ON DERIVATION OF METRIC FROM LIGHT DEFLECTION ANGLE IN THE STATIC, SPHERICALLY SYMMETRIC SPACETIME*

Szymon Sikora

Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Kraków, Poland

(Received September 14, 2009)

In this note general relativistic light deflection in the static, spherically symmetric spacetime is investigated as a means to determine the metric of the spacetime. It is shown that in this case derivation of spacetime metric is ambiguous from the light deflection angle only.

PACS numbers: 98.62.Sb, 04.20.Jb

1. Introduction

Gravitational light deflection can be used to estimate the mass of an astrophysical object, *e.g.* the galaxy cluster, in a method called gravitational lensing [2]. It is often assumed that gravitational field of deflecting object is relatively small, so it can be described in a linearised theory as a sum of point masses. In this paper we investigate more general situation in which we describe deflecting object as a static, spherically symmetric spacetime with metric:

$$ds^2 = -N^2(r)dt^2 + a(r)dr^2 + r^2 d\Omega^2 \,. \tag{1}$$

Note that for the metric (1), $4\,\pi\,r^2$ is the area of the sphere $r = \text{const.}$ which does not depend on coordinate choice, what gives r a geometrical meaning. It is not an easy problem to find deflection angle for a given metric. Sometimes advanced methods are needed [3]. An interesting question is the opposite one: can we derive the metric tensor from the deflection angle, in the case of static, spherically symmetric spacetime? Can we use the deflection angle to estimate unknown functions $N^2(r)$ and $a(r)$? If the answer was positive, we could find energy-momentum tensor of matter filling the spacetime by solving Einstein equations, which gives us the connection between deflection angle and the energy-momentum tensor. Unfortunately the answer is negative.

* Presented in the first part of the author's master thesis.

2. Calculations

To find the light trajectory one must solve the equation $U_\mu U^\mu = 0$ using the fact that in spacetime described by metric (1) the motion takes place in a plane (so one can determine the angle $\theta = \pi/2$), and exist two constant of motion:

$$E \equiv -N^2(r)\,\dot{t}\,, \qquad L \equiv r^2\,\dot{\phi}\,. \tag{2}$$

Using new variable $u := 1/r$ and the impact parameter $r_0 = L/E$ one can find equation:

$$\frac{du}{d\phi} = -\sqrt{\frac{1}{a(u)}\left(\frac{1}{r_0^2\,N^2(u)} - u^2\right)}\,. \tag{3}$$

Theorem 1 *In the static, spherically symmetric spacetime deflection angle as a function of impact parameter $\alpha(r_0)$ does not provide spacetime metric unambiguously.*

Proof Let us assume that $u(r_0,\phi)$ is the solution of the equation (3) for a given metric (1). Then the equation $u(r_0,\phi_\pm) = 0$ provides angles of trajectory asymptotes $\phi_\pm(r_0)$. The deflection angle is $\alpha(r_0) = \phi_+(r_0) - \phi_-(r_0) - \pi$. For an arbitrary function, $W(u) \neq 0$, the solution of equation $W(u(r_0,\phi))u(r_0,\phi) = 0$ gives identical angles $\phi_\pm(r_0)$. One must find the metric $d\widetilde{s}^2 = -\widetilde{N}^2(r)dt^2 + \widetilde{a}(r)dr^2 + r^2 d\Omega^2$ for which $W(u(r_0,\phi))u(r_0,\phi)$ satisfies equation (3):

$$\frac{d}{d\phi}(W(u)u) = -\sqrt{\frac{1}{\widetilde{a}(u)}\left(\frac{1}{r_0^2\,\widetilde{N}^2(u)} - (W(u)u)^2\right)}\,.$$

Therefore:

$$\frac{du}{d\phi} = -\sqrt{\frac{W^2(u)}{\widetilde{a}(u)\left(\frac{dW}{du}\,u + W(u)\right)^2}\left(\frac{1}{r_0^2\,W^2(u)\widetilde{N}^2(u)} - u^2\right)}\,.$$

The function $u(r_0,\phi)$ satisfies equation (3), as we assumed at the beginning, if and only if:

$$\widetilde{a}(u) = a(u)\frac{W^2(u)}{\left(\frac{dW}{du}\,u + W(u)\right)^2}\,, \qquad \widetilde{N}^2(u) = \frac{N^2(u)}{W^2(u)}\,. \tag{4}$$

If only denominator of the first expression is not zero, formulae (4) provide metric of spacetime $d\widetilde{s}^2$ in which deflection angle is identical to deflection angle in spacetime described by metric ds^2. □

3. Conclusions

The reason of an ambiguity of metric derivation is that deflection angle does not give information about the whole light trajectory but only about its asymptotes. Random choice of function $W(u)$ often leads to a spacetime filled by some exotic matter, *e.g.* with negative energy density. To use deflection angle in that kind of spacetimes as an useful observable, some additional assumptions about energy-momentum tensor are needed, *e.g.* energy conditions, fluid or dust form of matter.

REFERENCES

[1] S. Sikora, Master Thesis, Jagellonian University, Kraków 2009, in Polish.

[2] M. Bartelmann, P. Schneider, *Phys. Rep.* **340**, 291 (2001) `[astro-ph/9912508]`.

[3] P. Amore, S. Arceo, *Phys. Rev.* **D73**, 083004 (2006).

[4] N. Straumann, *General Relativity and Relativistic Astrophysics*, Springer–Verlag 1984, p. 173.

[5] V. Perlick, *Phys. Rev.* **D69**, 064017 (2004).