FLAT SPACETIME IN A CAPSULE

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We propose a parallel introduction to Galilean and Einsteinian relativity based on the causal structure and inertial motions. Galilean and Poincaré transformations, as objects secondary to the geometrical structure, are left aside.

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1. Introduction

In this article we propose a highly structured and logical approach to the fundamentals of Special Relativity (SR) based on its causal structure and relativity of inertial motions. For comparison and better understanding we parallelly build the Galilean Spacetime (GS) on similar ideas. We indicate that the causal structure determines the metric structure of SR spacetime uniquely, which is not the case for the choice of Euclidean metric in the Galilean case.

We want to stress the point that the Galilean and Lorentz (Poincaré) transformations are objects secondary to the geometric structure of spacetime: they are affine mappings leaving this structure invariant. We regard basing the introduction to SR on these transformations as a serious misconception and we do not discuss them in this article.

We are also of the opinion that introducing SR, for the sake of alleged simplicity, from the three-dimensional rather than full geometrical point of view, in fact makes understanding of SR more difficult, and can easily lead to misconceptions. We regard as especially harmful figures illustrating hypothetical relative motion of frames as depicted in Fig. 1. Whereas this is not the best, but correct picture in GS, it is completely wrong in SR. The reason for that is that the hyperplanes of constant time ('pure space') of observers in relative motion are not parallel, so they cannot be regarded as 'sliding' on each other.

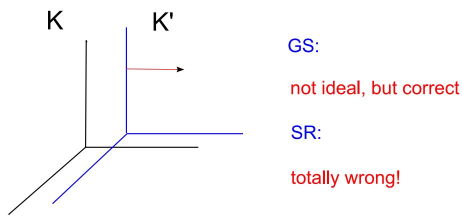

Fig. 1. Reference frames — a popular picture.

Elements of the programme sketched above appeared, of course, in many earlier publications and books (see *e.g.* Refs. [1–3]), thus the aims of this article are mainly pedagogical. However, we believe that our scheme adds some value to the clarity and logic of presentation.

In addition, we discuss some simple geometric effects in the present context. This will include a discussion of the view of the celestial sphere as seen by different observers [4, 5]. This point is particularly worth adding, as it is usually treated with the help of a rather indirect method of stereographic projection[1]. We discuss it directly on the celestial spheres of two observers.

In all discussions of effects involving different observers we consistently avoid, as mentioned above, the use of Galilean or Lorentz transformations. To relate the views on the spacetime as seen by two inertial observers one needs only to know the directional vectors of their world-lines. On the other hand, one needs complete bases attached to the observers to specify a transformation between them.

2. Homogeneity with respect to translations and the affine structure

It is fairly obvious from everyday experience that one needs four real numbers to place an event in space and time. For a given event the specific values of these numbers depend on an adopted system of labels, but they always form an element of the set $\mathbb{R}^4$. Our spacetime is a structure based on this set.

Another common experience points to the applicability of spacetime translations: if a physical occurrence takes place in a given region of space and within some time-span, an analogous occurrence may take place elsewhere and at another time. We include this property in our construction of a model of the spacetime in the following form: the group of four-dimensional translations acts transitively on the spacetime. This leads us to the following starting point for the construction of a spacetime model:

Flat spacetime is modeled by a real four-dimensional affine space $(\mathcal{M}, M)$.

[1] However, in the original article on the shape of moving sphere, Ref. [4], there is a short remark on the idea used in the present article.

Here $\mathcal{M}$ denotes the affine space based on the four-dimensional vector space M. We adopt the notation $P, Q, \ldots$ for points in $\mathcal{M}$ and $x, y, \ldots$ for vectors in M. We write $x = \overrightarrow{PQ}$ if $Q = P + x$. Moreover, if $P \in \mathcal{M}$ and $N \subset M$ is any subset then we use the usual shorthand: $P + N = \{P + x \mid x \in N\}$. In particular, straight lines are one-dimensional affine subspaces $P + L(x)$, where $L(x)$ denotes the one-dimensional vector subspace spanned by the vector x. Ordered vector bases in M will be denoted by (e_0, e_1, e_2, e_3). See Fig. 2 for a graphic representation (here, as in the following, one space dimension is omitted).

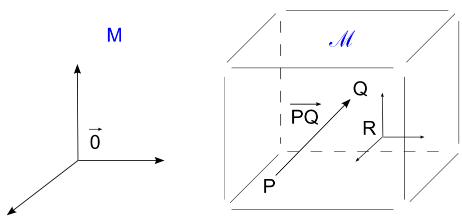

Fig. 2. Vector and affine space.

3. Causal structure and inertial motions

Of course, the affine space structure is still a very poor one, one needs further specification. The most obvious element needed is a one introducing the differentiation between physical time and space directions. This is achieved in the following way.

We shall say that the spacetime is equipped with the *causal structure* if in the accompanying vector space one has distinguished the following set (see Fig. 3):

GS: a three-dimensional subspace $S \subset M$,

SR: a homogeneous vector quadric $V \subset M$ (different from a subspace), with respect to which three dimensions of M are on equal footing, but not the fourth.

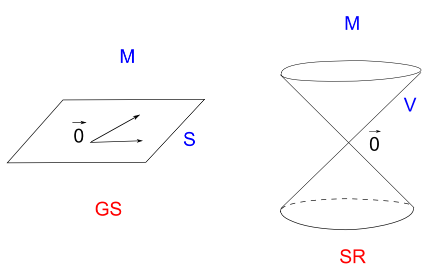

Fig. 3. Causal structure.

By a homogeneous vector quadric we mean here a set of vectors $x \in M$ whose coordinates x^0, x^1, x^2, x^3 in some (and then any) basis satisfy the equation $\sum_{i,j=0}^{3} \alpha_{ij} x^i x^j = 0$, with some basis-dependent numerical coefficients α_{ij}. We recall that for any such quadric there is a basis in which it takes one of the forms $\varepsilon_0 (x^0)^2 + \varepsilon_1 (x^1)^2 + \varepsilon_2 (x^2)^2 + \varepsilon_3 (x^3)^2 = 0$, where $\varepsilon_\mu = 0, \pm 1$ (uncorrelated values). The only possibility (up to a permutation of the basis vectors) to satisfy the demand imposed above on V is that in this canonical basis V is a cone given by:

$$x \in V \quad \Longleftrightarrow \quad (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0\,. \tag{1}$$

We shall say that a vector x lies inside (or outside) V if $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 > 0$ (< 0) respectively.

We say that a nonzero vector is a *causal vector* if it:

GS: does not lie in S,

SR: lies inside or on V.

In addition we introduce the notion of a *timelike vector* which

GS: is identical with a causal vector,

SR: lies inside V.

We shall say that two events P and Q are *causally related* if $\overrightarrow{PQ}$ is a causal vector, and they are *temporally related* if it is a timelike vector.

The causal structure makes contact with physics by the following identification. An *inertial motion* is a straight line in spacetime $\mathcal{M}$ with a timelike directional vector (thus any two events on this line are temporally related). Such lines will be called world-lines of the motion (see Fig. 4).

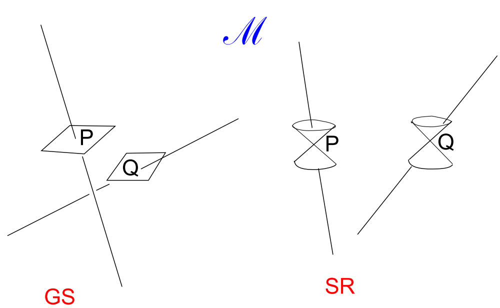

Fig. 4. Inertial motions.

If a point $Q \neq P$ is not causally related to P we say that it lies *elsewhere* with respect to P. One then cannot connect Q and P by an inertial motion.

4. The four orientations of the spacetime

Let us choose a basis of M in which

GS: the subspace S is given by $x^0 = 0$,

SR: the cone V has the canonical form.

The set of causal vectors splits into two disjoint sets: those for which $x^0 > 0$ or $x^0 < 0$ respectively in the distinguished basis. We denote one of these sets by C_+ and call it the future and the other by C_- and call it the past. (After this choice has been done we can adjust the sign of x^0 so that $x^0 > 0$ for $x \in C_+$.) Then the future (past) of any event P is the set $P + C_+$ $(P + C_-)$, and Q is in the future of P if, and only if, P is in the past of Q. Let us write $Q > P$ for 'Q is in the future of P', and $Q \geq P$ for: $Q > P$ or $Q = P$. Then the relation $Q \geq P$ defines a partial order in $\mathcal{M}$:

1. $P \geq P$,
2. if $Q \geq P$ and $P \geq Q$ then $Q = P$,
3. if $R \geq Q$ and $Q \geq P$ then $R \geq P$.

The only less obvious of these properties is the third one in the special relativity case. To prove it observe that $x \in C_+$ if in a canonical basis $0 < x^0 \geq \sqrt{(x^1)^2 + (x^2)^2 + x^3)^2}$. If y is another such vector then it is easily seen that the same relation is satisfied with x replaced by $x + y$, which was to be proved. See Fig. 5 for a graphic representation of causally defined regions.

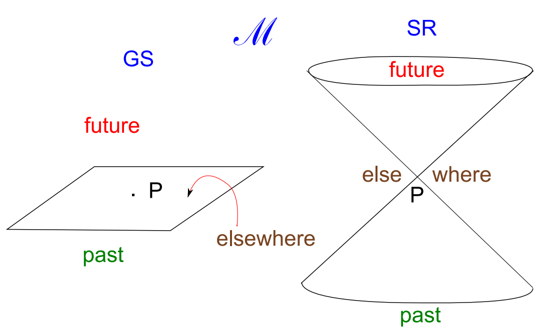

Fig. 5. Past, future, elswhere.

As there are two possible choices for the identification of the sets $C_\pm$ we say that there are two possible *causal orientations* of the spacetime $\mathcal{M}$.

At the same time M as a real vector space has two possible orientations defined as usually as the equivalence classes of bases. In combination with the causal orientation this gives four choices of the spacetime $\mathcal{M}$ orientations.

5. Relative rest, inertial observers, inertial frames

We do not have yet any metric tools, so we are unable to determine relative velocity of inertial motions, but we can already say what it means that two motions are in *relative rest*: their world-lines are parallel (*i.e.* have common directional vectors).

We decide that there is no need to differentiate between an inertial motion and an often used term of *inertial observer*; the difference, if any, is a rather psychological one.

Finally, by an *inertial frame* we mean the class of all inertial observers remaining in relative rest to each other. We do not see the need to make this notion more specific, as is often assumed, by demanding that a particular basis has been chosen with the timelike vector parallel to the world line of the motions in this family.

6. Metric structure, four-velocities

We recall two facts from linear algebra:

1. The kernel (zero space) of a nonzero linear form on a vector space is a subspace of codimension one. Conversely, any such subspace S determines uniquely up to a constant factor a linear form Dt such that
$$x \in S \qquad \Longleftrightarrow \qquad Dt(x) = 0\,. \tag{2}$$
2. A real vector quadric V (if different from a subspace) determines uniquely up to a constant factor a symmetric metric g such that
$$x \in V \qquad \Longleftrightarrow \qquad g(x,x) = 0\,. \tag{3}$$

A proof of the second fact for the case of our cone V is given for completeness in the Appendix.

6.1. Galilean spacetime

In the case of the Galilean spacetime we chose the sign of Dt by demanding that
$$Dt(x) > 0 \qquad \text{for} \qquad x \in C_+\,. \tag{4}$$
Then $Dt(\overrightarrow{PQ}) > 0$ if Q lies in the future of P. The remaining positive factor in the definition of Dt is fixed arbitrarily. For an arbitrarily chosen point P_0 we fix a real value $t(P_0)$. Then there is a unique affine form taking this value at P_0 and having Dt as its linear part. This means that for each pair of points P, Q there is
$$t(Q) = t(P) + Dt\left(\overrightarrow{PQ}\right)\,. \tag{5}$$

This form determines the universal time in the Galilean spacetime. The metric structure of this spacetime is now completed by choosing a Euclidean metric h on the subspace S. This metric then determines 'spatial' metric relations on each hyperplane $Q + S$ of constant time. One notes that there are no relations of this kind between points on different constant time planes. Note also that the relative scale of the metric tools Dt and h is arbitrary. See Fig. 6 for a graphic representation of the metric structure of GS.

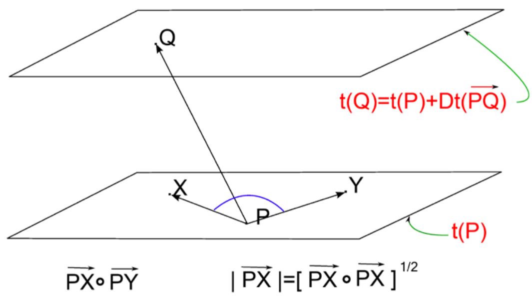

Fig. 6. Metric structure of GS.

The world-lines of inertial motions pierce precisely at one point each of the constant time hyperplanes. For each family of parallel inertial motions there is a unique directional vector u for which $Dt(u) = 1$. We shall call such vector a *unit timelike, future-pointing vector* or the *four-velocity* of these world-lines.

Having chosen a particular family of inertial parallel motions characterized by the four-velocity u one can split the vector space into time and space parts by

$$M = L(u) \oplus S\,, \tag{6}$$

where $L(u)$ denotes the one-dimensional subspace spanned by u. Observers in the chosen family decompose each vector x into the time and space parts by

$$x = Dt(x)u + x_u\,, \qquad \text{so} \qquad x_u \in S\,. \tag{7}$$

Note that while $Dt(x)$ does not depend on u, the space part x_u does depend on this vector, that is to say on the family of parallel inertial motions. The Euclidean scalar product h can be applied to the space parts of any two vectors x and y and we shall also write

$$h(x_u, y_u) = x_u \circ y_u\,. \tag{8}$$

6.2. Special relativity

In this case g is fixed up to a real factor by the cone V, as described above. We choose its sign by the convention that in the canonical basis of V

the metric has the signature $(+1, -1, -1, -1)$. The remaining positive factor is chosen arbitrarily. The metric structure of the spacetime is determined completely by g. The vector x is a timelike vector when $g(x,x) > 0$, and it is a causal vector when it is nonzero and $g(x,x) \geq 0$. In addition, we say that a vector is *spacelike* if $g(x,x) < 0$. We shall also use the notation

$$g(x,y) = x \cdot y \,, \qquad x \cdot x = x^2 \,. \tag{9}$$

See Fig. 7 for the metric properties of vector types.

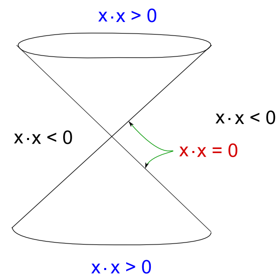

Fig. 7. Scalar product in SR.

If Q lies in the future of P then there is a unique inertial motion joining them. The *proper time interval* covered by this motion from P to Q is determined by

$$\Delta\tau(P,Q) = \left[g\left(\overrightarrow{PQ}, \overrightarrow{PQ}\right)\right]^{1/2} \,. \tag{10}$$

Let $u = \lambda\overrightarrow{PQ}$ with $\lambda > 0$ so that $u \in C_+$. If we demand that $g(u,u) = 1$ then u is fixed uniquely by these conditions and $\lambda = \left[g(\overrightarrow{PQ}, \overrightarrow{PQ})\right]^{-1/2}$. We call such u a *unit timelike, future-pointing vector* or a *four-velocity*.

A four-velocity u may be used to define a time variable correlated with the inertial frame defined by u. As in the Galilean case we fix $t_u(P_0)$ and then there is a unique affine form t_u taking this value at P_0 and having the linear form

$$Dt_u(x) = u \cdot x \tag{11}$$

as its linear part. This means that for each pair of points P, Q there is

$$t_u(Q) = t_u(P) + Dt_u\left(\overrightarrow{PQ}\right) \,. \tag{12}$$

Note that if P and Q lie on one u-world-line, Q in the future of P, then

$$Dt_u\left(\overrightarrow{PQ}\right) = \Delta\tau(P,Q) \tag{13}$$

so the definition of Dt_u is an extension of the proper time interval on a u-world-line, Eq. (10).

Let us denote by S_u the kernel of the form Dt_u, which is the subspace of vectors orthogonal to u with respect to the metric g. Then the hyperplanes $P + S_u$ are the sheets of constant t_u time. The metric g when restricted to S_u reduces to $-h_u$, where h_u is a Euclidean metric. Thus the objects Dt_u, t_u, S_u and h_u play a similar role as Dt, t, S and h in the Galilean case, but with several important differences:

1. Here these quantities are not universal as in the Galilean case, they are functions of the vector u; thus they depend on the choice of a family of inertial observers in relative rest.
2. This relative character implies weaker status of these quantities as compared to the Galilean case.
3. On the other hand, the form Dt_u and the metric h_u are uniquely determined by g, so their relative scale is unambiguous. This is to be contrasted with the Galilean case, where the scale of Dt and h could be fixed independently.

The decomposition of the vector space M into time and space parts takes now the form

$$M = L(u) \oplus S_u\,, \qquad x = Dt_u(x)u + x_u\,, \qquad x_u \in S_u\,. \tag{14}$$

Note that in this case both $Dt_u(x)$ and x_u depend on u, and for different choices of this four-velocity the space parts x_u lie in different subspaces. For $x_u, y_u \in S_u$ we shall write $x_u \circ y_u = -x_u \cdot y_u$ and also denote $|x_u| = \sqrt{x_u \circ x_u}$. Then

$$x \cdot y = Dt_u(x)Dt_u(y) - x_u \circ y_u\,, \qquad x^2 = (u \cdot x)^2 - |x_u|^2\,. \tag{15}$$

The scalar product, in contrast to the Galilean case, is applicable to any vectors. See Fig. 8 and 9 for a graphic representation of decompositions and four-velocities, and Fig. 10 for the dependence of S_u on u.

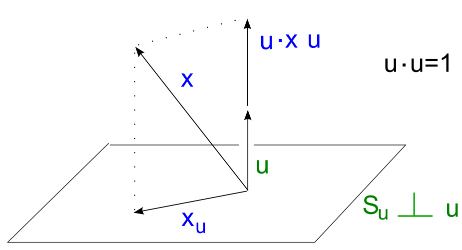

Fig. 8. Metric structure of SR.

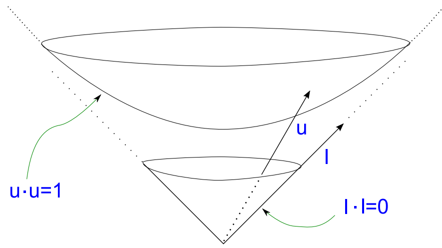

Fig. 9. Four-velocities and future-directed lightvectors.

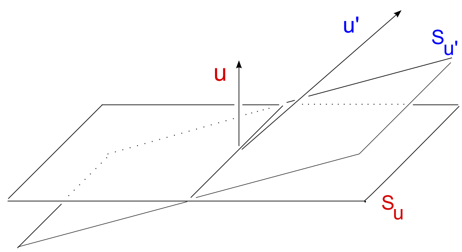

Fig. 10. Subspaces orthogonal to 4-velocities.

7. Equivalence of observers, light signals and their speed

The principle of relativity, *i.e.* of the equivalence of observers, can be now put in the following form:

1. Physical theories do not depend on the choice of an inertial observer, *i.e.* of the four-velocity u determining all inertial motions in a given frame or a particular world-line in the family.
2. The set of physical states conforming with physical theories does not distinguish any of the inertial observers.

In particular:

1. In SR the Maxwell equations imply that the light signals propagate along straight lines whose directional vectors lie on V, *i.e.* l is such a vector iff $g(l,l) = 0$. These vectors are called therefore *lightlike vectors* and V is called the *light-cone*. The Maxwell equations do not conform to the principle of relativity in the GS case. In this case the only way to avoid clash with the principle of relativity is to assume that light propagates with infinite speed, *i.e.* the directional vectors of light rays lie in S.
2. If one defines physical units of time and space in each inertial frame with the use of analogous physical phenomena then the proportion of these units to the geometrical units defined by Dt and h in the case of Galilean spacetime, and g in the case of SR, is the same for all observers.

3. In the SR case if l is lightlike and u is any four-velocity, then $|Dt_u(l)| = |l_u|$ — light covers in each inertial frame a unit distance in a unit time in geometrical units. If one determines physical units as in the preceding point their ratio gives the speed of light in all inertial frames in those units.

Note that the geometrical objects of the spacetime include beside metrical tools also the choice of one of the four orientations (as defined above). The principle of relativity in the above form does not require the independence of physics of this choice. As is well-known there are exceptions not conforming to this extended demand.

8. Relative velocities and their composition

To be precise the term 'four-velocity', although deeply rooted in the language usually used in SR, is somewhat misleading. In fact the vector u of an inertial frame simply points in the direction in which time flows but there is no space translation for all observers in this frame. To introduce a more justified notion of velocity one needs a reference observer which 'rests'. But 'all observers are equal', so one has to say with respect to which of them one makes the measurement.

Thus we assume there are given two four-velocities u and u' and we want to determine a velocity of the motion defined by u' with respect to that defined by u. We propose three candidates:

1. $\mathit{\Delta}\,(u', u) = u' - u\,,$
2. $v_{\mathrm{pr}}\,(u', u) = u'_u\,,$
3. $v\,(u', u) = u'_u / Dt_u\,(u')\,.$

The r.h.s. in 2. is formed as in (7) and (14) and the subscript 'pr' stands for 'proper'. In 3. Dt_u is independent of u in the Galilean case.

The first of these definitions satisfies the antisymmetry and chain properties:

$$\mathit{\Delta}\left(u', u\right) = -\mathit{\Delta}\left(u, u'\right)\,, \qquad \mathit{\Delta}\left(u'', u\right) = \mathit{\Delta}\left(u'', u'\right) + \mathit{\Delta}\left(u', u\right)\,, \tag{16}$$

which has obvious interpretational advantages.

8.1. Galilean spacetime

In this case $Dt(u') = 1$ and $u'_u = u' - Dt(u')u = u' - u$, so all three definitions coincide and we shall use notation $v(u', u)$ for this quantity (see Fig. 11). We have $v(u', u) \in S$ and point 3. above tells us that this vector gives the change of position of an observer with four-velocity u' with respect to one with four velocity u, undergone in unit time. The composition of velocities obeys simple vector addition law (16) (see Fig. 12).

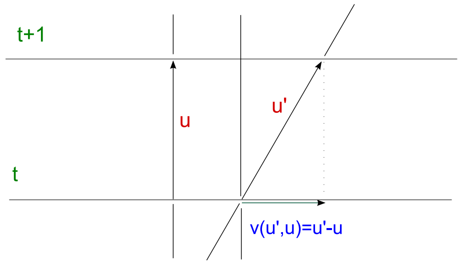

Fig. 11. Relative velocity in GS.

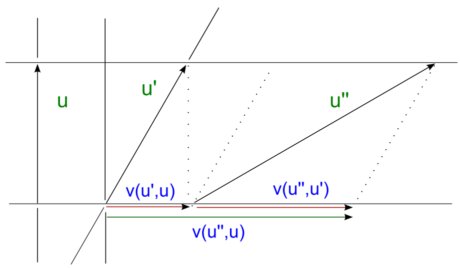

Fig. 12. Composition of velocities in GS.

8.2. Special relativity

In this case all three definitions are different (see Fig. 13). The first one has the advantage of the vector addition composition law (16) (see Fig. 14), but $\Delta(u', u)$ does not lie in any of the subspaces S_u or $S_{u'}$. Rather, it is in the subspace S_w of the observer with four-velocity 'half way' between u and u': $w = (u + u')/\sqrt{(u + u') \cdot (u + u')}$.

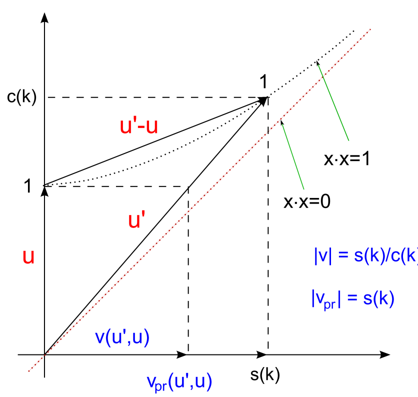

Fig. 13. Relative velocity in SR.

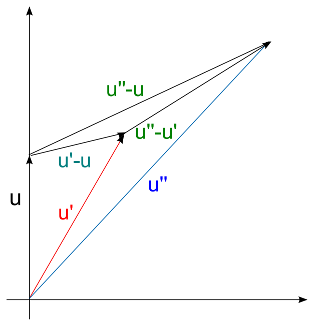

Fig. 14. Composition of velocities in SR.

The second and the third definitions give parallel vectors in S_u. The proper velocity $v_{\rm pr}(u', u)$ is the displacement of the motion along any world-line $P + L(u')$, as seen in the u-frame, undergone during unit time interval as measured on the world-line (proper time) (see Fig. 15). The velocity $v(u', u)$ is a similar displacement but scaled to unit time in u-frame. It is only this latter quantity which is bounded by 1 (light velocity as defined in Section 7).

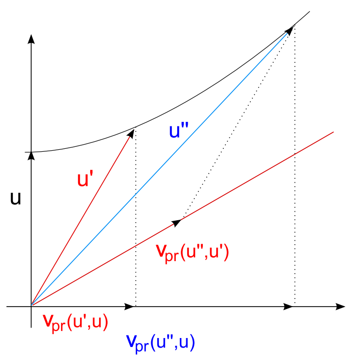

Fig. 15. Proper velocities in SR.

The explicit form of the two latter velocities is easily obtained:

$$v_{\rm pr}\left(u', u\right) = u' - u' \cdot u\, u\,, \tag{17}$$

$$v\left(u', u\right) = \frac{u'}{u' \cdot u} - u\,. \tag{18}$$

Neither of these velocities satisfies the antisymmetry or the chain rule properties (16). If we write the first of these equations in the form $u' = u' \cdot u\, u + v_{\rm pr}$ and take the scalar square of both sides we find

$$\left(u' \cdot u\right)^2 - |v_{\rm pr}|^2 = 1 \tag{19}$$

(from now on we write $v_{\rm pr} \equiv v_{\rm pr}(u',u)$, $v \equiv v(u',u)$). This tells us that the quantities $u'\cdot u$ and $|v_{\rm pr}|$ may be represented as the hyperbolic cosine and hyperbolic sine of some unique parameter $\psi \geq 0$. If we denote (after Bondi [1]) $k = \exp\psi \geq 1$ we get the representation

$$u'\cdot u = \tfrac{1}{2}\left(k+k^{-1}\right) \equiv c(k)\,, \qquad |v_{\rm pr}| = \tfrac{1}{2}\left(k-k^{-1}\right) \equiv s(k)\,, \qquad |v| = \frac{s(k)}{c(k)}\,. \tag{20}$$

Some other useful relations which follow are

$$c(k) \;=\; \sqrt{1+|v_{\rm pr}|^2} = \frac{1}{\sqrt{1-|v|^2}}\,, \qquad s(k) = \frac{|v|}{\sqrt{1-|v|^2}}\,, \tag{21}$$

$$k \;=\; |v_{\rm pr}| + \sqrt{1+|v_{\rm pr}|^2} = \left(\frac{1+|v|}{1-|v|}\right)^{1/2}\,. \tag{22}$$

We shall find the direct physical interpretation of k in the next section.

The magnitude of k is invariant with respect to the interchange of u and u', so if we denote $v'_{\rm pr} \equiv v_{\rm pr}(u,u')$ and $v' \equiv v(u,u')$ then we have

$$|v'_{\rm pr}| = |v_{\rm pr}|\,, \qquad |v'| = |v|\,. \tag{23}$$

The motion of an observer with respect to the u-frame is often defined rather in terms of $v_{\rm pr}$ or v than u', or similarly with the role of observers interchanged, and then

$$\begin{aligned} u' \;&=\; c(k)u + v_{\rm pr} = c(k)(u+v) = c(k)u + s(k)n\,,\\ u \;&=\; c(k)u' + v'_{\rm pr} = c(k)(u'+v') = c(k)u' + s(k)n'\,, \end{aligned} \tag{24}$$

where by n and n' we have denoted the unit spacelike vectors pointing in the direction of v and v' respectively. Although the use of $v_{\rm pr}$ or v instead of u' may seem better suited for the point of view of the u-frame, one has to be careful not to project Galilean properties of velocities to SR. For instance, we have $v' \neq -v$, in contrast to GS.

The composition of velocities of these types is rather complicated and not very illuminating. The special case of four-velocities u, u', u'' lying in one two-dimensional subspace will be discussed in the next section.

9. Time measurement

The problem one wants to address here is the following. Two events P and Q on a world line with four velocity u' are separated by the vector $\Delta t' u'$, so the time interval between them as measured directly by the inertial observer on this world-line is $\Delta t'$. What time-span Δt will be measured between these events in the frame defined by the four-velocity u?

9.1. Galilean spacetime

Here the answer is simple. The spacetime is equipped with the universal time interval form Dt, so there is no doubt how to measure this interval in any frame. One has

$$\Delta t = Dt(\Delta t' u') = \Delta t' \,. \tag{25}$$

9.2. Special relativity

If one employs the frame-dependent time interval form Dt_u described in Section 6.2, one finds

$$\Delta t = Dt_u(\Delta t' u') = u \cdot u' \; \Delta t' = c(k) \Delta t' \tag{26}$$

(notation as in the preceding section). This gives the famous 'time dilation' effect. However, one should be careful to interpret this result properly. No inertial observer from the u-frame can pass directly both events P and Q, thus the measurement in this frame is by necessity indirect. Observers on the world-lines $P + L(u)$ and $Q + L(u)$ to establish one frame-dependent time variable t_u need only to agree on a choice of a constant time hypersurface to synchronize their clocks (as the time-interval form Dt_u is known directly to both of them). After this has been settled (see below) the time $t_u(P)$ is measured directly by the first observer, and the time $t_u(Q)$ is measured directly by the other. The difference $t_u(Q) - t_u(P)$ gives Δt. See Fig. 16.

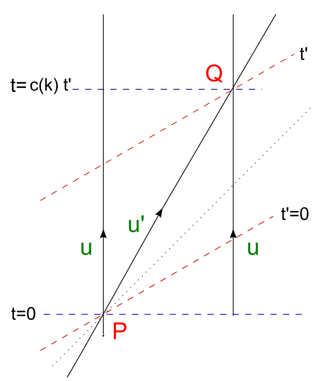

Fig. 16. Time measurement.

The synchronization of clocks can be done by the radar method. The first observer sends at his time t_1 a light signal towards the other one and receives it back reflected at t_2. Denote by X the event on the world-line of the first observer at his time $(t_1 + t_2)/2$, and by Y the event on the world-line of the second observer at which the reflection of the light ray takes place, see Fig. 17. If l_1 and l_2 are lightlike vectors as depicted in the figure, then

$(t_2 - t_1)u = l_1 + l_2$, $\overrightarrow{XY} = (l_1 - l_2)/2$, so $u \cdot \overrightarrow{XY} = 0$. Thus X and Y lie in one hyperplane of u-simultaneity and if the second observer agrees to set his clock for $(t_1 + t_2)/2$ at Y, the clocks will be synchronized.

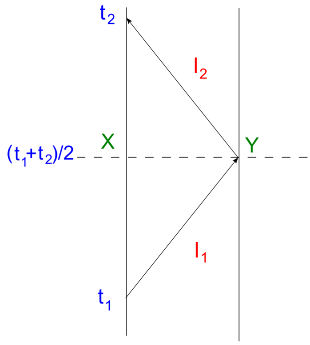

Fig. 17. Synchronization of clocks.

In real life the time dilation measurement is rarely, if at all, done this way. Probably the most famous instance of the dilation effect is the decay of muons produced by cosmic radiation coming to Earth. Muons are unstable particles with a characteristic lifetime (in their rest-frames). They are produced with known energy (so also known velocity) by scattered cosmic rays. One finds that their mean lifetime in the Earth-frame is much longer than the characteristic one. However, what is directly measured is not any time at all! One measures the distance they cover during their life; then knowing their relative velocity in the Earth-frame one *calculates* their lifetime in this frame.

Another type of time measurement is by registering the time of arrival of light signals. Suppose that two inertial observers travel along world-lines $P + L(u')$ and $P + L(u)$ respectively (thus we assume for simplicity that they meet at P). Let both of them set their clocks so as to show 0 at P. The u'-observer sends a light signal towards the u-observer at his time t', which arrives at the u-observer's world-line at the time t_+ on that line. Thus one has the equation $t'u' + l = t_+u$, where l is the lightlike, future-pointing vector connecting these two events (see Fig. 18). We write this as

$$l = t_+u - t'u' \,, \qquad l \cdot l = 0 \,, \qquad l \cdot u > 0 \,. \tag{27}$$

Solving the second equation for t_+ one obtains two values out of which the third condition selects only one:

$$t_+ = u \cdot u' \, t' + \sqrt{(u \cdot u')^2 - 1} \, \left|t'\right| = c(k)t' + s(k) \left|t'\right| \,. \tag{28}$$

Note that $t', t_+ < 0$ for observers approaching each other (parts of world-lines causally preceding P) and $t', t_+ > 0$ for observers moving away from each

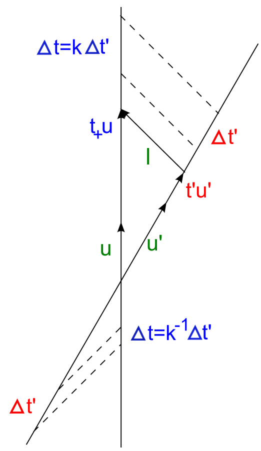

Fig. 18. Time of arrival of light signals.

other (parts of world-lines causally following P). Let now the u'-observer send two signals at times t'_1 and $t'_2 > t'_1$, either both negative or both positive, and denote $\Delta t' = t'_2 - t'_1$, $\Delta t_+ = t_{+2} - t_{+1}$. Then one finds from the above relation that

$$\begin{aligned} \Delta t_+ &= k^{-1}\Delta t' && \text{observers moving towards each other}\,, \\ \Delta t_+ &= k\Delta t' && \text{observers moving away from each other}\,. \end{aligned} \tag{29}$$

Note that the result is completely different from the 'dilation effect'.

The above connections have a directly observable physical consequence. The light is a wave phenomenon; the change of its phase from one ray to another is the same for each of the above observers. But the times corresponding to the given change of phase, say 2π, are related as above. Thus the frequencies of light ν' and ν for the two observers are related by

$$\begin{aligned} \nu &= k\,\nu' && \text{observers moving towards each other}\,, \\ \nu &= k^{-1}\nu' && \text{observers moving away from each other}\,. \end{aligned} \tag{30}$$

With the interpretation of k-coefficient given by the second equation in (29) we can now find a simple formula for the composition of velocities (or rather their lengths) in the special case of three co-planar four-velocities u, u', u''. Let the k-coefficients be denoted as in Fig. 19. This figure then also shows that $K = kk'$. Using the last equation in (20) and Eq. (22) one finds

$$\left|v\left(u'', u\right)\right| = \frac{\left|v\left(u', u\right)\right| + \left|v\left(u'', u'\right)\right|}{1 + \left|v\left(u', u\right)\right| \left|v\left(u'', u'\right)\right|}\,. \tag{31}$$

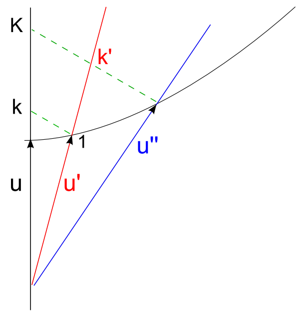

Fig. 19. Composition of k-coefficients for co-planar four-velocities: $k/1 = K/k'$, so $K = kk'$.

We end this section with a warning against a popular error in graphical representations of the time dilation found in many introductory texts on SR. One of many variants is this: an individual A is speeding in a rocket towards (or away from) another individual B, who is busy with some activity. Each of the individuals is equipped with a clock and A watches (by 'looking') B's activity. The claim then is that A will measure B's activity to last longer then it lasts for B in agreement with the time dilation formula. This, however, is wrong; in fact A receives light signals from B, so his measurement will give a result obeying one of the cases in Eqs. (29). In fact, for approaching observers, the time in question is shorter.

10. Space measurement

Here we pose the following question. Two parallel world-lines with four-velocity u' are separated by a vector z' which is a 'pure space' vector in the u'-frame. What is the 'pure space' vector z which separates them in the frame defined by u? These two vectors may be thought of as connecting two particles in a rigid body in these two frames. This latter notion has limitations in SR: it runs into difficulty when accelerations are involved, and then needs an input of dynamics to be modified. However, as long as only inertial motions are involved, a rigid body may be *identified* with some family of parallel world-lines. This body rests in the frame defined by these world-lines.

10.1. Galilean spacetime

Here again the answer is simple: the 'pure space' directions are universally determined by S, so

$$z = z' \in S\,. \tag{32}$$

10.2. Special relativity

In this case 'pure space' means that $u' \cdot z' = u \cdot z = 0$. The condition for z to connect the same two world-lines is $z = z' + \lambda u'$ with some real λ. Taking the scalar product of this equation with u we find this coefficient and obtain

$$z = z' - \frac{z' \cdot u}{u' \cdot u}\, u' \,. \tag{33}$$

These two vectors can be decomposed as

$$z' = z'_\perp + \alpha' n' \,, \qquad z = z_\perp + \alpha n \,, \tag{34}$$

where $z'_\perp$ is orthogonal to u' and n' (as defined at the end of Section 9), $z_\perp$ is orthogonal to u and n, and α, α' are numerical constants. Note that $z'_\perp$ and $z_\perp$ are equivalently identified as parts of z' and z orthogonal both to u and u'. Taking the scalar product of Eq. (33) with u' we find $z \cdot u' = -z' \cdot u / u' \cdot u$. Using now Eqs. (24) and (34) we find after some simple algebra

$$z_\perp = z'_\perp \,, \qquad \alpha = -\frac{\alpha'}{c(k)} \,. \tag{35}$$

The second of these equations describes the effect of the so-called 'length contraction', whose popular formulation could run as: 'the dimensions parallel to the relative velocity measured by the moving observer are by the factor $1/c(k)$ shorter then those measured by the observer in rest with respect to the object being measured'. However, one should note that this formulation and the term 'contraction' are somewhat misleading:

1. The vectors z' and z connect two different pairs of events on the two world-lines considered, nothing is being 'contracted'. Events separated by z' are simultaneous in the rest frame of the 'rigid body', while those separated by z are simultaneous for the moving observer.
2. The vectors n' and n (pointing in the directions of the two respective velocities) are not even parallel, so for each of the frames the term 'parallel to the velocity' means something different.

Figure 20 illustrates the situation for the special case $z'_\perp = z_\perp = 0$, which means that for the u-observer the rigid rod with ends on the two world-lines moves parallelly to its axis.

The proper understanding of the above dismisses various 'length contraction paradoxes' in SR (see *e.g.* [6]). The key to all of them is a cautious analysis of the relation between various vectors involved in the problem.

We illustrate this with a geometrical situation whose variants lie at the base of most of these effects. Suppose we have two pairs of parallel world-lines: $P + L(u')$, $Q + L(u')$, and $P + L(u)$, $Q + L(u)$, so that the first lines

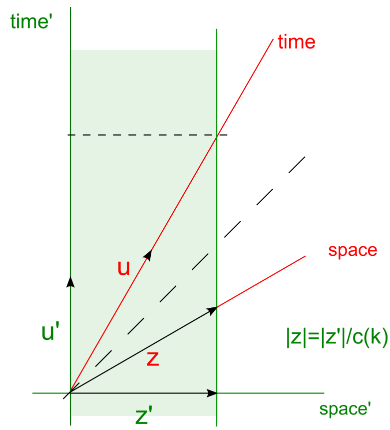

Fig. 20. Space measurement.

in these pairs intersect at P, and the second lines intersect at Q. Physically this may be thought of as modeling two rigid rods in relative motion, the ends of the first and the second rod described by the lines in the first and in the second pair respectively. The 'front' ends of the rods meet at some point and similarly the 'back' ends meet at some other point. Let z' and w be the 'pure space' vectors (in respective rest-frames) connecting the ends of rods and denote $x = \overrightarrow{PQ}$. (See Fig. 21. The picture might suggest that the rods are bound to clash and cannot 'go through'. This is because we lack in the picture the fourth dimension, which may be used to slightly detach the world-lines of the rods.)

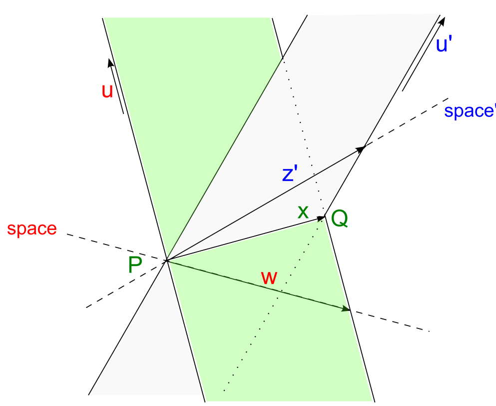

Fig. 21. Two rods with ends meeting at P and Q respectively.

Then one has

$$x = z' + \mu' u' = w + \nu u \tag{36}$$

with some constants μ', ν. We decompose z' as in the first Eq. (34) and similarly write

$$w = w_\perp + \beta n \,, \qquad w' = w_\perp - \frac{\beta}{c(k)}\, n' \,, \tag{37}$$

(the second formula obtained in analogy with Eqs. (34) and (35) is written down for later use). As n and n' can be expressed as linear combinations of u and u' (see Eq. (24)), the consistency condition for the second equality in (36) is

$$z'_\perp = w_\perp \,, \tag{38}$$

and then the constants μ' and ν have unique solutions, which we do not need to write down explicitly.

The geometry of the situation is clear and no interpretational difficulty arises if one insists on this four-dimensional picture. However, if one uses the 'length contraction' language 'paradoxes' easily arise. Suppose, for instance, that the vector x is spacelike (as in Fig. 21) and consider any frame with a four-velocity orthogonal to x. Then the intersecting of lines has this interpretation: in each of these frames the two rods pass each other parallelly, with both respective ends simultaneously coming into contact. But now the 'paradoxical' problem arises: if we go to some other frame not in this family, then due to different velocities of the two rods they will change their size in different way, so the ends cannot meet. The simple explanation is, of course, that what is simultaneous in one frame usually is not simultaneous in another, which falsifies the above conclusion. And even more, the rods moving parallelly in one frame usually do not remain parallel in another.

To illustrate the last point suppose that in the above geometrical setting $x = w$, *i.e.* the rods are parallel and of equal length in the u-frame. This means that $w = z$, and decomposing these vectors as before we find $\alpha' = -c(k)\beta$. Using this and Eq. (38) we find

$$w' = w_\perp - \frac{\beta}{c(k)}\, n' \,, \qquad z' = w_\perp - c(k)\beta\, n' \,. \tag{39}$$

These vectors are parallel if, and only if $w_\perp = 0$ or $\beta = 0$. In all other cases rods move in the u'-frame askew to each other. This is illustrated in Fig. 22.

11. Non-inertial motions, proper time, simultaneity

Inertial motions, as we have seen, have a special role to play for the interpretation of the geometry of spacetime. However, the picture would not be complete without mentioning other, non-inertial, motions. Straight lines are special examples in the more general class of curves. A *regular curve* may be defined as a set of points obtained as values of a differentiable

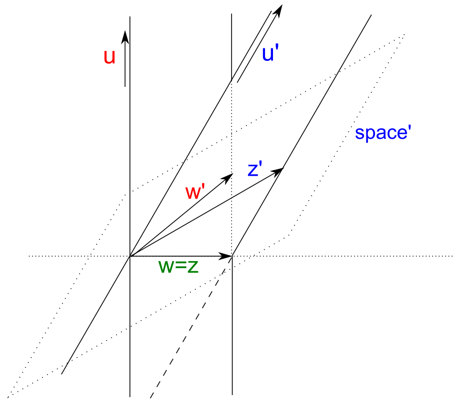

Fig. 22. Two rods moving parallelly in u-frame, and askew in u'-frame.

mapping $\lambda \mapsto P(\lambda)$, where λ is a real parameter taking values in some (finite or not) interval on the real axis. The curve is invariant under a change of parameter $\lambda = f(\lambda')$, where f is differentiable together with its inverse. Each regular curve has at each its point $P(\lambda)$ a *tangent vector* defined as $dP(\lambda)/d\lambda$. The extension of tangent vectors changes with the change of parameter (but the tangent straight lines they generate remain unchanged).

We now define a general *world-line* as a curve with a four-velocity as its tangent vector at each its point. We say that τ is a *proper time* of a world-line if it has the form $\tau \mapsto P(\tau)$ and the equation

$$\frac{dP(\tau)}{d\tau} = u(\tau) \tag{40}$$

defines at each point the tangent four-velocity $u(\tau)$. Physically proper time intervals are measured by clocks traveling along the world-line. Integrating the above equation one obtains

$$\overrightarrow{P_1P_2} = \int_{\tau_1}^{\tau_2} u(\tau)\, d\tau\,, \qquad \text{where} \qquad P_i = P(\tau_i)\,. \tag{41}$$

Note that sums of four-velocities are future-pointing timelike vectors, so P_2 is in the future of P_1. One introduces also the concept of the *four-acceleration*:

$$a(\tau) = \frac{du(\tau)}{d\tau}\,. \tag{42}$$

Note that acceleration, like relative velocity, points in a 'purely spatial' direction:

$$\text{GS:} \qquad Dt(a(\tau)) = \frac{d}{d\tau} Dt(u(\tau)) = 0\,,$$

$$\text{SR:} \qquad Dt_{u(\tau)}(a(\tau)) = u(\tau)\cdot a(\tau) = \frac{1}{2}\frac{d}{d\tau}[u(\tau)]^2 = 0\,. \tag{43}$$

However, unlike relative velocity, the acceleration is absolute — it does not need a reference observer.

We now want to find

1. what is the relation of the proper time to affine time functions defined earlier,
2. does the presence of acceleration influence the concept of simultaneity?

11.1. Galilean spacetime

We apply the linear form Dt to both sides of Eq. (41) and find

$$t(P_2) - t(P_1) = Dt\left(\overrightarrow{P_1P_2}\right) = \int\limits_{\tau_1}^{\tau_2} Dt(u(\tau))\,d\tau = \tau_2 - \tau_1\,. \tag{44}$$

Thus the proper time intervals are identical with the absolute time intervals. Also, the notion of simultaneity is in no way influenced by accelerations.

11.2. Special relativity

Here we take the form Dt_u and then proceed as in the Galilean case to find

$$t_u(P_2) - t_u(P_1) = \int\limits_{\tau_1}^{\tau_2} u\cdot u(\tau)\,d\tau \geq \tau_2 - \tau_1\,. \tag{45}$$

Therefore, the proper time interval is always smaller than any affine time function interval, except for the case when $u(\tau) \equiv u$. The latter case gives simply $P(\tau) = P(\tau_1) + (\tau - \tau_1)u$, which is an inertial motion; proper time intervals are then equal to the u-inertial time intervals on that line. In general this is not the case. However, put $\tau_1 = \tau$, $\tau_2 = \tau + d\tau$ and $u = u(\tau)$. Then we find

$$t_{u(\tau)}(P(\tau + d\tau)) - t_{u(\tau)}(P(\tau)) = d\tau\,, \tag{46}$$

so locally the proper time interval is equal to the time interval as defined earlier for inertial motions.

With accelerated motions in play it is now possible to let two general observers start from P_1, take different routes, and then meet again at P_2. In general their clocks will show different time intervals between these two events. In particular, let the first observer go straight from P_1 to P_2 along an inertial world-line, and let u be his four-velocity. Then his clock will show the interval $t_u(P_2) - t_u(P_1)$, which is always more than the reading of the proper time interval for any accelerated observer. There is no paradox here (the famous 'twin paradox') — the accelerations, as noted above, are absolute, so there is no symmetry between the observers.

Consider now simultaneity. Suppose that for an observer on the world-line $P(\tau)$ we can extend this notion in the way determined by his local position and four-velocity: an event X is from his point of view simultaneous with the event $P(\tau)$ iff $\overrightarrow{P(\tau)X} \cdot u(\tau) = 0$. However, this leads to conceptual difficulties. To see this suppose the observer crosses P_1 with a four-velocity u_1 and then P_2 with a different four-velocity u_2. The two corresponding simultaneity hyperplanes cross on the 2-plane of events X determined by the linear system

$$\overrightarrow{P_iX} \cdot u_i = 0\,, \qquad i = 1, 2\,. \tag{47}$$

Take any event X on this 2-plane and put $X'_i = X + \overrightarrow{P_iX}$. We have $\overrightarrow{P_iX'_i} = 2\overrightarrow{P_iX}$, so X'_i is simultaneous with P_i. At the same time there is $\overrightarrow{X'_1X'_2} = -\overrightarrow{P_1P_2}$. Therefore X'_2 is in the past of X'_1. Thus an event which according to the above definition is simultaneous with P_1 turns out to be in the future of an event simultaneous with a later event P_2 (see Fig. 23).

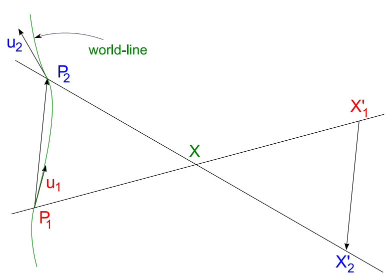

Fig. 23. Accelerated motion and simultaneity.

This difficulty should by no means be interpreted as an argument against the objectivity of the 'direction of time flow'. This latter notion should be simply identified with the choice of the causal orientation and the emerging partial order $Q \geq P$, as discussed in Section 4. The difficulty rather points to the weakness of the notion of simultaneity, its restricted applicability and, to some degree, its conventional character. It also shows that the strict 'dilation' and 'contraction' problems are of rather academic nature.

12. Four-momentum, four-angular momentum and their conservation

The four-momentum of a particle with mass m_1 and four-velocity u_1 is given by

$$p_1 = m_1 u_1 \,. \tag{48}$$

If one chooses a reference point O and x_1 is a vector from this point to the position of the particle then the four-angular momentum tensor is defined by

$$L_1 = 2x_1 \wedge p_1 \,. \tag{49}$$

Let $p_1, \ldots, p_k$ be the initial and $p'_1, \ldots p'_l$ the final four-momenta in a conservative mechanical process. The invariant laws of momentum and angular momentum conservation say

$$\sum_{i=1}^{k} p_i = \sum_{j=1}^{l} p'_j \,, \qquad \sum_{i=1}^{k} L_i = \sum_{j=1}^{l} L'_j \,. \tag{50}$$

12.1. Galilean spacetime

Here the mass is an invariant of the four-momentum given by $m_1 = Dt(p_1)$. The decomposition of the four-momentum with respect to the frame defined by the four-velocity u is thus

$$p_1 = m_1 u + p_{1u} \,, \tag{51}$$

see Fig. 24. We see thus that the law of conservation of mass and the law of conservation of momentum are aspects of one observer-invariant law of conservation of four-momentum.

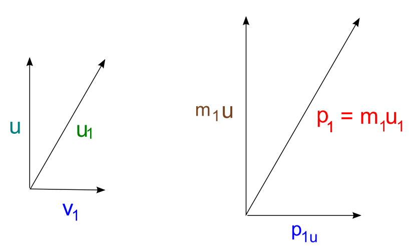

Fig. 24. Four-momentum in GS.

12.2. Special relativity

The mass again is an invariant, but formed in another way: $p_1 \cdot p_1 = m_1^2$. Then in the u-frame we have

$$p_1 = E_{1u} u + p_{1u} \,, \qquad E_{1u}^2 - |p_{1u}|^2 = m_1^2 \,, \tag{52}$$

see Fig. 25. E_{1u} has the interpretation of the energy as seen in the chosen frame. Now the aspects of the observer-invariant law of conservation of four-momentum are laws of energy and momentum conservation, while the sum of masses needs not to be conserved.

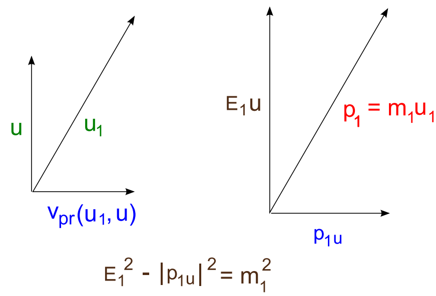

Fig. 25. Four-momentum in SR.

We observe that geometrical analogy is:

$$\text{Galilean mass} \qquad \leftrightarrow \qquad \text{Einsteinian energy}$$

(and not energy $\leftrightarrow$ energy). This analogy is further confirmed when one considers the time-space part of the conservation of four-angular momentum. For freely moving particles one obtains the law of uniform motion of center of mass in the Galilean case, and of center of energy in the SR case.

13. Galilean kinetic energy

The question then arises what is the geometrical status of the Galilean kinetic energy and does its conservation have an invariant character.

To answer this observe that while there is no geometrical numerical invariant formed out of space-part of a single timelike vector, one can form a respective invariant for a pair of such vectors. Let $Dt(p_i) = m_i$, $i = 1, 2$, and let u be any four-velocity. Then $p_i = m_i u + p_{iu}$, so that

$$\frac{p_1}{m_1} - \frac{p_2}{m_2} = \frac{p_{1u}}{m_1} - \frac{p_{2u}}{m_2} \in S\,. \tag{53}$$

Thus the number

$$d(p_1, p_2) = \frac{m_1 m_2}{2} \left| \frac{p_{1u}}{m_1} - \frac{p_{2u}}{m_2} \right|^2 \geq 0 \tag{54}$$

does not depend on u (see Fig. 26). For momenta $p_1, \ldots, p_k$ it is now easy to show, that

$$\sum_{i,j=1}^{k} d(p_i, p_j) = 2ME - |P_u|^2 \geq 0\,, \tag{55}$$

where

$$P = \sum_{i=1}^{k} p_i\,, \qquad P = Mu + P_u\,, \qquad E = \sum_{i=1}^{k} \frac{|p_{iu}|^2}{2m_i}\,. \tag{56}$$

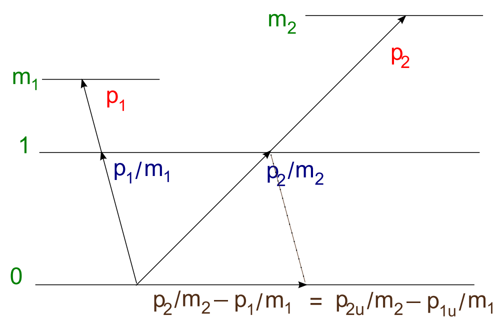

Fig. 26. Galilean invariant of two causal vectors: $|p_{2u}/m_2 - p_{1u}/m_1|$.

We learn two facts:

1. If the total four-momentum is conserved, then the condition of energy conservation is Galilean invariant.
2. There is always $E \geq |P_u|^2/2M$, and the equality holds if, and only if, all momenta are parallel.

14. Celestial sphere

We fix a reference point O and consider all light rays coming into this point. Imagine a world-line of an inertial observer with four-velocity u passes through this point. At this point the observer positions the space directions from which all light rays arrive. We want to find how the picture obtained in this way depends on the four-velocity u of the observer.

14.1. Galilean spacetime

Here we assume that the light rays propagate with infinite speed. Thus the straight lines of the rays lie in the hyperplane $O + S$, and their directional vectors are in S. But for such vectors the decomposition (7) is trivial and independent of u. Therefore the picture formed by light on the celestial sphere is independent of the choice of particular observer crossing the point O.

14.2. Special relativity

A light ray with the directional past-pointing vector $-l \in V$ comes from the space direction pointed by the unit spacelike vector

$$r(l,u) = \frac{-l_u}{|l_u|} = -\frac{l - u \cdot l\, u}{u \cdot l} = -\frac{l}{u \cdot l} + u\,, \tag{57}$$

where we have used the fact that $|l_u|^2 = -l_u \cdot l_u = (u \cdot l)^2$ (see Fig. 27). If u' is the four-velocity of another observer passing O and we denote for brevity $r = r(l,u)$, $r' = r(l,u')$ then we find

$$\frac{u' \cdot l}{u \cdot l} = (u - r) \cdot u' = c(k) + s(k)\, n \circ r\,. \tag{58}$$

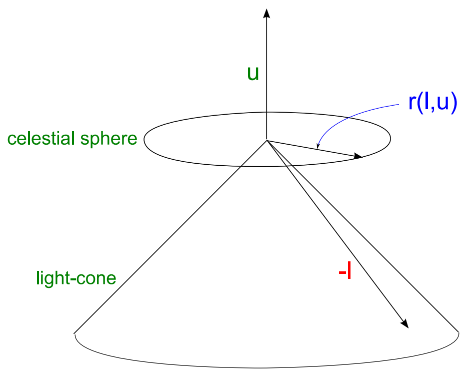

Fig. 27. Celestial sphere.

Using this and Eq. (57) for r and r' we find the transformation $r \mapsto r'$ of the celestial sphere of the u-observer to the sphere of the u'-observer:

$$r' = u' + \frac{r - u}{c(k) + s(k)\, n \circ r}\,. \tag{59}$$

Taking the scalar product of this equation with u we find, in particular, the well-known aberration formula:

$$n' \circ r' = -\frac{s(k) + c(k)\, n \circ r}{c(k) + s(k)\, n \circ r} \tag{60}$$

(the difference in signs is due to the direction of n and n').

A small variation of the direction of the light ray induces small variations δr and $\delta r'$, which are tangent to the two respective celestial spheres. The linear transformation $\delta r \mapsto \delta r'$ is found by varying Eq. (59):

$$\delta r' = \frac{\delta r}{c(k) + s(k)\, n \circ r} + \frac{s(k)\, n \circ \delta r}{[c(k) + s(k)\, n \circ r]^2}\,(u - r)\,. \tag{61}$$

Taking now two different variations δ_1 and δ_2 and using the constraints $u \cdot \delta r = r \cdot \delta r = 0$ we find

$$\delta_1 r' \circ \delta_2 r' = \frac{\delta_1 r \circ \delta_2 r}{[c(k) + s(k)\, n \circ r]^2}\,. \tag{62}$$

This equation tells us that the linear transformation $\delta r \mapsto \delta r'$ differs only by the factor $[c(k) + s(k)\, n \circ r]^{-1}$ from an isometric transformation. Thus locally (in the first order in δr) the picture registered on the celestial sphere scales by this factor without a change of the shape (the angles) [5].

Larger areas on the celestial sphere lose this scaling property and undergo more complicated transformations. However, one feature of the local transformation survives. To find it chose a spacelike vector z, $z^2 < 0$, and consider among vectors $-l$ all those which satisfy the equation

$$z \cdot l = 0\,. \tag{63}$$

Using the geometrical quantities correlated to u the spacelike character of z is written down as $(u \cdot z)^2 < |z_u|^2$ and the above condition on l's takes the form

$$r(l, u) \circ \frac{z_u}{|z_u|} = -\frac{u \cdot z}{|z_u|} = \cos[\phi(z, u)]\,, \tag{64}$$

where the last equality defines the angle $\phi(z, u)$. This equation tells us that the vectors $r(l, u)$ are all those which form the angle $\phi(z, u)$ with the vector $z_u/|z_u|$. Thus they form a circle on the celestial sphere. This fact is independent of the choice of a particular observer (its vector u) crossing the point O. However, the angle $\phi(z, u)$ does depend on this choice. Note in particular that if Eq. (63) determines a 'great circle' for the observer with four-velocity u (*i.e.* $\phi(z, u) = \pi/2$), this circle will in general cease to be 'great' for the one with the four-velocity u'. The exceptional cases when 'great' goes to 'great' are those determined by z orthogonal both to u and u'.

To summarize, the picture obtained on the celestial sphere undergoes deformation from one observer to another, but in such a way that angles are conserved and circles become circles, although the 'greatness' property is usually not conserved. This is illustrated in Figs. 28 and 29.

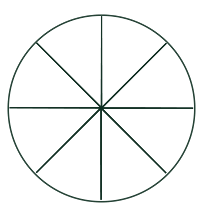

Fig. 28. A bicycle wheel in rest.

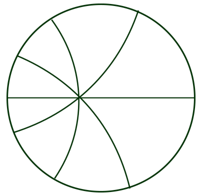

Fig. 29. The same wheel as seen by a fast moving observer.

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Appendix

Theorem. *The cone V determines uniquely up to a constant factor a symmetric bilinear form g such that* $x \in V \iff g(x,x) = 0$.

Proof. In a canonical basis V takes the form given in Eq. (1), which is equivalent to $x^0 = \pm\sqrt{\sum_{i=1}^3 (x^i)^2}$. If this implies $g(x,x) = \sum_{\mu,\nu=0}^3 g_{\mu\nu}x^\mu x^\nu = 0$, then the conditions

$$(g_{ik} + g_{00}\delta_{ik})\, x^i x^k \pm 2g_{0i}\sqrt{\sum_{i=1}^3 (x^i)^2}\, x^i = 0$$

must be satisfied identically (for any numbers x^i, $i = 1,2,3$). Thus $g_{0i} = 0$, $i = 1,2,3$, and $g_{ik} + g_{00}\delta_{ik} = 0$, $i,k = 1,2,3$. Therefore in this frame $g(x,y) = g_{00}\left(x^0y^0 - \sum_{i=1}^3 x^iy^i\right)$. $\square$

REFERENCES

[1] H. Bondi, *Relativity and Common Sense*, Doubleday & Company, New York 1964.

[2] W. Kopczyński, A. Trautman, *Spacetime and Gravitation*, John Wiley & Sons, 1992 [Polish original publication: Ossolineum, 1971].

[3] R. Geroch, *General Relativity from A to B*, The University of Chicago Press, Chicago & London 1978.

[4] R. Penrose, *Proc. Cambridge Phil. Soc.* **55**, 137 (1959).

[5] J. Terrell, *Phys. Rev.* **116**, 1041 (1959).

[6] W. Rindler, *Am. J. Phys.* **29**, 365 (1961); R. Shaw, *Am. J. Phys.* **30**, 72 (1962).