SOLVABLE POTENTIALS WITH POSITION-DEPENDENT EFFECTIVE MASS AND CONSTANT MASS SCHRÖDINGER EQUATION

H. Panahi†, Z. Bakhshi

Department of Physics, University of Guilan
Rasht 51335-1914, Iran

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Using the point canonical transformation method, we show that a large class of solvable potentials with Position-Dependent Effective Mass (PDEM) can be obtained by using the internal functions which are introduced by Levai for solvable potentials with constant mass. We also obtain the explicit expressions for some of these solvable potentials and show that their eigenfunctions can be obtained in terms of the known special functions such as Jacobi, generalized Laguerre and Hermit polynomials.

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1. Introduction

In recent years, quantum mechanical systems with a Position-Dependent Effective Mass (PDEM) have attracted a lot of attention due to their applications in condensed matter physics, nuclear physics, semiconductor theory and other related fields [1–8]. In theoretical researches, many different methods have been used in the study of systems with constant mass such as the Lie algebraic techniques [9, 10], point canonical transformation [11–13], factorization method [14] and supersymmetric quantum mechanics together with shape invariance techniques [11, 15–17]. During the last few years, some of these developments have been generalized to the systems with PDEM and a number of interesting results has been produced [18–29]. For systems with constant mass, Levai used the point canonical transformation approach and calculated eigen spectrum of a large class of exactly solvable potentials, by transforming the Schrödinger equation into the second order differential equation which has solutions of the special functions [12]. By using of the

† Corresponding author: t-panahi@guilan.ac.ir

Levai approach in systems with PDEM, it is seen that there are two unknown functions instead of only one unknown function in the constant mass case, which complicate its study [21, 27–29]. The first unknown function is the position dependent effective mass $M(x)$ and the second is the coordinate transformation function $g(x)$ which is named as internal function. In the above references, the authors have tried to obtain solvable potentials with PDEM based on simplified choices between effective mass function and internal function as $M(x) = \lambda {g'}^2$, $M(x) = \lambda g'$ and $M(x) = \frac{\lambda}{g'}$, where λ is a constant parameter.

Now, the main purpose of this work is to obtain a large class of solvable potentials by introducing some choices between $M(x)$ and $g(x)$ such that potential functions and energy spectrum for each specific choice can be calculated by orthogonal polynomials where are the same as those involved in constant mass formalism. In other word, we establish some connections between the derived differential equation with position dependent effective mass and the Schrödinger equation of constant mass by using of the corresponding internal functions which introduced in Ref. [12] for constant mass.

This paper is organized as follows: in Section 2, we will show that the Schrödinger equation with position-dependent effective mass can be solved by solutions of the Schrödinger equation with constant mass which are as orthogonal polynomials. We will introduce some relations between the effective mass $M(x)$ and the internal function $g(x)$ for obtaining a large class of solvable potentials with PDEM. In the next sections, some of these potentials will be obtained and the spectrum of them will be written in terms of Jacobi general Laguerre and Hermit polynomials. The paper ends with a brief conclusion in Section 6.

2. Position-dependent effective mass equation and point canonical transformation

The general Hermitian PDEM Hamiltonian, initially proposed by Von Roos [1], in terms of three ambiguity parameters α, β, γ and in natural units ($\hbar = 2m_0 = 1$) is given by

$$\left[-\tfrac{1}{2}\left(M^\alpha(x)\frac{d}{dx}M^\beta(x)\frac{d}{dx}M^\gamma(x) + M^\gamma(x)\frac{d}{dx}M^\beta(x)\frac{d}{dx}M^\alpha(x)\right) + V(x)\right] \times \psi(x) = E\psi(x)\,, \tag{2.1}$$

where $M(x)$ is the dimensionless form of the function $m(x) = m_0 M(x)$. The ambiguity parameters are constrained by the condition $\alpha + \beta + \gamma = -1$ and we get the following time-independent Schrödinger equation from (2.1)

$$H\psi(x) \equiv \left[-\frac{d}{dx}\frac{1}{M(x)}\frac{d}{dx} + V_{\rm eff}(x)\right]\psi(x) = E\psi(x)\,, \tag{2.2}$$

where the effective potential

$$V_{\rm eff}(x) = V(x) + \frac{1}{2}(\beta+1)\frac{M''}{M^2} - [\alpha(\alpha+\beta+1)+\beta+1]\,\frac{M'^2}{M^3} \qquad (2.3)$$

depends on $M(x)$ and its first and second derivatives. Using the following coordinate transformation in (2.2) as [12]

$$\psi(x) = f(x)F(g(x))\,, \qquad (2.4)$$

where $F(g(x))$ is some special function and satisfies the following second order differential equation

$$\frac{d^2F}{dg^2} + Q(g)\frac{dF}{dg} + R(g)F(g) = 0\,, \qquad (2.5)$$

then we can calculate $Q(x)$ and $R(x)$, with respect to $E - V_{\rm eff}(x), M(x)$, $f(x), g(x)$ and their derivatives by comparing (2.5) with the result of inserting (2.4) into (2.2). The $f(x)$ function can be also calculated by $Q(x)$ and $R(x)$ expressions as [29]

$$f(x) \propto \left(\frac{M}{g'}\right)^{1/2} \exp\left(\frac{1}{2}\int (Q(u)du)\right), \qquad (2.6)$$

where the usual square-integrablity condition for bound-state wave functions should indeed be satisfied by the additional restriction $\frac{|\psi(x)|^2}{\sqrt{M(x)}} \to 0$ at the end points of definition interval of $V(x)$ to ensure the Hermiticity of H in the Hilbert space spanned by its eigenfunction [25]. Therefore, the $E - V_{\rm eff}(x)$ is obtained as

$$E - V_{\rm eff}(x) = \frac{g'^2}{M}\left(R - \frac{1}{2}Q' - \frac{1}{4}Q^2\right) + \frac{g'''}{2Mg'} - \frac{3}{4M}\left(\frac{g''}{g'}\right)^2 - \frac{M''}{2M^2} + \frac{3M'^2}{4M^3}\,. \qquad (2.7)$$

Existing E in the left-hand side of (2.7) induces a constant value in the right-hand side hence one should define the functions of $M(x)$ and $g(x)$ simultaneously for ensuring the presence of a constant on the right-hand side.

The choice of $M(x) = \lambda g'^2(x)$ has been used in Refs. [21,27,28] where λ is a constant parameter, also the possibilities of $M(x) = \lambda g'(x)$ and $M(x) = \frac{\lambda}{g'(x)}$ have been studied in Ref. [29] for generating of exact and quasi-exact solvable potentials, respectively. In present work, we use the constant mass formalism according to [12], and show that for differential equation (2.5) which can be solved in terms of known special functions, one can obtain the new relations between $M(x)$ and $g(x)$ such that a constant term is induced in right-hand side of (2.7). Therefore, we can obtain a large class of solvable potentials with PDEM by using the internal function which is introduced in [12]. In other word, for each internal function $g(x)$, we obtain the related effective mass $M(x)$, and then, the effective potential $V_{\text{eff}}(x)$, energy spectrum and wave function $\psi(x)$ can be calculated in terms of known special functions. We will use this formalism for Jacobi, generalized Laguerre and Hermit differential equation in Sections 3, 4 and 5, respectively.

3. Jacobi polynomial and solvable potentials

The expression of $Q(x)$ and $R(x)$ for differential equation (2.5) corresponded to Jacobi polynomial $P_n^{(\alpha,\beta)}$ are given by [12]

$$Q(g) = \frac{-\alpha+\beta}{1-g^2} - \frac{(2+\alpha+\beta)g}{1-g^2}, \tag{3.1}$$

$$R(g) = \frac{n(1+\alpha+\beta+n)}{1-g^2}, \tag{3.2}$$

where $\alpha, \beta > -1$ and $n = 0, 1, 2, \ldots$. Substituting (3.1) and (3.2) into (2.7), we get

$$\begin{aligned} E - V_{\text{eff}}(x) &= \frac{g'^2}{M(1-g^2)}[n(n+\alpha+\beta+1)] \\ &+ \frac{g'^2}{M(1-g^2)^2}\left[\frac{1}{2}(\alpha+\beta+2) - \frac{1}{2}(\beta-\alpha)^2\right] \\ &+ \frac{gg'^2}{M(1-g^2)^2}\left[\frac{1}{2}(\beta-\alpha)(\beta+\alpha)\right] \\ &- \frac{g^2g'^2}{M(1-g^2)^2}\left[\frac{1}{4}(\alpha+\beta)(\alpha+\beta+2)\right] \\ &+ \frac{g'''}{2Mg'} - \frac{3}{4M}\left(\frac{g''}{g'}\right)^2 - \frac{M''}{2M^2} + \frac{3M'^2}{4M^3}. \end{aligned} \tag{3.3}$$

In (3.3), as discussed before, we have to find some relations between $M(x)$ and $g(x)$ for obtaining a constant term in right-hand side. For instance, substituting $M(x) = \frac{1}{1-g^2(x)}$ in (3.3), we can generate a constant term on right-hand side of (3.3) by assuming

$$\frac{g'^2}{(1-g^2)} = C, \tag{3.4}$$

where (3.4) is the same differential equation of type PI which is obtained by Levai for constant mass solvable models [12].

Here, by choosing $C = -a^2$ in (3.4), where a is a real positive parameter we get

$$g(x) = i \sinh(ax)\,, \qquad -\infty < x < +\infty \tag{3.5}$$

which leads us to an effective mass from $M(x) = \frac{1}{1-g^2(x)}$ as

$$M(x) = \frac{1}{\cosh^2(ax)}\,. \tag{3.6}$$

Substituting the $g(x)$ and $M(x)$ into (3.3) we get

$$\begin{aligned}
&E - V_{\rm eff}(x) = a^2\left[\tfrac{3}{2} - \tfrac{1}{2}(\alpha+\beta+2) - \tfrac{1}{4}(\beta-\alpha)^2 - n(n+\alpha+\beta+1)\right] \\
&+a^2\left[\tfrac{3}{4} - n(n+\alpha+\beta+1) - \tfrac{1}{4}(\alpha+\beta)(\alpha+\beta+2)\right]\sinh^2(ax) \\
&-ia^2\left[\tfrac{1}{2}(\beta-\alpha)(\beta+\alpha)\right]\sinh(ax)\,.
\end{aligned} \tag{3.7}$$

Now we will obtain a term in (3.7) that is independent of x, it should be also noticed that the constant term has to contain n. When the constant term in (3.7) is different from the one containing n, we have to shift the n dependence to the constant term. This can be carried out by a transformation of the parameters so that these transformations determine the n dependence of the spectrum such that $E_{n=0} = 0$. After some calculations, we obtain the following expressions for energy eigenvalue and effective mass potential from (3.7)

$$E_n = a^2(s-n)^2 + \frac{\lambda^2 a^2}{(s-n)^2} - a^2 s^2 - \frac{\lambda^2 a^2}{s^2}\,, \tag{3.8}$$

$$\begin{aligned}
V_{\rm eff}(x) &= \left(s - \frac{1}{2}\right)a^2 - \frac{\lambda^2 a^2}{s^2} \\
&+ \left[\left(s + \frac{1}{2}\right)^2 a^2 - a^2\right]\sinh^2(ax) + 2a^2\lambda\sinh(ax)\,,
\end{aligned} \tag{3.9}$$

where $\alpha = s - n + i\bar{a}$, $\beta = s - n - i\bar{a}$ and $\bar{a} = \frac{\lambda}{s-n}$. Also, substituting $g(x)$, $M(x)$ and $Q(x)$ associated to Jacobi differential equation in the Eq. (2.6), we obtain the expression $f(x)$, and then using the Eq. (2.4), the eigenfunction of the above solvable potential based on Jacobi polynomials is given by

$$\begin{aligned}
\psi(x) &\propto \frac{1}{(-a^2)^{1/4}}\cosh^{s-n-1/2}(ax) \\
&\times \exp\left[-i\bar{a}\tanh^{-1}(i\sinh(ax))\right] P_n^{(s-n+i\bar{a},\,s-n-i\bar{a})}(i\sinh(ax))\,.
\end{aligned} \tag{3.10}$$

We have checked two conditions mentioned below the Eq. (2.6) and it is seen that the eigenfunction of (3.10) is a square integrable function at the end points of interval provided that s is restricted to the range of $s < 0$. Therefore the effective mass potential (3.9) has bound states and the choice of $M(x)$ function (3.6), yielding Eq. (3.10), ensures hermiticity of the Hamiltonian.

Of course, for $M(x) = \frac{1}{1-g^2(x)}$, we can consider in (3.3) other terms as a constant value and get different kinds of $g(x)$ functions, but it should be noticed that this does not mean that we can find all the possible functions, in fact, this is a convenient way to find some of them. Besides, one can deduce the other solvable models related to Jacobi polynomials by assuming $M(x) = 1 - g^2(x)$, $M(x) = \frac{g(x)}{1-g^2(x)}$, $M(x) = \frac{g^2(x)}{1-g^2(x)}$ and $M(x) = g(x)$. However, it should be pointed out that not every choice of $M(x)$ function, inducing a constant term on the r.h.s. of Eq. (2.7), automatically produces Hermitian solvable models. To see this, substituting $M(x) = 1 - g^2(x)$ in (3.3), we can make sure that the first term in the right-hand side of (3.3) gives a constant term as

$$\frac{g'^2(x)}{(1-g^2(x))^2} = C\,. \tag{3.11}$$

This equation is the same differential equation of type PI for constant mass cases [12], but it does not give an usual square-integral function or a Hermitian solvable model.

Therefore, a large class of solvable potentials with PDEM, corresponding to Jacobi polynomials, can be obtained by $g(x)$ functions listed in table 1 of Ref. [12], and one can obtain the explicit expressions for effective mass $M(x)$, effective potential $V_{\text{eff}}(x)$, energy eigenvalue E_n and the eigenfunction of the solvable potential corresponding to Jacobi polynomials in PDEM.

4. Generalized Laguerre polynomial and solvable potentials

When we choose $F(g(x))$ to be the generalized Lagurre polynomial $L_n^{\alpha}(x)$, the expressions of $Q(g)$ and $R(g)$ in (2.5) are given by [12]

$$Q(g) = \frac{1 - g(x) + \alpha}{g(x)}\,, \qquad R(g) = \frac{n}{g}\,, \tag{4.1}$$

where $n = 0, 1, 2, \ldots$ and $\alpha > -1$. As in previous section, substituting (4.1) into (2.7), we obtain

$$
\begin{aligned}
E - V_{\text{eff}}(x) \;=\; & \frac{(2n+\alpha+1)}{2M}\frac{g'^{\,2}}{g} - \frac{(\alpha+1)(\alpha-1)}{4M}\frac{g'^{\,2}}{g^2} \\
& - \frac{g'^{\,2}}{4M} + \frac{g'''}{2Mg'} - \frac{3}{4M}\left(\frac{g'''}{g'}\right)^2 - \frac{M''}{2M^2} + \frac{3M'^{\,2}}{4M^3}\,. \qquad (4.2)
\end{aligned}
$$

One can make a constant term in the right-hand side of Eq. (4.2), when $g(x)$ satisfies the equation

$$\frac{g'^{\,2}(x)}{g^2(x)} = C\,, \qquad (4.3)$$

with the assumption of $M(x) = g(x)$. As a special example, if $C = a^2$ with $a > 0$, we have

$$g(x) = \exp(-ax)\,, \qquad M(x) = \exp(-ax)\,, \qquad -\infty < x < +\infty\,. \qquad (4.4)$$

Inserting (4.4) into (4.2) we get

$$E - V_{\text{eff}}(x) = \frac{a^2(2n+\alpha+1)}{2} - \frac{a^2(\alpha+1)(\alpha-1)}{4\exp(-ax)} - \frac{a^2\exp(-ax)}{4}\,, \qquad (4.5)$$

by choosing $l = \alpha - \frac{1}{2}$, the energy eigenvalue and potential of PDEM system can be obtained as

$$
\begin{aligned}
E_n \;&=\; na^2\,, && (4.6)\\
V_{\text{eff}}(x) \;&=\; -\left(l+\frac{3}{2}\right)\frac{a^2}{2} - \frac{a^2\left(l+\frac{3}{2}\right)\left(l-\frac{1}{2}\right)}{4\exp(-ax)} - \frac{a^2\exp(-ax)}{4}\,. && (4.7)
\end{aligned}
$$

After some calculations, we obtain

$$\psi(x) \propto \frac{1}{(-a^2)^{\frac{1}{2}}}\exp\left[(-a)\frac{(l+\frac{3}{2})}{2}x\right]\exp\left(-\frac{1}{2}e^{-ax}\right)L_n^{(l+\frac{1}{2})}(\exp(-ax)). \qquad (4.8)$$

Similarily to Eq. (3.16), we have checked two conditions mentioned below the Eq. (2.6) and so, the Eq. (4.8) manifestly represents a square integrable function at the end points of interval and the hermiticity of Hamiltonian is also automatically satisfied, therefore the effective mass potential (4.7) has bound states.

As mentioned before, we can consider the other terms in Eq. (4.2), as a constant for $M(x) = g(x)$. Therefore we can obtain other solvable models with PDEM corresponding to generalized Lagurre polynomial. Other possibilities such as $M(x) = \frac{1}{g(x)}$, $M(x) = \frac{1}{g^2(x)}$ and $M(x) = g^2(x)$ can also be chosen which induce a constant term in right-hand side of (4.2). Of course, it should be noticed that, some of choices lead to the same differential equation for $g(x)$. For example, when $M(x)=\frac{1}{g(x)}$ is chosen and the second term in (4.2) is a constant, then the differential equation (4.3) is the same as for the choice $M(x)=g(x)$, or as the differential equation $g'^2(x)=C$ obtained for both assumptions $M(x)=\frac{1}{g(x)}$ and $M(x)=\frac{1}{g^2(x)}$ when the first and the second term in (4.2) are constant, respectively. Hence in the reminding of this section, we obtain another example of solvable potential with PDEM associated to generalized Lagurre polynomial by assuming $M(x)=\frac{1}{g(x)}$. When the second term in right-hand side of Eq. (4.2) suppose to be a constant value then we have

$$\frac{g'^2(x)}{g(x)}=C\,, \tag{4.9}$$

where for $C=2\omega$ (ω is a real positive parameter), it yields

$$g(x)=\frac{1}{2}\omega x^2\,,\qquad M(x)=\frac{2}{\omega x^2}\,,\qquad -\infty<x<+\infty\,. \tag{4.10}$$

Substituting Eqs. (4.10) into (4.2) we get

$$E-V_{\rm eff}(x)=\frac{17}{8}\omega+\frac{(2n+\alpha+1)}{2}\omega^2x^2-\frac{(\alpha+1)(\alpha-1)\omega}{2}-\frac{1}{8}\omega^3x^4\,. \tag{4.11}$$

Using the parameter definition $s=n+\frac{1}{2}\alpha$, we have

$$E_n = 2\omega\left[s^2-(s-n)^2\right], \tag{4.12}$$

$$V_{\rm eff}(x) = \frac{17}{8}\omega+2\omega s^2-\left(\frac{2s+1}{2}\right)\omega^2x^2+\frac{1}{8}\omega^3x^4\,, \tag{4.13}$$

$$\psi(x) \propto \frac{1}{(\omega x)^{1/2}}\left(\frac{\omega x^2}{2}\right)^{s-n}\exp\left(-\frac{1}{4}\omega x^2\right)L_n^{(2s-2n)}\left(\frac{1}{2}\omega x^2\right). \tag{4.14}$$

By checking two conditions mentioned below the Eq. (2.6), it is seen that the square integrability is satisfied for (4.14), therefore there are the bound states for (4.13).

5. Hermit polynomials and solvable potentials

In Eq. (2.5), the expressions of $Q(g)$ and $R(g)$ correspondent to Hermit polynomials $H_n(g)$ are given by [12]

$$Q(g)=-2g\,,\qquad R(g)=2n\,,\qquad (n=0,1,2,\ldots)\,. \tag{5.1}$$

Substituting (5.1) into (2.7) we get

$$E - V_{\rm eff}(x) = \frac{(2n+1)g'^2}{M} - \frac{g^2 g'^2}{M} + \frac{g'''}{2Mg'} - \frac{3}{4M}\left(\frac{g''}{g'}\right)^2 - \frac{M''}{2M^2} + \frac{3M'^2}{4M^3}\,. \tag{5.2}$$

By assuming $M(x) = \frac{1}{g^2(x)}$ and the first term in the right-hand side of Eq. (5.2) to be a constant value, we have the following differential equation

$$g^2 g'^2 = C\,. \tag{5.3}$$

Solving this equation yields

$$g(x) = \sqrt{2x}C^{1/4}\,, \tag{5.4}$$

$$M(x) = \frac{1}{2\sqrt{C}x}\,, \qquad 0 < x < +\infty\,. \tag{5.5}$$

Inserting the Eqs. (5.4) and (5.5) into (5.2) we get

$$E - V_{\rm eff}(x) = (2n+1)C - 2C^{3/2}x - \frac{C^{1/2}}{8x}\,. \tag{5.6}$$

Therefore energy eigenvalue, effective mass potential and its eigenfunction for $C = \omega$, where ω is a real positive parameter, are obtained as

$$E_n = 2n\omega\,, \tag{5.7}$$

$$V_{\rm eff}(x) = 2\omega^{3/2}x + \frac{\omega^{1/2}}{8x} - \omega\,, \tag{5.8}$$

$$\psi(x) \propto \frac{1}{\left(2\omega^{3/2}x\right)^{1/4}} \exp(-\sqrt{\omega}x) H_n\left[\omega^{1/4}\sqrt{2x}\right]. \tag{5.9}$$

In the same way, if $M(x) = g^2(x)$ and the second term in the right-hand side of the Eq. (5.2) suppose to be a constant value then the obtained effective potential does explicitly depend on quantum number n and we cannot obtain one solvable model.

6. Conclusion

Here, we have used the point canonical transformation method to obtain the solvable potentials with PDEM by assuming some relations between the effective mass $M(x)$ and the internal function $g(x)$ which is introduced for the constant mass case. We have shown that the eigenfunctions of all these potentials can be expressed in terms of known special functions. We have also shown that for ensuring the presence of a constant term on the right-hand side of Eq. (2.7), the possibilities such as $M(x) = 1 - g^2(x)$, $M(x) = \frac{1}{1-g^2(x)}$, $M(x) = \frac{g(x)}{1-g^2(x)}$, $M(x) = \frac{g^2(x)}{1-g^2(x)}$ and $M(x) = g(x)$ for Jacobi polynomials, $M(x) = \frac{1}{g(x)}$, $M(x) = \frac{1}{g^2(x)}$, $M(x) = g(x)$ and $M(x) = g^2(x)$ for generalized Laguerre polynomials, and $M(x) = g^2(x)$, $M(x) = \frac{1}{g^2(x)}$ for Hermit polynomials must be chosen. By choosing these relations, the internal function $g(x)$ is obtained by those differential equations which are given for the constant mass case.

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