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ELEMENTARY DERIVATION OF THE LENSE–THIRRING PRECESSION

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An elementary pedagogical derivation of the Lense–Thirring precession is given based on the use of Hamilton vector. The Hamilton vector is an extra constant of motion of the Kepler/Coulomb problem related simply to the more popular Runge–Lenz vector. When a velocity-dependent Lorentz-like gravitomagnetic force is present, the Hamilton vector, as well as the canonical orbital momentum are no longer conserved and begin to precess. It is easy to calculate their precession rates, which are related to the Lense–Thirring precession of the orbit.

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1. Introduction

Every relativistic theory of gravitation must include a Lorentz-like force induced by a magnetic-type component of the gravitational field; the general theory of relativity by Einstein does so. More precisely, in its weak-field and slow-motion approximation the highly nonlinear Einstein field equations get linearized, thus resembling the linear equations of the Maxwellian electromagnetism. As a consequence, a magnetic-type component of the gravitational field appears, induced by the off-diagonal components $g_{0i}, i = 1, 2, 3$ of the spacetime metric tensor [1]. The role of the electric currents is played by mass-energy currents: in the case of a slowly rotating mass, a test particle geodesically moving far from it is acted upon by a non-central velocity-dependent Lorentz-like force. For other effects induced by the gravitomagentic field on the motion of gyroscopes, test particles, moving clocks and atoms, light rays see, for example, [2].

Recent years have seen increasing efforts, both from theoretical and observational points of view, towards a better comprehension of the post-Newtonian gravitomagnetic field: for a comprehensive recent overview see, for example, [3]. Thus, we feel it is not worthless to offer to the reader an elementary derivation of the precessional effects induced by the gravitomagnetic field on some Keplerian orbital elements of a test-particle, *i.e.* the so called Lense–Thirring effect [4], although a recent historical analysis [5] suggests that it should be more appropriately named Einstein–Thirring–Lense effect. For other derivations of such an effect see, for example, [6–10].

Our presentation is aimed for the first year physics students with limited experience in mathematics and theoretical methods of physics. We, therefore, give somewhat detailed exposition which an experienced physicist may find unnecessary long, but as our teaching experience shows such an exposition is helpful and necessary for newcomers in the field. We hope the material presented here will help students which just begin their physics education not only master some simple mathematical methods learned during the first year mechanics course but also feel the beauty of advanced topics which they will learn in detail later on.

2. Gravitomagnetism

Nowadays it is widely accepted that the correct theory of gravitation is provided by the Einstein's general theory of relativity [11] (for an excellent introduction for beginners see [12]; a classic textbook is [13]). However, the general theory of relativity is a highly non-linear theory. The spacetime in it is not merely a static background for physical processes; it is dynamic and affected by any contribution to the energy-momentum tensor of the system under investigation. Such a tensor enters in a prescribed manner the Einstein's equations which determine the ten components of the metric tensor. This metric tensor enters by itself the equations of motion of the system under study. There is no hope for a neophyte to struggle his/her way into this impossibly tangled up mess of non-linear jungles "unless months of study on the specialized terminology, procedures, and conventions of the general relativity theorist have been completed" [14].

However, virtually the large part of all the performed and/or proposed tests, aiming to observationally scrutinize the post-Newtonian regime, deal with weak fields and non-relativistic velocities. Therefore, we may expect that the full machinery of general relativity is not necessarily required to estimate the relevant post-Newtonian effects. Some linearized version of the theory will suffice to do the job much faster and physically in a more transparent manner [2, 14–16].

Although, in general, spacetime is dynamical and there is no natural way to split it into space plus time, for stationary spacetimes, like the one around the Earth, stationarity dictates a preferred way of how this splitting can be performed. There is no approximation in such slicing of spacetime into three-dimensional space plus one-dimensional time. It is just a new mathematical language convenient in stationary situations [16].

Under $3+1$ slicing, the spacetime metric tensor $g_{\mu\nu}$ naturally decomposes into several parts. In the case of weak gravity and non-relativistic velocities, this decomposition allows one to set up a remarkable analogy with electromagnetism [2, 14–16]. In particular, gravitational analogies of the electromagnetic scalar and vector potentials, Φ and $\vec{A}$, respectively, are determined by the time–time and time–space components of the spacetime metric:

$$\Phi = \tfrac{1}{2}\,(g_{00}-1)\,c^2\,, \qquad A_i = g_{0i}c^2\,,$$

where c is the speed of light. Gravitoelectric field $\vec{E}$ and gravitomagnetic field $\vec{H}$ are related to these potentials (in the Lorentz gauge [2]) in the usual way (up to an extra factor 4 which also appears in the Lorentz gauge condition):

$$\vec{E} = -\nabla\Phi - \frac{1}{4c}\frac{\partial\vec{A}}{\partial t}\,, \qquad \vec{H} = \nabla\times\vec{A}$$

and satisfy a gravitational analog of Maxwell's equations [2, 16, 17]

$$\begin{aligned} &\nabla\cdot\vec{E} = -4\pi G\rho\,, && \nabla\cdot\vec{H} = 0\,,\\ &\nabla\times\vec{E} = 0\,, && \nabla\times\vec{H} = 4\left[-4\pi G\,\frac{\rho\vec{v}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right]. \end{aligned} \tag{1}$$

Similarity with Maxwell's electromagnetic equations is apparent. The role of charge density is played by mass density, ρ, times Newton's gravitation constant, G. The mass current density, $G\rho\vec{v}$, with $\vec{v}$ as the velocity of the source mass, plays the role of the charge current density.

However, there are several important differences [16]. Gravity is mediated by a spin-two field and is attractive. In contrast, electromagnetism is mediated by a spin-one field and can be both attractive and repulsive. This difference leads to the extra minus signs in the source terms in (1). Another remnant of the tensor character of gravity is an extra factor, 4, in the equation for $\nabla\times\vec{H}$. No gravitational analog of the Faraday induction is present in (1). However, this is an artifact of restricting to the only first order terms in v/c of the gravitating masses in the above equations. At the second order, $(-1/c)(\partial\vec{H}/\partial t)$ term reappears in the equation for $\nabla\times\vec{E}$, but also some non-Maxwell-like terms are introduced elsewhere in the field equations [15, 16].

Once the gravitoelectric and gravitomagnetic fields are known, the force acting on a small test body of mass m is given by the formula which is analogous to the Lorentz force law [16]

$$\vec{F} = m\vec{E} + \frac{m}{c}\,\vec{v}\times\vec{H}\,.$$

More precisely, there is still another difference from electromagnetism as far as the equation of motion is concerned. So far nothing was said about the space–space components, g_{ij}, of the metric tensor. In general, they do not correspond to Euclidean space but to that of curved space. Just the space curvature effects are responsible for the classic general relativistic predictions for Mercury's perihelion precession and bending of light rays. In the presence of space curvature, we have the curvilinear equation of motion [14]

$$\frac{1}{\Gamma}\,\frac{d(\Gamma v_i)}{dt}-\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}\right)v^j v^k=E_i+\frac{1}{c}\,\left(\vec{v}\times\vec{H}\right)_i-\frac{1}{c}\frac{\partial H_i}{\partial t}\,, \tag{2}$$

where

$$\Gamma\approx\left(1+\frac{2\Phi}{c^2}-\frac{v^2}{c^2}\right)^{-1/2}$$

and, as usual, summation over repeated indexes is assumed.

In this paper, we are interested in tiny secular effects of Earth's rotation on a satellite orbit. Therefore, we will neglect the space curvature effects, as well as terms of the order of $(v/c)^2$ other than gravitomagnetic. Although these curvature effects are even more prominent than the effects we are interested in, they are well known. Therefore, for our goals, equation (2) takes the form

$$m\,\frac{d\vec{v}}{dt}=-\frac{\alpha}{r^2}\,\vec{n}+\frac{m}{c}\,\vec{v}\times\vec{H}\,, \tag{3}$$

where $\vec{r}=r\vec{n}$ is the radius-vector of the satellite of mass m, $\alpha=GmM$, where M is the Earth's mass, and $\vec{H}$ is the gravitomagnetic field caused by Earth's rotation.

The gravito-electromagnetic analogy implied by (1) allows us to find $\vec{H}$ [17]. It is well known that the magnetic moment

$$\vec{\mu}=\frac{1}{2c}\int\left[\vec{r}\times\vec{j}\right]\,dV\,,$$

related to the electric current density $\vec{j}$, creates a dipole magnetic field

$$\vec{H}=\frac{3\vec{n}\,(\vec{n}\cdot\vec{\mu})-\vec{\mu}}{r^3}\,.$$

Equations (1) indicate that the gravitational analog of the electric current density is $-4G\rho\vec{v}$. Therefore, the gravitational analog of the magnetic moment is

$$\vec{\mu}=-4G\,\frac{1}{2c}\int\rho\,\left[\vec{r}\times\vec{v}\right]\,dV=-2G\,\frac{\vec{S}}{c}\,,$$

where $\vec{S}=\int[\vec{r}\times\vec{v}]\,\rho dV$ is the rotating body's proper angular momentum.

Thus the gravitomagnetic field of the Earth is given by

$$\vec{H} = \frac{2G}{c}\,\frac{\vec{S} - 3\vec{n}\,(\vec{n}\cdot\vec{S})}{r^3}\,. \tag{4}$$

3. Larmor precession

It is instructive first to consider in detail a simpler case of a non-relativistic Coulomb atom slightly perturbed by a weak and uniform constant magnetic field $\vec{H}$. The equation of motion is

$$m\dot{\vec{v}} = -\frac{\alpha}{r^2}\,\vec{n} + \frac{e}{c}\,\vec{v}\times\vec{H}\,, \tag{5}$$

where now m and e are electron mass and charge respectively, and α is the strength of the corresponding electromagnetic coupling between the electron and the atom nucleus.

In the absence of perturbation, $\vec{H}=0$, equation (5) describes the electron motion on the ellipse. It is convenient to characterize the orientation of this ellipse by the angular momentum vector $\vec{L}=\vec{r}\times\vec{p}$, which is perpendicular to the orbit plane, and the Runge–Lenz vector

$$\vec{A} = \vec{v}\times\vec{L} - \alpha\,\vec{n}\,, \tag{6}$$

which is directed towards perihelion of the orbit and equals in magnitude to the orbit eccentricity times electromagnetic coupling α. The Runge–Lenz vector is an extra constant of motion of the Coulomb problem and its existence is related to the so called hidden symmetry of the problem [18, 19] which makes the Kepler/Coulomb problem an interesting testing ground for various algebraic and geometric methods [20–22]. Note that first integrals for the Kepler Problem, equivalents of the Runge–Lenz vector, were first obtained by Ermanno and Bernoulli many years before Gibbs devised his vectorial notation [23–25]. "The lack of recognition of Ermanno and Bernoulli parallels that of Aristarchos and makes one wonder if there is some sinister influence for those who are initiators in the field of the motions of the planets" [23].

The Runge–Lenz vector $\vec{A}$ and the angular momentum vector $\vec{L}$ are mutually perpendicular. Therefore, we can find the third vector $\vec{u}$ such that

$$\vec{A} = \vec{u}\times\vec{L}\,. \tag{7}$$

This vector is called the Hamilton vector [26–29] and it has the form

$$\vec{u} = \vec{v} - \frac{\alpha}{L}\,\vec{e}_\varphi\,, \tag{8}$$

where

$$\vec{e}_\varphi = \frac{1}{L}\,\vec{L}\times\vec{n} = \vec{l}\times\vec{n}\,, \qquad \text{with} \quad \vec{l} = \frac{\vec{L}}{L}\,, \tag{9}$$

is the unit vector in the direction of the polar angle φ in the orbit plane. Sometimes it is more convenient to characterize the orbit orientation by vectors $\vec{u}$ and $\vec{L}$.

Now let us return to (5) with the non-zero magnetic field $\vec{H}$. As the Lorentz force, in general, has a component outside the unperturbed orbital plane, it is evident that the motion will no longer be planar. However, for weak magnetic fields, the Lorentz force component in the orbital plane is much smaller than the binding Coulomb force. Therefore, it can cause only small perturbations of the elliptical orbit. The Lorentz force component perpendicular to the orbital plane is expected to cause this plane to turn slowly around the Coulomb field center. This intuitive picture of the perturbed motion suggests us to decompose the electron velocity in such a way

$$\vec{v} = \vec{v}\,' + \vec{\Omega}\times\vec{r}\,. \tag{10}$$

Here $\vec{v}\,'$ is the electron velocity relative to the instantaneous orbital plane, and the second term, $\vec{\Omega}\times\vec{r}$, is due to slow revolution of this plane with a small angular velocity $\vec{\Omega}$.

If we want the angular momentum $\vec{L}$ to be perpendicular to the instant orbital plane, we should replace $\vec{v}$ by $\vec{v}\,'$ in its definition:

$$\vec{L} = \vec{r}\times\vec{P}\,, \qquad \vec{P} = m\vec{v}\,'\,. \tag{11}$$

Analogously, the appropriately generalized Hamilton vector is

$$\vec{u} = \vec{v}\,' - \frac{\alpha}{L}\,\vec{e}_\varphi = \frac{\vec{P}}{m} - \frac{\alpha}{L}\,\vec{e}_\varphi\,, \tag{12}$$

where $\vec{e}_\varphi$ is still given by equation (9).

The Runge–Lenz vector, defined as earlier by (7), is no longer conserved. However, at instances when the particle is at perihelion of the instant ellipse, the Runge–Lentz vector still points towards the perihelion, as the following simple argument shows [30]. At the perihelion of the instant ellipse, the intrinsic velocity $\vec{v}\,'$ has no radial components and, therefore, is perpendicular to the radius vector $\vec{r}$. Then (12) shows that the Hamilton vector is also perpendicular to $\vec{r}$ and, according to (7), the Runge–Lenz vector $\vec{A}$ will be parallel to $\vec{r}$, that is pointing towards the perihelion.

If our intuitive picture of the orbit plane precession is correct, we should have

$$\dot{\vec{L}} = \vec{\Omega}\times\vec{L}\,. \tag{13}$$

On the other hand,

$$\dot{\vec{L}} = \dot{\vec{r}} \times \vec{P} + \vec{r} \times \dot{\vec{P}}\,. \tag{14}$$

As for the velocity $\vec{v} = \dot{\vec{r}}$, we have from (10)

$$\dot{\vec{r}} = \frac{\vec{P}}{m} + \vec{\Omega} \times \vec{r}\,. \tag{15}$$

While, up to the first order terms in $\vec{H}$ and $\vec{\Omega}$,

$$\dot{\vec{P}} = \frac{d}{dt}\left(m\vec{v} - m\vec{\Omega} \times \vec{r}\right) \approx -\frac{\alpha}{r^2}\vec{n} + \frac{e}{mc}\vec{P} \times \vec{H} - \vec{\Omega} \times \vec{P} - m\dot{\vec{\Omega}} \times \vec{r}\,. \tag{16}$$

Substituting (15) and (16) in (14), we get

$$\dot{\vec{L}} = \left(\vec{\Omega} \times \vec{r}\right) \times \vec{P} + \frac{e}{mc}\vec{r} \times \left(\vec{P} \times \vec{H}\right) - \vec{r} \times \left(\vec{\Omega} \times \vec{P}\right) - m\vec{r} \times \left(\dot{\vec{\Omega}} \times \vec{r}\right)\,,$$

or, after using $(\vec{\Omega} \times \vec{r}) \times \vec{P} = -\vec{r} \times (\vec{\Omega} \times \vec{P}) - \vec{\Omega} \times (\vec{P} \times \vec{r})$,

$$\dot{\vec{L}} = \vec{\Omega} \times \vec{L} - \vec{r} \times \left[\left(2\vec{\Omega} + \frac{e}{mc}\,\vec{H}\right) \times \vec{P}\right] - m\vec{r} \times \left(\dot{\vec{\Omega}} \times \vec{r}\right)\,.$$

As we see, to get (13), it is sufficient to take

$$\vec{\Omega} = -\frac{e}{2mc}\,\vec{H}\,. \tag{17}$$

Now let us calculate (note that $\dot{r} = \vec{v} \cdot \vec{n}$)

$$\dot{\vec{n}} = \frac{\vec{v}}{r} - \frac{\vec{r}(\vec{v} \cdot \vec{n})}{r^2} = \frac{1}{r}\,(\vec{n} \times \vec{v}) \times \vec{n}\,.$$

Substituting

$$\vec{v} = \frac{\vec{P}}{m} + \vec{\Omega} \times \vec{r}$$

and using

$$\left(\vec{n} \times \vec{P}\right) \times \vec{n} = \frac{L}{r}\,\vec{e}_\varphi\,, \qquad \left[\vec{n} \times \left(\vec{\Omega} \times \vec{n}\right)\right] \times \vec{n} = \vec{\Omega} \times \vec{n}\,,$$

we get

$$\dot{\vec{n}} = \vec{\Omega} \times \vec{n} + \frac{L}{mr^2}\,\vec{e}_\varphi\,. \tag{18}$$

Analogously,

$$\dot{\vec{e}}_\varphi = \dot{\vec{l}} \times \vec{n} + \vec{l} \times \dot{\vec{n}} = \left(\vec{\Omega} \times \vec{l}\right) \times \vec{n} + \vec{l} \times \left(\vec{\Omega} \times \vec{n}\right) + \frac{L}{mr^2}\,\vec{l} \times \vec{e}_\varphi\,.$$

But $(\vec{\Omega}\times\vec{l})\times\vec{n}+\vec{l}\times(\vec{\Omega}\times\vec{n})=-\vec{\Omega}\times(\vec{n}\times\vec{l})=\vec{\Omega}\times\vec{e}_\varphi$ and $\vec{l}\times\vec{e}_\varphi=-\vec{n}$. Therefore,

$$\dot{\vec{e}}_\varphi=\vec{\Omega}\times\vec{e}_\varphi-\frac{L}{mr^2}\vec{n}\ . \tag{19}$$

At last, by using (5) for $\dot{\vec{v}}$, we get

$$\dot{\vec{v}}'=-\frac{\alpha}{mr^2}\,\vec{n}+\frac{e}{mc}\,\vec{v}\times\vec{H}-\vec{\Omega}\times\vec{v}=\vec{\Omega}\times\vec{v}-\frac{\alpha}{mr^2}\vec{n}\,.$$

But, up to first order terms in $\vec{\Omega}$, $\vec{\Omega}\times\vec{v}\approx\vec{\Omega}\times\vec{v}'$. Therefore,

$$\dot{\vec{v}}'\approx\vec{\Omega}\times\vec{v}'-\frac{\alpha}{mr^2}\vec{n}\,. \tag{20}$$

Having (20) and (19) at hand, it is easy to find

$$\dot{\vec{u}}\approx\vec{\Omega}\times\vec{v}'-\frac{\alpha}{L}\,\vec{\Omega}\times\vec{e}_\varphi=\vec{\Omega}\times\vec{u}\,.$$

Therefore, the Hamilton vector precesses with the same angular velocity (17) as the angular momentum, and this angular velocity can be considered as related to the precession of the instantaneous ellipse as the whole.

As the final remark, note that $\vec{\mathcal{A}}=1/2\,\vec{H}\times\vec{r}$ is the vector-potential corresponding to the uniform magnetic field $\vec{H}$. Therefore,

$$\vec{P}=m\vec{v}-m\,\vec{\Omega}\times\vec{r}=m\vec{v}+\frac{e}{c}\vec{\mathcal{A}}$$

is just the canonical momentum of the non-relativistic electron in the magnetic field $\vec{H}$.

4. Lense–Thirring precession

It is well known [31] that the vector-potential created by a magnetic moment $\vec{\mu}$ is

$$\vec{\mathcal{A}}^{(\mu)}=\frac{\vec{\mu}\times\vec{r}}{r^3}\,. \tag{21}$$

Therefore, by making the change $\vec{\mu}\to-\frac{2G}{c}\,\vec{L}'$ in (21), we get the gravitational analog of the vector potential for the gravitomagnetic field (4) of the Earth with angular momentum $\vec{L}'$:

$$\mathcal{A}=\frac{2G}{cr^3}\,\vec{r}\times\vec{L}'\,. \tag{22}$$

To recast the canonical momentum,

$$\vec{P}=m\,\vec{v}+\frac{m}{c}\mathcal{A}\,,$$

in the form $\vec{P} = m\,\vec{v} - m\,\vec{\Omega} \times \vec{r}$, which corresponds to the decomposition of velocity (10), we could take

$$\vec{\Omega} = \frac{2G}{c^2 r^3}\,\vec{L}\,' \,. \tag{23}$$

Then the gravitomagnetic field (4) can be rewritten as

$$\frac{\vec{H}}{c} = \vec{\Omega} - 3\vec{n}\,\left(\vec{\Omega}\cdot\vec{n}\right)\,. \tag{24}$$

By using (24) and equation of motion (3), we get

$$\dot{\vec{P}} = -\frac{\alpha}{r^2}\,\vec{n} + 2m\,\vec{v}\times\vec{\Omega} - 3m\,\left(\vec{\Omega}\cdot\vec{n}\right)\,\vec{v}\times\vec{n} - m\,\dot{\vec{\Omega}}\times\vec{r}\,. \tag{25}$$

From (23), we find

$$\dot{\vec{\Omega}} = -3\frac{\vec{\Omega}}{r}\,(\vec{v}\cdot\vec{n})\,,$$

and (25) takes the form

$$\dot{\vec{P}} = -\frac{\alpha}{r^2}\,\vec{n} + 2m\,\vec{v}\times\vec{\Omega} - 3m\,\left[\vec{v}\,\left(\vec{\Omega}\cdot\vec{n}\right) - \vec{\Omega}\,(\vec{v}\cdot\vec{n})\right]\times\vec{n}\,.$$

However,

$$\left[\vec{v}\,\left(\vec{\Omega}\cdot\vec{n}\right) - \vec{\Omega}\,(\vec{v}\cdot\vec{n})\right]\times\vec{n} = \left[\vec{n}\times\left(\vec{v}\times\vec{\Omega}\right)\right]\times\vec{n} = \vec{v}\times\vec{\Omega} - \vec{n}\,\vec{n}\cdot\left(\vec{v}\times\vec{\Omega}\right)\,,$$

and we end up with the equation

$$\dot{\vec{P}} = -\frac{\alpha}{r^2}\,\vec{n} + m\,\vec{\Omega}\times\vec{v} + 3m\,\vec{n}\,\vec{n}\cdot\left(\vec{v}\times\vec{\Omega}\right)\,. \tag{26}$$

Up to the first order in the small parameter $\vec{\Omega}$, (26) can be rewritten as

$$\dot{\vec{P}} \approx -\frac{\alpha}{r^2}\,\vec{n} + \vec{\Omega}\times\vec{P} + 3\,\vec{n}\;\vec{\Omega}\cdot\left(\vec{n}\times\vec{P}\right)\,. \tag{27}$$

Now it is easy to find

$$\dot{\vec{L}} = \left(\frac{\vec{P}}{m} - \frac{1}{c}\,\mathcal{A}\right)\times\vec{P} + \vec{r}\times\left(\vec{\Omega}\times\vec{P}\right) = -\left(\vec{r}\times\vec{\Omega}\right)\times\vec{P} + \vec{r}\times\left(\vec{\Omega}\times\vec{P}\right)\,. \tag{28}$$

But $\vec{r}\times(\vec{\Omega}\times\vec{P}) + \vec{P}\times(\vec{r}\times\vec{\Omega}) = -\vec{\Omega}\times(\vec{P}\times\vec{r})$ and (28) indicates that the canonical angular momentum vector $\vec{L}$ really precesses with angular velocity $\vec{\Omega}$:

$$\dot{\vec{L}} = \vec{\Omega}\times\vec{L}\,. \tag{29}$$

We are interested in the secular changes of the orbital parameters. Therefore, it makes sense to average $\vec{\Omega}$ in (29) over fast orbital motion:

$$\vec{\Omega} \to \left\langle \vec{\Omega} \right\rangle = \frac{2G}{c^2}\,\vec{L}'\left\langle \frac{1}{r^3} \right\rangle . \tag{30}$$

For the desired accuracy, we can average $1/r^3$ in (30) over the unperturbed orbit

$$\frac{p}{r} = 1 + e\cos\varphi\,,$$

where p and e are the semi-latus rectum and eccentricity of the orbit. But for the unperturbed orbit

$$dt = \frac{mr^2}{L}\,d\varphi\,, \tag{31}$$

and we get

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{T}\int\limits_0^T \frac{dt}{r^3} = \frac{m}{LTp}\int\limits_0^{2\pi}(1+e\cos\varphi)\,d\varphi = \frac{2\pi m}{LTp}\,.$$

Integrating (31) over the complete orbital period T, we get

$$\frac{LT}{m} = 2S = 2\pi a^2\sqrt{1-e^2}\,,$$

where S is the area of the ellipse and a is its semi-major axis. On the other hand, $p = a(1-e^2)$ and we finally get

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{a^3(1-e^2)^{3/2}}\,. \tag{32}$$

Therefore, the averaged angular velocity of the precession is

$$\left\langle \vec{\Omega} \right\rangle = \frac{2G}{c^2a^3(1-e^2)^{3/2}}\,\vec{L}'\,. \tag{33}$$

It is not difficult to check that relations (18) and (19) remain valid if $\vec{\Omega}$ in these formulas is given by (23). Then, after using (27) and (19), we easily find

$$\dot{\vec{u}} \approx \vec{\Omega}\times\vec{u} + 3\,\frac{\vec{n}}{mr}\,\vec{\Omega}\cdot\vec{L} = \vec{\Omega}\times\vec{u} + 3\frac{2G}{mc^2}\,\frac{\vec{n}}{r^4}\,\vec{L}'\cdot\vec{L}\,. \tag{34}$$

Therefore, the Hamilton vector does not precess with angular velocity $\vec{\Omega}$. However, we should average (34) over the fast orbital motion. The first term simply gives $\langle \vec{\Omega} \times \vec{u} \,\rangle = \langle \vec{\Omega} \rangle \times \; \vec{u}$ because $\vec{u}$ is a slowly changing vector and we can assume that it does not change over time scales comparable to the orbital period T. As for the second term, we can use $\vec{n} = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}$ and get

$$\left\langle \frac{\vec{n}}{r^4} \right\rangle = \frac{m}{TLp^2} \int\limits_0^{2\pi} \left(\cos\varphi\,\vec{i} + \sin\varphi\,\vec{j} \right) (1 + e\cos\varphi)^2 \, d\varphi = \frac{2\pi m e}{TLp^2}\,\vec{i}\,. \tag{35}$$

However,

$$\vec{i} = \vec{j} \times \vec{k} = \frac{\vec{u}}{u} \times \frac{\vec{L}}{L}\,, \qquad \frac{e}{puL} = \frac{1}{p\alpha} = \frac{m}{L^2}\,,$$

and (35) takes the form

$$\left\langle \frac{\vec{n}}{r^4} \right\rangle = \left\langle \frac{1}{r^3} \right\rangle \frac{m}{L^2}\,\vec{u} \times \vec{L}\,. \tag{36}$$

Therefore, after averaging, (34) changes to

$$\dot{\vec{u}} \approx \vec{\Omega}_{\rm LT} \times \vec{u}\,, \tag{37}$$

where

$$\vec{\Omega}_{\rm LT} = \frac{2G}{c^2 a^3 (1 - e^2)^{3/2}} \left[\vec{L}' - 3\vec{l} \left(\vec{l} \cdot \vec{L}' \right) \right]. \tag{38}$$

As we see, the secular precession of the elliptic orbit contains two terms: the precession of the orbital plane with the angular velocity $\langle \vec{\Omega} \rangle$ around the central body's angular momentum $\vec{L}'$, and the precession within the orbital plane with the angular velocity $\vec{\Omega}_{\rm LT} - \langle \vec{\Omega} \rangle$ around the angular momentum $\vec{L}$. The magnitudes of the angular momentum $\vec{L}$ and the Hamilton vector $\vec{u}$, and therefore, the orbital parameters such as the eccentricity and the semi-major axis, remain unchanged to the first order of the perturbation theory.

5. Concluding remarks, Skovoroda's principle and all that

Lev Borisovich Okun cites [32] eighteenth century Ukrainian philosopher Grigory Skovoroda as the author of the remarkable principle, which Okun's PhD adviser Isaak Yakovlevich Pomeranchuk used to quote: "Thanks God: All what is relevant is simple, all what is not simple is not relevant." The formal analogy between weak field low velocity general relativity and Maxwellian electrodynamics is a simple and elegant way to illuminate a whole class of interesting physical phenomena dubbed gravitomagnetism. Lense–Thirring precession is one such example. However, it should be kept in mind that the analogy is only formal and sometimes can lead to strange and erroneous conclusions if we forgot about rather strong limitations under which the approximation underlying the analogy is valid [33].

Frame dragging is maybe more appropriate interpretation of the Lense–Thirring effect not restricted to the weak field limit. However, "whereas 'frame dragging' is a very catchy appellation" [34] its meaning is not easy to explain to introductory level students. Therefore, for them and not only explaining the Lense–Thirring effect by analogy with Larmor precession remains a relevant and simple option.

The use of Hamilton or Runge–Lenz vector greatly simplifies the discussion. However, a great deal of vector algebra is still needed in either cases, as the previous chapters illustrate. This is in contrast with situation in perihelion precession under central force perturbations where the use of the Hamilton vector trivializes the problem [35]. In fact, the amount of vector algebra is a price we should pay for our desire to keep the approach elementary and accessible in introductory mechanics course. If more advanced background in analytical mechanics is assumed, much more elegant and technically simple way is provided by the use of the Poisson brackets [36] or, alternatively, by Hamilton equations of motion [37].

In terms of the standard Poisson brackets, we have

$$\dot{L}_i = \{L_i, \mathcal{H}\}\,, \qquad \dot{u}_i = \{u_i, \mathcal{H}\}\,, \tag{39}$$

where the Hamiltonian has the form $\mathcal{H} = \mathcal{H}_0 + \delta\mathcal{H}$ with $\delta\mathcal{H} = \vec{\Omega}\cdot\vec{L}$ [37] and $\{L_i, \mathcal{H}_0\} = \{u_i, \mathcal{H}_0\} = 0$.

The Poisson brackets are easy to calculate by using the Leibniz rule $\{f, gh\} = \{f, g\}h + g\{f, h\}$ and the fundamental Poisson brackets

$$\begin{aligned}
&\{r_i, r_j\} = 0\,, && \{P_i, P_j\} = 0\,, && \{r_i, P_j\} = \delta_{ij}\,,\\
&\{L_i, r_j\} = \varepsilon_{ijk}\, r_k\,, && \{L_i, P_j\} = \varepsilon_{ijk}\, P_k\,, && \{L_i, L_j\} = \varepsilon_{ijk}\, L_k\,,\\
&\{L_i, f(\vec{r}, \vec{p})\} = 0\,, && \{P_i, f(\vec{r}, \vec{p})\} = -\frac{\partial f(\vec{r}, \vec{p})}{\partial r_i}\,, &&
\end{aligned} \tag{40}$$

where $f(\vec{r}, \vec{p})$ is any scalar function of its arguments. It follows from (40) that

$$\begin{aligned}
&\{L_i, \Omega_j\} = 0\,, && \{L_i, e_{\varphi j}\} = \varepsilon_{ijk}\, e_{\varphi k}\,,\\
&\{\Omega_i, e_{\varphi j}\} = 0\,, && \{P_i, \Omega_j\} = 3\,\frac{\Omega_j}{r}\, n_i\,.
\end{aligned} \tag{41}$$

Therefore,

$$\dot{L}_i = \{L_i, \Omega_j\}\, L_j + \{L_i, L_j\}\, \Omega_j = \varepsilon_{ijk}\, L_k \Omega_j\,,$$

which is equivalent to (29).

Besides, for any vector of the form $\vec{B} = f_1(\vec{r}, \vec{p})\,\vec{r} + f_2(\vec{r}, \vec{p})\,\vec{P}$, with any scalar functions f_1 and f_2, and in particular for vectors $\vec{e}_\varphi$ and $\vec{u}$, we will have

$$\{L_i, B_j\} = \varepsilon_{ijk}\, B_k\,,$$

and we get immediately

$$\dot{u}_i = \{u_i, L_j\}\varOmega_j + \frac{1}{m}\{P_i, \varOmega_j\}L_j = \varepsilon_{ijk}\, u_k \varOmega_j + \frac{3n_i}{mr}\,\vec{\varOmega}\cdot\vec{L}\,,$$

which is equivalent to (34).

As we see, in combination with some background in Hamiltonian mechanics, the use of Hamilton vector again makes the exposition rather trivial, in complete agreement with the Skovoroda's principle.

In our discussions canonical variables played an important role, as they are in accord with the intuitive physical picture of precessing orbital plane. However, the use of canonical variables are not mandatory for the perturbation theory discussion of the Lense–Thirring effect. The standard perturbation theory in celestial mechanics is based on variation-of-parameters method which has some inherent gauge freedom [38]. Indeed, the unperturbed Keplerian ellipse is determined by six parameters

$$\vec{r} = \vec{f}(C_1, C_2, C_3, C_4, C_5, C_6, t)\,. \tag{42}$$

In the role of these parameters one can take, for example, three Euler angles which determine the orbit plane orientation, the semi-major axis and eccentricity of the orbit, and the so called mean anomaly at an epoch, which determines the initial position of the body. Another obvious choice is the initial values of the position and velocity vectors of the body, and many other sets (Delaunay, Poincare, Jacobi, Hill) can be found in the literature [39].

To solve the perturbed equation

$$m\ddot{\vec{r}} = -\frac{\alpha}{r^2}\,\vec{n} + \vec{F}\,, \tag{43}$$

with $\vec{F}$ as a small perturbation, by the variation-of-parameters method, one can assume that the solution still has the form (42) but C_i are no longer constant. However, three scalar equations (43) are not sufficient to determine six unknown functions $C_i(t)$. We need three auxiliary conditions on them and the freedom in choosing of these auxiliary conditions is just the gauge freedom mentioned above. Usually the gauge fixing is achieved by the Lagrange constraint

$$\sum_{i=1}^{6} \dot{C}_i\,\frac{\partial \vec{f}}{\partial C_i} = 0\,. \tag{44}$$

Under this condition

$$\dot{\vec{r}} = \frac{\partial \vec{f}}{\partial t}\,.$$

That is, the velocity is the same function of the parameters C_i as in the absence of perturbations and the instantaneous ellipse, determined by these parameters, is tangent to the real trajectory. Correspondingly, the instantaneous orbital parameters C_i so determined are called osculating.

However, not all choices of orbital parameters C_i are compatible to the Lagrange constraint (44). For example, in previous chapters we have choose the canonical angular momentum and the Hamilton vector to characterize the orbit shape and orientation. When velocity dependent perturbations are present, such parameters are not osculating, as is evident from the ascribed physical picture of the combined motion.

The canonical perturbation theory in celestial mechanics was developed in [8]. Again the corresponding orbital elements are not osculating if the perturbation depends on velocity. In fact this is a general property: under velocity-dependent disturbances, canonicity and osculation are not compatible [38].

The problem of orbit perturbations can be solved in any gauge. At that some gauges are more convenient, because if the gauge corresponds to the real physical picture of the perturbed motion, the formalism simplifies. Some examples are given in [8] when the standard perturbation theory with osculating elements is cumbersome while the canonical perturbation theory is more elegant.

The Lense–Thirring effect can be considered either in the canonical perturbation theory, or with the osculating orbital elements [8], or in any other convenient gauge. At that the corresponding orbital elements can differ considerably. Of course, the real physical quantities, like position and velocity, do not depend on the gauge used after the initial values are accounted correctly [8]. However, sometimes the orbital elements, like inclination angle, are also considered as physically real. In such cases care should be taken to relate the measured quantities to the orbital parameters used in the perturbation theory. Without this care a confusion can arise when two different mathematically correct approaches give seemingly different results which are in fact equivalent [6].

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