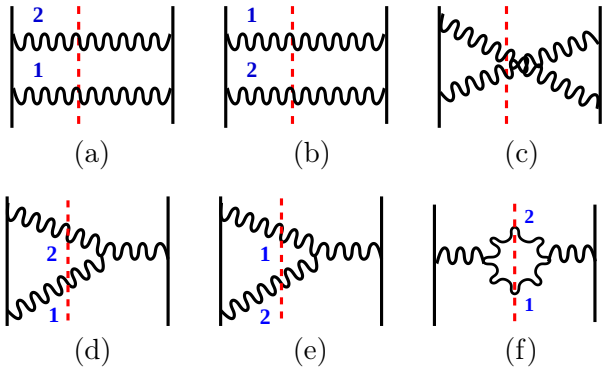


Fig. 1. Two gluon Feynman diagrams contributing to non-singlet NLO kernels.

Let us now introduce notation. The two gluon phase-space is parameterized using Sudakov variables:

$$\begin{aligned} k_i &= \alpha_i p + \alpha_i^- n + k_{i\perp}\,, \qquad && i = 1,2\,, \\ k &= k_1 + k_2\,, && q = p - k\,, \end{aligned} \tag{2}$$

with p being the four-momentum of the incoming quark and n a light-cone vector. Four-vectors k_1 and k_2 denote four-momenta of the emitted gluons, with their transverse parts being $k_{1\perp}$ and $k_{2\perp}$ respectively, and

$k^2 = (k_1 + k_2)^2$ being their effective mass. The sum of α_i is fixed

$$\alpha_1 + \alpha_2 = 1 - x\,. \tag{3}$$

We shall examine the distributions of two gluons in the soft limit:

$$\left\{ \begin{array}{l} \alpha_1 \to 0\,, \\ \alpha_2 \to 1 - x\,, \end{array} \right. \qquad \text{or} \qquad \left\{ \begin{array}{l} \alpha_1 \to 1 - x\,, \\ \alpha_2 \to 0\,. \end{array} \right. \tag{4}$$

We shall also use the "eikonal minus variables" v_i of the emitted gluons defined as

$$v_i = \frac{k_{i\perp}}{\sqrt{\alpha_i}}\,. \tag{5}$$

The kernel is then extracted according to the scheme [3] as a single pole in Γ_0:

$$\Gamma_1 = \frac{C}{2\varepsilon}\Big(\frac{\alpha}{\pi}\Big)^2 \int d\Psi \delta(1-x-\alpha_1-\alpha_2)\Theta(Q-\max\{v_1,v_2\})\rho(\alpha_1,\alpha_2,v_1,v_2,x)\,, \tag{6}$$

where C is the colour factor and the function ρ represents contribution from each Feynman diagram (trace and kinematics). We shall also use the dimensionless "eikonal phase-space" defined as follows:

$$d\Psi = \frac{d\alpha_1}{\alpha_1}\frac{dv_1}{v_1}d\phi_1\frac{d\alpha_2}{\alpha_2}\frac{dv_2}{v_2}d\phi_2\,. \tag{7}$$

The two-gluon phase space is cut from below by means of geometrical regulator δ, namely the factors $1/\alpha_i$ are regulated by principal value prescription: $\frac{1}{\alpha_i} \to \frac{\alpha_i}{\alpha_i^2+\delta^2}$. The closing of the phase space from above is ensured by the Θ function[1]. For the gluon pair mass we shall also use technical cut $k^2 > \kappa$. In the numerical exercises we shall typically integrate (6) over the azimuthal angles ϕ_i of the gluons, while concentrating on the dependence on v_i and α_i. Also, because of the constraint in Eq. (3), if we say that we examine the distribution in α_1/α_2 it really means that we use $\alpha_1/(1-x-\alpha_1)$. Also, due to a simple dimensional argument one of the variables v_i can be always factored out from ρ function and the essential dependence of the distributions in variables v_i can be reduced to the dependence on the ratio $y = v_1/v_2$ only.

[1] The choice of the variable in Θ function closing the phase space from above can be different. We use $\max\{v_1, v_2\}$ or $\max\{a_1, a_2\}$ which is different from CFP choice $q^2 = -(p - k_1 - k_2)^2$.

2. IR cancellations among bremsstrahlung diagrams

In the first part we analyse "QED-like" bremsstrahlung diagrams of Fig. 1(a)–1(c). We shall first show infrared cancellations among these diagrams in the Monte Carlo exercise and later on analyse the same cancellations analytically.

In the following numerical exercises we keep variable x fixed and equal 0.3. This will ensure that at least one gluon is relatively hard.

The distributions

$$f(\alpha_i, v_i) = \int d\phi_1 d\phi_2 \delta(1 - x - \alpha_1 - \alpha_2) \delta\left(Q - \max\{v_1, v_2\}\right) \rho(\alpha_i, v_i) \quad (8)$$

are plotted on Sudakov plane parametrized using variables $\log(\alpha_1/\alpha_2)$ and $\log(v_1/v_2)$, see Eq. (5) for definition of v_i.

Plots in Fig. 2 show contributions from the two ladder diagrams. They are obtained using Monte Carlo program FOAM [5]. As we see, their contributions appear to be strongly ordered in virtuality variables v_i of the emitted gluons, see for instance the left part of Fig. 2. The interference diagram is shown in the upper right plot of Fig. 3. It contributes in the region of the phase-space where both bremsstrahlung diagrams are comparable, which is exactly the line of equal virtualities $v_1 = v_2$. The crossed-ladder interference diagram has a singly-logarithmic singularity along the same line $v_1 = v_2$.

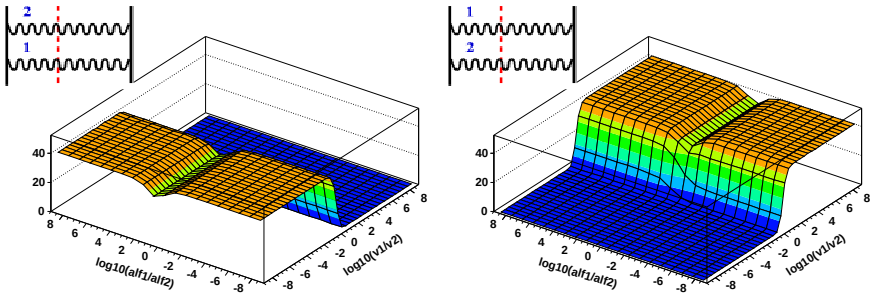


Fig. 2. Two double gluon emission ladders.

Fig. 3 represents both contributions: from symmetrised and crossed ladder (upper plots) together with their sum (lower plot). The singly-logarithmic structure, visible on the upper-left plot, disappears completely. The dominant contribution is an infinite "plateau", with a long "valley" along the line $\alpha_1 \simeq \alpha_2$, This is not an internal infra-red singularity, however, but a leading-log structure. The ladder diagram is not 2PI, but quadratic in the leading-order kernel and, therefore, requires a soft counterterm to cancel this doubly-logarithmic plateau, see [2]. This counterterm is necessary to construct the NLO kernel and its construction is justified by the CFP

scheme. We do not introduce it in this paper, as our aim is to present the universality of gauge cancellations. Therefore we often refer to: "contribution to the NLO kernel" rather than a kernel itself, having in mind that this is not a complete NLO kernel in a strict sense. Despite this dominant LO contribution, in the Fig. 3 one sees no structure in the soft Sudakov limit (4).

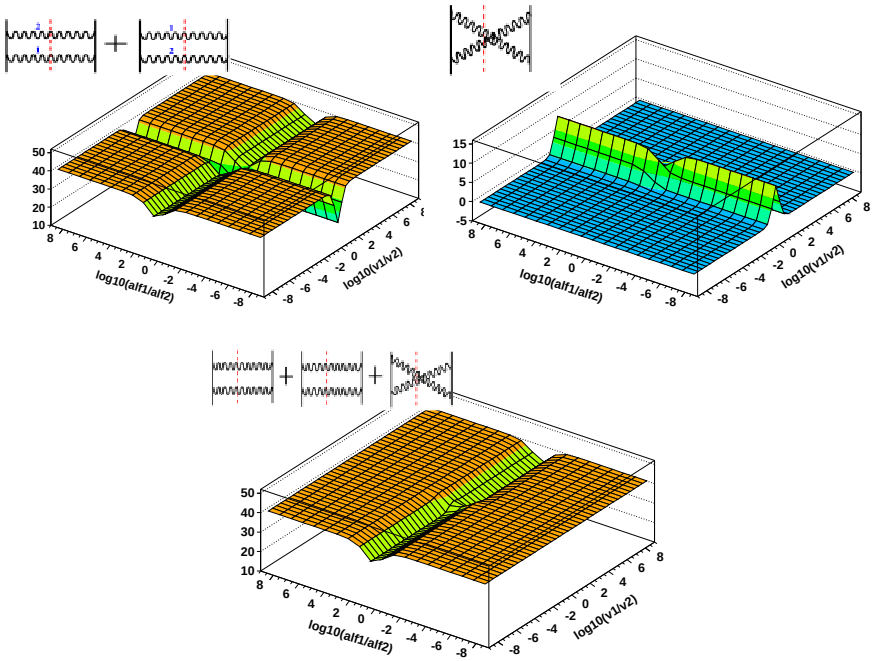


Fig. 3. Symmetrised bremsstrahlung diagram (left), crossed-ladder diagram (right) and their sum (down).

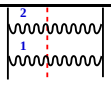
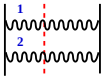
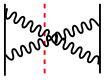
Results of the above numerical exercise can be also understood analytically. Contributions from diagrams in Figs. 1(a)–1(c) in the soft limit are proportional to simple expressions shown in Table I. Two columns in this table refer to two possible different Sudakov limits, with either first or second gluon being soft.

The common denominator $\frac{1}{(1+xy)^2}$ is the (rescaled) square of the virtual quark propagator $\frac{1}{q^4} = \frac{1}{((p-k_1-k_2)^2)^2}$ after emitting two gluons. Other factors come from γ-traces[2]. The last row is the sum of the three, see also the lower plot in Fig. 3. It is finite in the soft Sudakov limit, as the denominator cancels out exactly with the spinorial part of the matrix element squared.

[2] Only "$C_{\rm F}^2$" part of the crossed ladder is taken here, hence the colour coefficient is shown explicitly. Other unimportant factors are omitted.

TABLE I

Contributions from ladders in the Sudakov limit (up to a constant factor).

		$\alpha_1 \to 0$	$\alpha_2 \to 0$
Br1		$C_{\mathrm{F}}^2 \frac{(1+x^2)}{(1+xy)^2}$	$C_{\mathrm{F}}^2 \frac{(1+x^2)}{(y+x)^2}\, x^2$
Br2		$C_{\mathrm{F}}^2 \frac{(1+x^2)}{(1+xy)^2}\, x^2y^2$	$C_{\mathrm{F}}^2 \frac{(1+x^2)}{(y+x)^2} y^2$
Bx		$2C_{\mathrm{F}}^2 \frac{(1+x^2)}{(1+xy)^2}\, xy$	$2C_{\mathrm{F}}^2 \frac{(1+x^2)}{(y+x)^2}\, xy$
	SUM	$C_{\mathrm{F}}^2 \left(1+x^2\right)$	$C_{\mathrm{F}}^2 \left(1+x^2\right)$

In the above warm-up exercise we have examined in a fine detail how in the NLO kernel calculations, in the axial gauge, quantum interference cancellations work in practice. In fact they were exactly the same as in QED in axial gauge. Let us now turn to the genuine non-abelian gauge cancellations of the same kind.

3. IR cancellations among genuine non-abelian contributions

In this section we will discuss contributions proportional to $C_{\mathrm{F}}C_{\mathrm{A}}$ from the diagrams (c)–(f) of Fig. 1. They are of the genuine non-abelian character, as certified by the presence of C_{A}. It is interesting to see how they all "communicate" in the soft Sudakov limit defined below.

We shall start with the overview of the IR cancellations in analytical form and next we shall illustrate them with 2-dimensional plots coming from Monte Carlo numerical exercises. In the following we find useful to use rapidity-related variables a_i in order to parametrise phase-space of two gluons:

$$a_i = \frac{k_{i\perp}}{\alpha_i}, \qquad d\Psi = \frac{d\alpha_1}{\alpha_1}\frac{da_1}{a_1}d\phi_1 \frac{d\alpha_2}{\alpha_2}\frac{da_2}{a_2}d\phi_2\,. \tag{9}$$

The angular dependence enters only through the relative angle between $k_{1\perp}$ and $k_{2\perp}$ namely ϕ_{12}, the remaining angle can be integrated out since nothing depends on it.

By the soft Sudakov limit we understand that

$$\begin{cases} \alpha_i \to 0\,, \\ k_{i\perp} \to 0\,, \end{cases} \tag{10}$$

while a_i is finite. The Sudakov plane in the plots will be parametrised using variables $\log(\alpha_1/\alpha_2)$ and $\log(a_1/a_2)$.

3.1. General structure of the IR non-abelian cancellations

In Table II we summarize the IR cancellations of the non-abelian origin between the two real gluon diagrams contributions to NLO kernel in analytical form. Formulas in Table II are the leading divergences extracted from two real gluon distriburions in a maximally simplified form. Functions F_i are relatively mild and defined as follows:

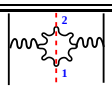
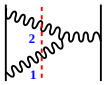
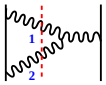
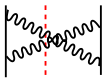
$$\begin{aligned} F_1 &= \frac{\alpha_2}{\alpha_1}\frac{1}{u^2}F_0\,, \qquad F_2 = \frac{\alpha_1}{\alpha_2}u^2 F_0\,, \\ \sqrt{F_0} &= \frac{\sqrt{\alpha_1\alpha_2}u}{\alpha_1(1-\alpha_2)u^2+\alpha_2(1-\alpha_1)+2\alpha_1\alpha_2 u\cos\phi_{12}} = \frac{\sqrt{\alpha_1\alpha_2}a_1a_2}{q^2}\,, \end{aligned} \tag{11}$$

where $u = a_1/a_2$. Moreover, we define $a^2 = 1 + u^2 - 2u\cos\phi_{12}$, which is up to the $\alpha_1\alpha_2 a_2^2$ factor, the effective mass of the gluon pair squared $k^2 = (k_1 + k_2)^2$. Function F_0 is proportional to rescaled square of the propagator $1/q^4$, which is regular in the soft limit.

The pattern of the IR cancellations in Table II is manifest.

TABLE II

Cancellations of the non-abelian contributions from various diagrams due to gauge invariance (colour coherence). Singular part for each diagram is shown in analytical form. Factors F_i are relatively mild, see text for their definition.

Vg		$C_{\rm F}C_{\rm A}(1+x^2)\left[\frac{u^2}{a^2}F_1 + \frac{1}{a^2}F_2\right]$
Yg1		$C_{\rm F}C_{\rm A}(1+x^2)\left[\frac{u\cos\phi_{12}-u^2}{a^2}F_1 + x\,\frac{\alpha_2}{1-x}F_0\right]$
Yg2		$C_{\rm F}C_{\rm A}(1+x^2)\left[\frac{u\cos\phi_{12}-1}{a^2}F_2 + x\,\frac{\alpha_1}{1-x}F_0\right]$
Bx		$-C_{\rm F}C_{\rm A}(1+x^2)\,xF_0$
	SUM	$C_{\rm F}C_{\rm A}(1+x^2)\,\frac{u\cos\phi_{12}}{a^2}\,(F_1+F_2)$

The most evident singularity is associated with u/a^2 factor, infinite when the effective mass of the gluon pair is zero. In logarithmic variables this u/a^2 divergence is seen as a thin infinite ridge along $u = 1$. It is a function of u and ϕ_{12} only, strongly peaked at $u = 1$. In the gluonic vacuum polarization diagram this singularity is dominant. This is, however, a collinear singularity, not a soft one. It remains uncancelled but it is not relevant in our

discussion of cancellation of soft singularities. We only mention it to understand the full singularity structure of the diagrams of interest. In Yg1 and Yg2 diagrams the u/a^2 factor is present, too but the singularity is cancelled out by the numerators.

The factors uF_1 and F_2/u are functions of both u and α_1/α_2. They "soften" the sharp fall of u/a^2 in the limits $u \to 0$ and $u \to \infty$. They are non-zero in the soft Sudakov limit (10), giving rise to a doubly-logarithmic IR divergence[3]. These terms are present in both Vg and Yg (Yg = Yg1 + Yg2) diagrams, with opposite signs. This is checked and discussed below in the context of numerical exercises.

In the last row of Table II the sum of all aforementioned contributions is presented. The terms proportional to u^2F_1/a^2 and F_2/a^2 cancel explicitly among diagram with gluonic vacuum polarization and its interference with bremsstrahlung. The remaining factor $F_1 + F_2$ is equal to 1 in the soft Sudakov limit (10), leaving out u/a^2 factor free from doubly-logarithmic divergences.

Let us stress that this particular cancellation of the doubly logarithmic Sudakov structure of the non-abelian origin in the two-gluon distribution in QCD is usually referred to in the literature as the "colour coherence effect", see for instance Ref. [6].

The other IR divergence is caused by the presence of terms $\sim F_0$ in diagrams Yg1, Yg2 and Bx, see again Table II. F_0 is nonzero along $a_1^2/a_2^2 = \alpha_1/\alpha_2$, giving rise to a single-log singularity after phase space integration. This has been discussed already in the case of bremsstrahlung diagrams. Here, however, the analytical cancellation among Yg1, Yg2 and Bx, ensured by $\alpha_1 + \alpha_2 = 1 - x$, occurs in the whole phase-space, not only in the soft limit.

Finally, the only singularities that remain are associated with u/a^2 term, as discussed before in this section.

3.2. Numerical illustration of IR non-abelian cancellations

In Fig. 4 we show the distribution $f(a_i, \alpha_i)$ for two gluons (averaged over the gluon azimuthal angles) for fixed $x = 1 - \alpha_1 - \alpha_2 = 0.3$, from gluonic pair production graph (Vg) and its interference with bremsstrahlung (Yg = Yg1 + Yg2), see also Table II. Contributions from all diagrams are written explicitly in Table II. In the plots on Figs. 4–7 we omit their colour coefficients. The gluonic pair production graph (Vg) has a strong peak along the line of equal rapiditites $a_1 = a_2$ originating from u/a^2 factor (off-shell gluon propagator). In addition, this diagram features in the plot triangular

[3] The doubly-logarithmic IR divergence after phase space integration is $\sim \ln^2 \frac{1}{\delta}$, where δ is cut-off variable, see *Introduction.*

infinite plateau between the lines of equal rapiditites $a_1 = a_2$ and the line of equal virtualities $v_1 = v_2$. It is exactly this triangular plateau which upon integration leads to $\ln^2(1/\delta)$. Note, that in Figs. 4–7 we use variables $\log(a_1/a_2)$, contrary to the previous section, where we had $\log(v_1/v_2)$. This is why the line $v_1 = v_2$ is now diagonal in the plots.

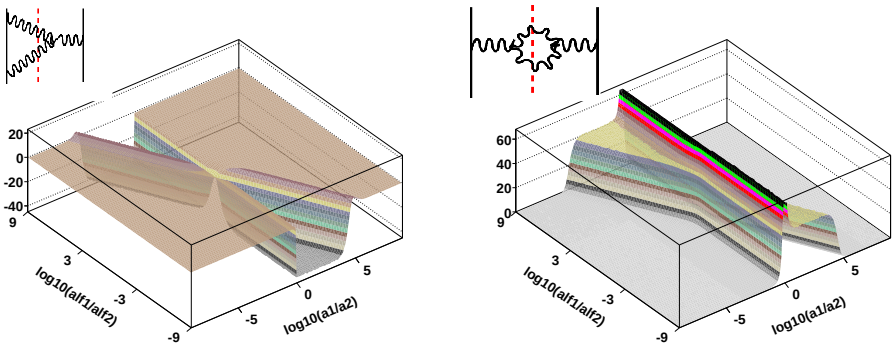


Fig. 4. Triple-gluon-vertex diagrams.

Very similar, but with opposite sign, doubly-logarithmic structure is present in the left plot in Fig. 4 from the interference graphs Yg1+Yg2, see also Table II. After adding the contributions from the above diagrams, see Fig. 5, the doubly-logarithmic structure between the lines: $a_1 = a_2$ and $v_1 = v_2$ disappears. What remains, is the single-log singularity, appearing in a familiar shape along the diagonal line of $v_1 = v_2$ (barely visible in the plots of Fig. 4) and collinear singularity represented here as an infinite ridge along the line of equal rapiditites. The latter is associated with zero effective mass of the gluon pair, k^2.

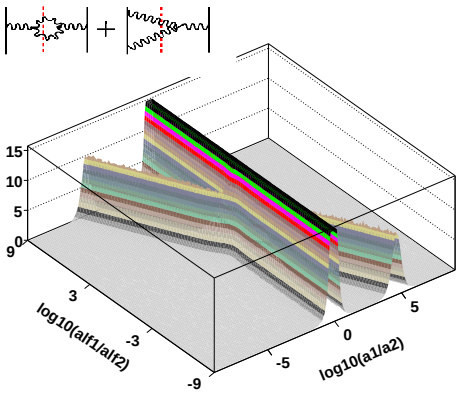


Fig. 5. Full cancellation of double-logs between the vacuum polarisation diagram and bremsstrahlung-vacuum polarisation interference.

The above cancellation is the well known "colour coherence effect", see for instance Ref. [6]. It reflects the fact that the gluonic pair production graph (Vg) pretends that in the triangular region it is the harder gluon carrying octet colour charge which is the "emitter", with the emission strength $\sim C_{\mathrm{F}}C_{\mathrm{A}}$. In reality, however, the emitter should be quark carrying triplet colour charge, with the almost twice weaker emission strength $\sim C_{\mathrm{F}}^2$. The role of the interference diagram Yg1+Yg2 is to correct for that and we see this to happen in the plot and in the formulas in Table II. One has to remember that this part of the plot is already populated with the bremsstrahlung diagrams of the previous section proportional to $\sim C_{\mathrm{F}}^2$.

What still remains in Fig. 5 is a singly-logarithmic structure along the diagonal line $v_1 = v_2$. Its presence is in principle allowed in the doubly-logarithmic Sudakov approximation. A more subtle analysis of the soft limit in QCD shows that it should also vanish and this phenomenon is often referred to as "eikonalisation", see for instance Ref. [7, 8]. The job of bringing back the proper soft limit of the two gluon distribution and eliminating remaining single-log structure is done by the crossed bremsstrahlung diagram Bx (in fact its $C_{\mathrm{F}}C_{\mathrm{A}}$ part), as it is shown in Figs. 6 and 7, see also Table II.

On right hand side of Fig. 6 the crossed-ladder diagram Bx is presented again. On the left hand side of Fig. 6 we show the result of adding the triple-gluon vertex diagrams (Vg+Yg1+Yg2). Both plots have a characteristic single-log structure, seen as an infinite ridge along the diagonal line of equal virtualities $v_1 = v_2$. However, Bx has opposite signs in its $C_{\mathrm{F}}C_{\mathrm{A}}$ colour coefficients. In Fig. 7 we see the result of adding all the above " non-abelian" diagrams. Bx enters with a minus sign. We see that the singly-logarithmic structure $\sim C_{\mathrm{F}}C_{\mathrm{A}}$ disappears completely in the soft limit $\alpha_i \to 0$. What is still present in the picture is the dominant $1/k^2 \sim u/a^2$ gluon pair mass singularity.

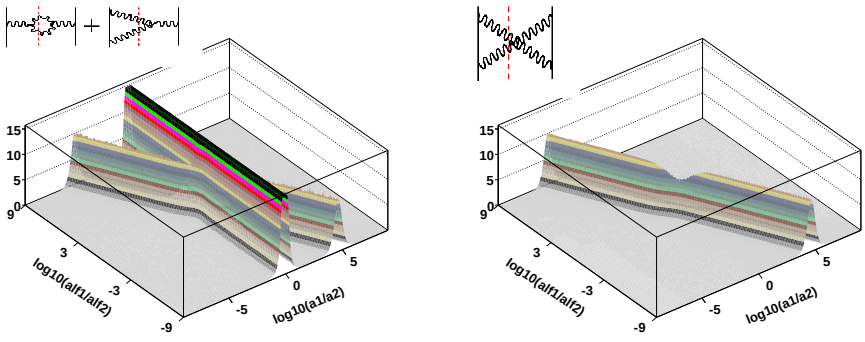


Fig. 6. The previous plot (left) and bremsstrahlung diagram (right).

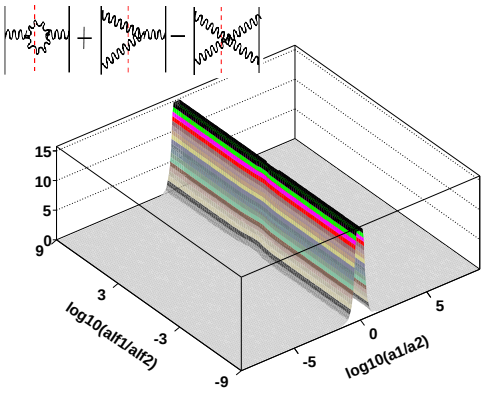


Fig. 7. $C_{\mathrm{F}}C_{\mathrm{A}}$ part of the sum (Vg+Yg − Bx).

The concluding plots are shown in Fig. 8. We presented there Feynman diagrams entering the kernel, both $\sim C_{\mathrm{F}}^2$ and $\sim C_{\mathrm{F}}C_{\mathrm{A}}$. In the left plot there are solely amplitude-squared diagrams (Br1, Br2 and Vg) and in the right plot all diagrams including interferences. We see explicitly the crucial role of "colour coherence effects" (the interference diagrams) in the cancellation of IR singularities. In the sum of all diagrams of interest we see that remaining structure lies on top of the LO doubly-logarithmic plateau. The plateau does not enter into the NLO kernel. It is cancelled by the counterterm required by the kernel definition, as discussed in Section 2.

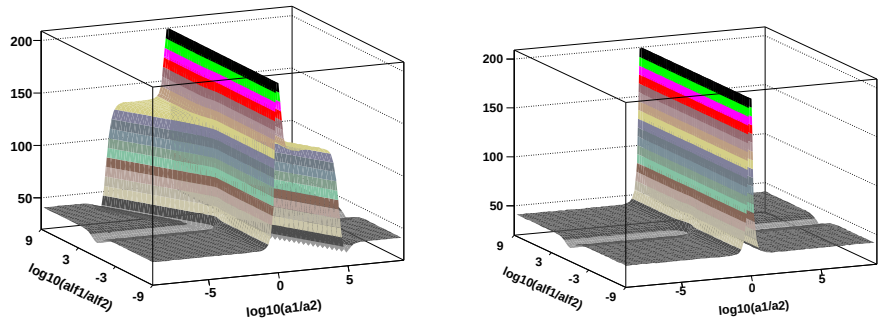


Fig. 8. Infra-red cancellations among NLO non-singlet diagrams. Br1, Br2 and Vg (left), Br1, Br2, Bx, Vg, Yg1 and Yg2 (right).

4. Conclusions

We examined the infra-red structure of the diagrams contributing to NLO non-singlet kernel in the unintegrated form.

We have shown the mechanisms of gauge cancellations occurring among different diagrams and the importance of "colour coherence effects" for this cancellations.

These effects in soft Sudakov limit are examined/discussed in both analytical and numerical form.

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