EXACT SOLUTIONS FOR A QUANTUM-MECHANICAL PARTICLE WITH SPIN 1 AND ADDITIONAL INTRINSIC CHARACTERISTICS IN A HOMOGENEOUS MAGNETIC FIELD

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With the use of the general covariant matrix 10-dimensional Petiau–Duffin–Kemmer formalism in cylindrical coordinates exact solutions of the quantum-mechanical equation for a particle with spin-1 in the presence of an external homogeneous magnetic field are constructed. Three linearly independent types of solutions are separated; in each case the formula for the energy levels has been found. Within similar technique for the quantum-mechanical equation for a particle with spin-1 and additional intrinsic electromagnetic characteristics — polarizability, exact solutions are found in the presence of an external homogeneous magnetic field.

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1. Introduction

The problem of a quantum-mechanical particle in an external homogeneous magnetic field is well-known in theoretical physics. In fact, only two cases are considered: a scalar (Schrödinger's) non-relativistic particle with spin-0, and fermions (non-relativistic Pauli's and relativistic Dirac's) with spin-1/2 (the first investigation were [1–4]). In the case of spin-1 particle, the most popular quantum-mechanical problem is the Coulomb one [4].

In the first part of the paper (Sections 1–3), exact solutions for an ordinary vector particle will be constructed. In the second part (Sections 4–6), the exact solutions for a particle with spin-1 and an additional intrinsic electromagnetic parameter (polarizability) will be also explicitly constructed. In principle, these results provide us with a possibility for experimental testing of this characteristics — polarizability of the spin-1 particle.

To treat the problem for an ordinary vector particle we take the matrix Petiau–Duffin–Kemmer approach extended to a general covariant form on the basis of the tetrad formalism (recent consideration and references see *e.g.*, in [5,6]). The main equation in the tetrad form reads [6]

$$\left[i\beta^{\alpha}(x) \left(\partial_{\alpha} + B_{\alpha} - i\frac{e}{\hbar}A_{\alpha} \right) - \frac{Mc}{\hbar} \right] \Psi(x) = 0\,,$$
$$\beta^{\alpha}(x) = \beta^{a} e^{\alpha}_{(a)}(x)\,, \qquad B_{\alpha}(x) = \tfrac{1}{2} J^{ab} e^{\beta}_{(a)} \nabla_{\alpha} e_{(b)\beta}\,, \tag{1}$$

$e^{\alpha}_{(a)}(x)$ is a tetrad, J^{ab} stands for generators for 10-dimensional representation of the Lorentz group referred to 4-vector and anti-symmetric tensor (for brevity we note $Mc/\hbar$ as M). The homogeneous magnetic field $\boldsymbol{B} = (0,0,B)$ corresponds to 4-potential $A^{a} = (0, \frac{1}{2}\vec{B} \times \vec{r})$; in the cylindric coordinates, the last is given by

$$dS^2 = c^2dt^2 - dr^2 - r^2\,d\phi^2 - dz^2\,, \qquad A_{\phi} = -\frac{B\,r^2}{2}\,. \tag{2}$$

Choosing a diagonal cylindric tetrad

$$e^{\alpha}_{(0)} = (1,0,0,0)\,, \quad e^{\alpha}_{(1)} = (0,1,0,0)\,, \quad e^{\alpha}_{(2)} = \left(0, 0, \frac{1}{r}, 0\right)\,, \quad e^{\alpha}_{(3)} = (0,0,0,1)\,,$$

after simple calculations, the main equation (1) reduces to the form

$$\left[i\beta^0\partial_0 + i\beta^1\partial_r + i\frac{\beta^2}{r}\left(\partial_{\phi} + \frac{ieB}{2\hbar}r^2 + J^{12} \right) + i\beta^3\partial_z - M \right] \Psi = 0\,. \tag{3}$$

For brevity we will note $(eB/2\hbar)$ as B. It is better to choose the matrices β^a in the so-called cyclic form, where the generator J^{12} has a diagonal structure. These matrices are given in [6].

2. Separation of variables

With the use of a special substitution (it corresponds to diagonalization of the third projections of momentum P_3 and angular momentum J_3 for a particle with spin-1, specified to the cylindric tetrad basis)

$$\Psi = e^{-i\epsilon t} e^{im\phi} e^{ikz} \left| \begin{array}{c} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{array} \right|\,, \tag{4}$$

the main equation reads

$$\left[\epsilon\beta^0 + i\beta^1\partial_r - \frac{\beta^2}{r}\left(m + Br^2 - S_3 \right) - k\beta^3 - M \right] \left| \begin{array}{c} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{array} \right| = 0\,,$$

after calculations we arrive at the radial system of 10 equations

$$\begin{aligned} -\hat{b}_{m-1}E_1 - \hat{a}_{m+1}E_3 - ikE_2 &= M\Phi_0\,, \\ -i\hat{b}_{m-1}H_1 + i\hat{a}_{m+1}H_3 + i\epsilon E_2 &= M\Phi_2\,, \\ i\hat{a}_m H_2 + i\epsilon E_1 - kH_1 &= M\Phi_1\,, \\ -i\hat{b}_m H_2 + i\epsilon E_3 + kH_3 &= M\Phi_3\,, \end{aligned} \tag{5}$$

$$\begin{aligned} \hat{a}_m\Phi_0 - i\epsilon\Phi_1 &= ME_1\,, & -i\hat{a}_m\Phi_2 + k\Phi_1 &= MH_1\,, \\ \hat{b}_m\Phi_0 - i\epsilon\Phi_3 &= ME_3\,, & i\hat{b}_m\Phi_2 - k\Phi_3 &= MH_3\,, \\ -i\epsilon\Phi_2 - ik\Phi_0 &= ME_2\,, & i\hat{b}_{m-1}\Phi_1 - i\hat{a}_{m+1}\Phi_3 &= MH_2\,, \end{aligned} \tag{6}$$

where special abbreviations were used for first order differential operators

$$\frac{1}{\sqrt{2}}\left(\frac{d}{dr} + \frac{m+Br^2}{r}\right) = \hat{a}_m\,, \qquad \frac{1}{\sqrt{2}}\left(-\frac{d}{dr} + \frac{m+Br^2}{r}\right) = \hat{b}_m\,.$$

From (5) and (6) it follows 4 equations for the components Φ_a

$$\begin{aligned} &\left(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - k^2 - M^2\right)\Phi_0 - \epsilon k\Phi_2 + i\epsilon\left(\hat{b}_{m-1}\Phi_1 + \hat{a}_{m+1}\Phi_3\right) = 0\,, \\ &\left(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - M^2\right)\Phi_2 + \epsilon k\Phi_0 - ik\left(\hat{b}_{m-1}\Phi_1 + \hat{a}_{m+1}\Phi_3\right) = 0\,, \\ &\left(-\hat{a}_m\hat{b}_{m-1} + \epsilon^2 - k^2 - M^2\right)\Phi_1 + \hat{a}_m\hat{a}_{m+1}\Phi_3 + i\epsilon\hat{a}_m\Phi_0 + ik\hat{a}_m\Phi_2 = 0\,, \\ &\left(-\hat{b}_m\hat{a}_{m+1} + \epsilon^2 - M^2 - k^2\right)\Phi_3 + \hat{b}_m\hat{b}_{m-1}\Phi_1 + i\epsilon\hat{b}_m\Phi_0 + ik\hat{b}_m\Phi_2 = 0\,. \end{aligned} \tag{7}$$

3. General analysis of the radial equations

Eqs. (7) can be transformed to the form

$$\begin{aligned} &\left[-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - M^2 - k^2\right](k\Phi_0 + \epsilon\Phi_2) = 0\,, \\ &\left[-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2 - M^2\right](\epsilon\Phi_0 + k\Phi_2) \\ &= \left(\epsilon^2 - k^2\right)\left[(\epsilon\Phi_0 + k\Phi_2) - \left(i\hat{b}_{m-1}\Phi_1 + i\hat{a}_{m+1}\Phi_3\right)\right]\,, \end{aligned} \tag{8}$$

$$\begin{aligned} &\left(-\hat{a}_m\hat{b}_{m-1} + \epsilon^2 - k^2 - M^2\right)\Phi_1 + \hat{a}_m\hat{a}_{m+1}\Phi_3 + i\epsilon\hat{a}_m\Phi_0 + ik\hat{a}_m\Phi_2 = 0\,, \\ &\left(-\hat{b}_m\hat{a}_{m+1} + \epsilon^2 - M^2 - k^2\right)\Phi_3 + \hat{b}_m\hat{b}_{m-1}\Phi_1 + i\epsilon\hat{b}_m\Phi_0 + ik\hat{b}_m\Phi_2 = 0\,. \end{aligned} \tag{9}$$

Let us introduce new variables

$$F(r) = k\Phi_0(r) + \epsilon\Phi_2(r)\,, \qquad G(r) = \epsilon\Phi_0(r) + k\Phi_2(r)\,, \tag{10}$$

then Eqs. (8) and (9) read

$$\left[-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - M^2 - k^2\right] F = 0\,,$$
$$\left[-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - M^2\right] G = -\left(\epsilon^2 - k^2\right)\left(i\hat{b}_{m-1}\Phi_1 + i\hat{a}_{m+1}\Phi_3\right)\,, \quad (11)$$
$$\left(-\hat{a}_m\hat{b}_{m-1} + \epsilon^2 - k^2 - M^2\right)\Phi_1 + \hat{a}_m\hat{a}_{m+1}\Phi_3 + i\hat{a}_m G = 0\,,$$
$$\left(-\hat{b}_m\hat{a}_{m+1} + \epsilon^2 - M^2 - k^2\right)\Phi_3 + \hat{b}_m\hat{b}_{m-1}\Phi_1 + i\hat{b}_m G = 0\,. \quad (12)$$

For equations (12), let us multiply the first one (from the left) by $\hat{b}_{m-1}$ and the second one by the $\hat{a}_{m+1}$, that results in

$$\begin{aligned} -\hat{b}_{m-1}\hat{a}_m\left(\hat{b}_{m-1}\Phi_1\right) + \left(\epsilon^2 - k^2 - M^2\right)\left(\hat{b}_{m-1}\Phi_1\right) & \\ +\hat{b}_{m-1}\hat{a}_m\left(\hat{a}_{m+1}\Phi_3\right) + i\hat{b}_{m-1}\hat{a}_m G &= 0\,, \\ -\hat{a}_{m+1}\hat{b}_m\left(\hat{a}_{m+1}\Phi_3\right) + \left(\epsilon^2 - M^2 - k^2\right)\left(\hat{a}_{m+1}\Phi_3\right) & \\ +\hat{a}_{m+1}\hat{b}_m\left(\hat{b}_{m-1}\Phi_1\right) + i\hat{a}_{m+1}\hat{b}_m G &= 0\,. \end{aligned} \quad (13)$$

Again, let us introduce two new field variables

$$\hat{b}_{m-1}\Phi_1 = Z_1\,, \qquad \hat{a}_{m+1}\Phi_3 = Z_3\,. \quad (14)$$

Eqs. (13) read as follows

$$\begin{aligned} -\hat{b}_{m-1}\hat{a}_m Z_1 + \left(\epsilon^2 - k^2 - M^2\right) Z_1 + \hat{b}_{m-1}\hat{a}_m Z_3 + i\hat{b}_{m-1}\hat{a}_m G &= 0\,, \\ -\hat{a}_{m+1}\hat{b}_m Z_3 + \left(\epsilon^2 - M^2 - k^2\right) Z_3 + \hat{a}_{m+1}\hat{b}_m Z_1 + i\hat{a}_{m+1}\hat{b}_m G &= 0\,. \end{aligned} \quad (15)$$

With the help of new functions $f(r)$, $g(r)$

$$Z_1 = \frac{f+g}{2}\,, \qquad Z_3 = \frac{f-g}{2}\,, \qquad Z_1 + Z_3 = f\,, \qquad Z_1 - Z_3 = g \quad (16)$$

the system (15) is transformed to the following form

$$\begin{aligned} -\hat{b}_{m-1}\hat{a}_m g + \left(\epsilon^2 - k^2 - M^2\right)\frac{f+g}{2} + i\hat{b}_{m-1}\hat{a}_m G &= 0\,, \\ \hat{a}_{m+1}\hat{b}_m g + \left(\epsilon^2 - M^2 - k^2\right)\frac{f-g}{2} + i\hat{a}_{m+1}\hat{b}_m G &= 0\,. \end{aligned} \quad (17)$$

Combining these equations we get

$$\left[-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2 - M^2\right] g + i\left(\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m\right) G = 0\,,$$
$$\left(-\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m\right) g + \left(\epsilon^2 - k^2 - M^2\right) f + i\left(\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m\right) G = 0\,. \quad (18)$$

In turn, Eqs. (11) can be presented as

$$\left(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - M^2 - k^2\right) F = 0\,,$$
$$\left(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - M^2\right) G = -i\left(\epsilon^2 - k^2\right) f\,. \tag{19}$$

Further, with the use of identities

$$-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m = \mathit{\Delta}\,, \qquad -\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m = 2B\,. \tag{20}$$

Eqs. (19) and (18) can be written as follows

$$\begin{aligned} &\left(\mathit{\Delta} + \epsilon^2 - M^2 - k^2\right) F = 0\,,\\ &\Delta G = M^2 G - i\left(\epsilon^2 - k^2\right) f\,,\\ &\left(\mathit{\Delta} + \epsilon^2 - k^2 - M^2\right) g = 2iBG\,,\\ &\left(\epsilon^2 - k^2 - M^2\right) f - i\,\Delta G + 2Bg = 0\,. \end{aligned} \tag{21}$$

With the help of the second equation, from the forth one it follows

$$f = -i\,G + \frac{2B}{M^2}g\,. \tag{22}$$

Now, one excludes the function f in the second equation in (21) and gets

$$\left(\mathit{\Delta} + \epsilon^2 - k^2 - M^2\right) G = -i\left(\epsilon^2 - k^2\right)\frac{2B}{M^2}g\,. \tag{23}$$

Thus, the general problem is reduced to the system of four equations

$$\begin{aligned} &\left(\mathit{\Delta} + \epsilon^2 - M^2 - k^2\right) F = 0\,,\\ &f = -i\,G + \frac{2B}{M^2}g\,,\\ &\left(\mathit{\Delta} + \epsilon^2 - k^2 - M^2\right) g = 2iB\,G\,,\\ &\left(\mathit{\Delta} + \epsilon^2 - k^2 - M^2\right) G = -2iB\frac{\epsilon^2 - k^2}{M^2}g\,. \end{aligned} \tag{24}$$

The structure of this system allows to separate an evident, linearly independent solution as follows

$$\begin{aligned} f(r) &= 0\,, \qquad g(r) = 0\,, \qquad H(r) = 0\,,\\ F(r) &\neq 0\,, \qquad \left(\mathit{\Delta} - k^2 - M^2 + \epsilon^2\right) F = 0\,. \end{aligned} \tag{25}$$

Corresponding functions and energy spectrum are known. We are to solve the system of two last equations in (24); in the matrix form it reads (let $\gamma = (\epsilon^2 - k^2)/M^2$)

$$\left(\Delta + \epsilon^2 - M^2 - k^2\right) \begin{vmatrix} g \\ G \end{vmatrix} = \begin{vmatrix} 0 & 2iB \\ -2iB\gamma & 0 \end{vmatrix} \begin{vmatrix} g \\ G \end{vmatrix} . \tag{26}$$

Let us construct the transformation changing the matrix on the right to a diagonal form

$$\left(\Delta + \epsilon^2 - M^2 - k^2\right) \begin{vmatrix} g' \\ G' \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} \begin{vmatrix} g' \\ G' \end{vmatrix} ,$$
$$\begin{vmatrix} g' \\ G' \end{vmatrix} = S \begin{vmatrix} g \\ G \end{vmatrix} , \qquad S = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix} . \tag{27}$$

The problem reduces to linear systems

$$\begin{cases} -\lambda_1 s_{11} - 2iB\gamma s_{12} = 0 \,, \\ 2iBs_{11} - \lambda_1 s_{12} = 0 \,, \end{cases} \qquad \begin{cases} -\lambda_2 s_{21} - 2iB\gamma s_{22} = 0 \,, \\ 2iBs_{21} - \lambda_2 s_{22} = 0 \,. \end{cases}$$

The values of λ_1 and λ_2 are given by

$$\begin{aligned} \lambda_1 &= +2B\sqrt{\gamma} \,, \qquad \lambda_2 = -2B\sqrt{\gamma} \,, \\ is_{11} - \sqrt{\gamma} s_{12} &= 0 \,, \qquad is_{21} + \sqrt{\gamma} s_{22} = 0 \,, \end{aligned}$$
$$s_{12} = 1 \,, \qquad s_{22} = 1 \,, \qquad S = \begin{vmatrix} -i\sqrt{\gamma} & 1 \\ +i\sqrt{\gamma} & 1 \end{vmatrix} . \tag{28}$$

In the new (primed) basis, Eq. (26) takes the form of two separated differential equations

$$\begin{aligned} \left(\Delta + \epsilon^2 - k^2 - M^2 - 2B\sqrt{\gamma}\right) g' &= 0 \,, \\ \left(\Delta + \epsilon^2 - k^2 - M^2 + 2B\sqrt{\gamma}\right) G' &= 0 \,. \end{aligned} \tag{29}$$

Recalling the meaning of Δ, let us specify the second order differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\left(m + B\,r^2\right)^2}{r^2} + \lambda^2 \right) \varphi(r) = 0 \,,$$
$$\lambda^2 = \epsilon^2 - k^2 - M^2 \pm 2B\sqrt{\gamma} \,, \qquad \sqrt{\gamma} = \frac{\sqrt{\epsilon^2 - k^2}}{M} \,. \tag{30}$$

It is convenient to introduce a new variable $x = B r^2$, then Eq. (30) reads[1]

$$x\frac{d^2\varphi}{dx^2} + \frac{d\varphi}{dx} - \left(\frac{m^2}{4x} + \frac{x}{4} + \frac{m}{2} - \frac{\lambda^2}{4B}\right)\varphi = 0\,. \tag{31}$$

With the substitution $\varphi(x) = x^A e^{-Cx} f(x)$, for $f(x)$ we get

$$x\frac{d^2 f}{dx^2} + (2A + 1 - 2Cx)\,\frac{df}{dx}$$
$$+\left[\frac{A^2 - m^2/4}{x} + \left(C^2 - \frac{1}{4}\right)x - 2AC - C - \frac{m}{2} + \frac{\lambda^2}{4B}\right] f = 0\,.$$

When A, C are taken as $A = + \mid m \mid /2\,,\ C = +1/2$ the previous equation becomes simpler

$$x\frac{d^2 R}{dx^2} + (2A + 1 - x)\,\frac{dR}{dx} - \left(A + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B}\right) R = 0\,,$$

which is of confluent hypergeometric type

$$x\,Y'' + (\gamma - x)\,Y' - \alpha Y = 0\,,$$

$$\alpha = \frac{\mid m \mid}{2} + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B}\,, \qquad \gamma = \mid m \mid +1\,.$$

To obtain polynomials we must impose an additional condition, $\alpha = -n$, which provides us with the following quantization rule for λ^2

$$\lambda^2 = 4B\left(n + \frac{1}{2} + \frac{\mid m \mid + m}{2}\right)\,. \tag{32}$$

Thus, we have arrived at two formulas for the energy

$$\sqrt{\epsilon^2 - k^2} = \frac{+B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M}\,,$$
$$\sqrt{\epsilon^2 - k^2} = \frac{-B + \sqrt{B^2 + M^2\,(M^2 + \lambda^2)}}{M}\,. \tag{33}$$

In turn, the energy spectrum for the case (25) is given by

$$\epsilon^2 = M^2 + k^2 + \lambda^2\,. \tag{34}$$

Thus, on the base of the use of general covariant formalism in the Petiau–Duffin–Kemmer theory of the vector particle, the exact solutions for such a particle are constructed in the presence of an external homogeneous magnetic field. There are separated three types of linearly independent solutions, and the corresponding energy spectra are found.

[1] For definiteness let us consider B to be positive, which does not affect the generality of the analysis. So, to infinite values of r correspond infinite and positive values of x.

4. On a spin-1 particle with intrinsic structure — polarizability

In [7, 13], it was described a generalized equation for spin-1 particle possessing in addition to electric charge the special electromagnetic characteristics named polarizability. In the framework of the first order relativistic wave equations, such a particle requires a 15-dimensional wave function, consisting of a 4-vector $\Phi_a(x)$, 4-tensor $\Phi_{ab}(x)$, and subsidiary scalar and 4-vector fields, $C(x)$ and $C_a(x)$.

To treat the problem, we take the matrix approach in the theory of the generalized $S = 1$ particle extended to a general covariant form on the base of tetrad formalism (recent consideration, notation and list of references see in [15, 16]). The use of cylindric tetrad permits to take account of the cylindric symmetry of the problem. The main equation in tetrad form is [12]

$$\left[\Gamma^0\partial_0 + \Gamma^1\partial_r + \frac{1}{r}\Gamma^2\left(\partial_\phi + \frac{ieB}{2\hbar}r^2 + J^{12}\right) + \Gamma^3\partial_z - M\right]\Psi = 0\,. \qquad (35)$$

It is better to choose the matrices β^a in the so-called cyclic form, where the generator J^{12} has a diagonal structure. These matrices Γ^a are given in [6].

5. Separation of variables

With the use of special substitution

$$\begin{aligned}
\Psi &= \left\{C, C_0, \vec{C}, \Phi_0, \vec{\Phi}, \vec{E}, \vec{H}\right\}, \qquad C(x) = e^{-i\epsilon t/\hbar}e^{ikz}e^{im\phi}C(r)\,,\\
C_0 &= e^{-i\epsilon t/\hbar}e^{ikz}e^{im\phi}\,C_0(r)\,, \qquad \vec{C} = e^{-i\epsilon t/\hbar}e^{ikz}e^{im\phi}\begin{vmatrix} C_1(r)\\ C_2(r)\\ C_3(r)\end{vmatrix},\\
\Phi_0 &= e^{-i\epsilon t/\hbar}e^{ikz}e^{im\phi}\Phi_0(r)\,, \qquad \vec{\Phi} = e^{-i\epsilon t/\hbar}e^{ikz}e^{im\phi}\begin{vmatrix} \Phi_1(r)\\ \Phi_2(r)\\ \Phi_3(r)\end{vmatrix},\\
\vec{E} &= e^{-i\epsilon t/\hbar}e^{ikz}e^{im\phi}\begin{vmatrix} E_1(r)\\ E_2(r)\\ E_3(r)\end{vmatrix}, \qquad \vec{H} = e^{-i\epsilon t/\hbar}e^{ikz}e^{im\phi}\begin{vmatrix} H_1(r)\\ H_2(r)\\ H_3(r)\end{vmatrix}, \qquad (36)
\end{aligned}$$

after calculations we arrive at the radial system of 15 equations

$$\begin{aligned}
-i\epsilon\,C_0 - \hat{b}_{m-1}C_1 - \hat{a}_{m+1}C_3 - ik\,C_2 &= M\,C\,, \qquad (37)\\
-\hat{b}_{m-1}E_1 - \hat{a}_{m+1}E_3 - ik\,E_2 &= M\,C_0\,,\\
i\epsilon\,E_1 + i\hat{a}_m H_2 - ikH_1 &= M\,C_1\,,\\
i\epsilon\,E_2 - i\hat{b}_{m-1}H_1 + i\hat{a}_{m+1}H_3 &= M\,C_2\,,
\end{aligned}$$

$$i\epsilon\, E_3 - i\hat{b}_m H_2 + kH_3 = M\, C_3\,, \tag{38}$$

$$\begin{aligned}
-i\epsilon\,\sigma C - \hat{b}_{m-1}E_1 - \hat{a}_{m+1}E_3 - ik\, E_2 &= M\Phi_0\,,\\
i\epsilon\, E_1 - \sigma\hat{a}_m C + i\hat{a}_m H_2 - kH_1 &= M\Phi_1\,,\\
i\epsilon\, E_2 - i\hat{b}_{m-1}H_1 + i\hat{a}_{m+1}H_3 + i\,k\sigma C &= M\Phi_2\,,\\
i\epsilon\, E_3 - \sigma\hat{b}_m C - i\hat{b}_m H_2 + kH_3 &= M\Phi_3\,,
\end{aligned} \tag{39}$$

$$\begin{aligned}
-i\epsilon\,\Phi_1 + \hat{a}_m\Phi_0 = ME_1\,, &\qquad -i\epsilon\,\Phi_2 - ik\Phi_0 = ME_2\,,\\
-i\epsilon\,\Phi_3 + \hat{b}_m\Phi_0 = ME_3\,, &\qquad -i\hat{a}_m\Phi_2 + k\Phi_1 = MH_1\,,\\
i\hat{b}_{m-1}\Phi_1 - i\hat{a}_{m+1}\Phi_3 = MH_2\,, &\qquad i\hat{b}_m\Phi_2 - k\Phi_3 = MH_3\,.
\end{aligned} \tag{40}$$

6. Solution of the radial system

With the use of (38), Eqs. (39) give

$$\begin{aligned}
C_0 &= \Phi_0 + i\frac{\epsilon\sigma}{M}C\,, \qquad C_1 = \Phi_1 + \frac{\sigma}{M}\hat{a}_m C\,,\\
C_2 &= \Phi_2 - i\frac{k\sigma}{M}C\,, \qquad C_3 = \Phi_3 + \frac{\sigma}{M}\hat{b}_m C\,.
\end{aligned} \tag{41}$$

Substituting these formulas for C_a into (37)

$$\begin{aligned}
&-i\epsilon\left(\Phi_0 + i\frac{\epsilon\sigma}{M}C\right) - \hat{b}_{m-1}\left(\Phi_1 + \frac{\sigma}{M}\hat{a}_m C\right)\\
&-\hat{a}_{m+1}\left(\Phi_3 + \frac{\sigma}{M}\hat{b}_m C\right) - ik\left(\Phi_2 - i\frac{k\sigma}{M}C\right) = M\,C\,,
\end{aligned}$$

we further get

$$\begin{aligned}
&M\left(\hat{b}_{m-1}\Phi_1 + \hat{a}_{m+1}\Phi_3\right) = -iM\left(\epsilon\Phi_0 + k\Phi_2\right)\\
&+\sigma\left(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2\right)C - M^2 C\,.
\end{aligned} \tag{42}$$

This equation will be required below.

Note that Eqs. (39) and (40) include the main field variables, 4-vector and 4-tensor, and also the scalar C obeying Eq. (42)

$$\begin{aligned}
-i\epsilon\sigma C - \hat{b}_{m-1}E_1 - \hat{a}_{m+1}E_3 - ikE_2 &= M\Phi_0\,,\\
i\epsilon E_1 - \sigma\hat{a}_m C + i\hat{a}_m H_2 - kH_1 &= M\Phi_1\,,\\
i\epsilon E_2 - i\hat{b}_{m-1}H_1 + \hat{a}_{m+1}H_3 + ik\sigma C &= M\Phi_2\,,\\
i\epsilon E_3 - \sigma\hat{b}_m C - i\hat{b}_m H_2 + kH_3 &= M\Phi_3\,.
\end{aligned} \tag{43}$$

$$\begin{aligned} -i\epsilon\Phi_1 + \hat{a}_m\Phi_0 &= M\,E_1\,, & -i\epsilon\Phi_2 - ik\Phi_0 = M\,E_2\,, \\ -i\epsilon\Phi_3 + \hat{b}_m\Phi_0 &= M\,E_3\,, & -i\hat{a}_m\Phi_2 + k\Phi_1 = M\,H_1\,, \\ i\hat{b}_{m-1}\Phi_1 - i\hat{a}_{m+1}\Phi_3 &= M\,H_2\,, & i\hat{b}_m\Phi_2 - k\Phi_3 = M\,H_3\,. \end{aligned} \tag{44}$$

By means of (44), we are to eliminate tensor components in (43). Then we obtain two equations

$$\begin{aligned} -i\epsilon\sigma M\,C + \left(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - k^2 - M^2\right)\Phi_0 & \\ +i\epsilon\left(\hat{b}_{m-1}\Phi_1 + \hat{a}_{m+1}\Phi_3 + ik\Phi_2\right) &= 0\,, \end{aligned} \tag{45}$$

$$\begin{aligned} ik\sigma M\,C + \left(\epsilon^2 - \hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - M^2\right)\Phi_2 & \\ -ik\left(i\epsilon\Phi_0 + \hat{b}_{m-1}\Phi_1 + \hat{a}_{m+1}\Phi_3\right) &= 0\,. \end{aligned} \tag{46}$$

Multiplying the first one (45) by $+ik$, and the second one (46) by $i\epsilon$, and summing the results, we get

$$\left(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - k^2 - M^2 + \epsilon^2\right)(k\Phi_0 + \epsilon\Phi_2) = 0\,. \tag{47}$$

In the same manner, combining Eqs. (45) and (46) with other coefficients, we arrive at

$$\begin{aligned} &\left(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - M^2\right)(\epsilon\Phi_0 + k\Phi_2) \\ &= -i\left(\epsilon^2 - k^2\right)\left(\hat{b}_{m-1}\Phi_1 + \hat{a}_{m+1}\Phi_3\right) + i\sigma M\left(\epsilon^2 - k^2\right)C\,. \end{aligned} \tag{48}$$

Thus, two second order equations have been found

$$\left(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - k^2 - M^2 + \epsilon^2\right)(k\Phi_0 + \epsilon\Phi_2) = 0\,, \tag{49}$$

$$\begin{aligned} &\left(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m - M^2\right)(\epsilon\Phi_0 + k\Phi_2) \\ &= -i\left(\epsilon^2 - k^2\right)\left(\hat{b}_{m-1}\Phi_1 + \hat{a}_{m+1}\Phi_3\right) + i\sigma M\left(\epsilon^2 - k^2\right)C\,. \end{aligned} \tag{50}$$

Now, let us turn to equations in (43), containing functions $m\Phi_1$ and $m\Phi_3$

$$\begin{aligned} &\left(-\hat{a}_m\hat{b}_{m-1} + \epsilon^2 - k^2 - M^2\right)\Phi_1 \\ &+\hat{a}_m\hat{a}_{m+1}\Phi_3 + i\hat{a}_m(\epsilon\Phi_0 + k\Phi_2) - M\sigma\hat{a}_m C = 0 \end{aligned} \tag{51}$$

and

$$\begin{aligned} &\left(-\hat{b}_m\hat{a}_{m+1} + \epsilon^2 - k^2 - M^2\right)\Phi_3 \\ &+\hat{b}_m\hat{b}_{m-1}\Phi_1 + i\hat{b}_m(\epsilon\Phi_0 + k\Phi_2) - M\sigma\hat{b}_m C = 0\,. \end{aligned} \tag{52}$$

In two last equations, (51) and (52), multiplying the first one by $\hat{b}_{m-1}$ (from the left) and the second one by $\hat{a}_{m+1}$ (from the left), we produce

$$\begin{aligned} & \left(-\hat{b}_{m-1}\hat{a}_m + \epsilon^2 - k^2 - M^2\right)\hat{b}_{m-1}\Phi_1 \\ +\hat{b}_{m-1}\hat{a}_m\hat{a}_{m+1}\Phi_3 + i\hat{b}_{m-1}\hat{a}_m\left(\epsilon\Phi_0 + k\Phi_2\right) - M\sigma\hat{b}_{m-1}\hat{a}_m C &= 0\,, \quad (53) \\ & \left(-\hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2 - M^2\right)\hat{a}_{m+1}\Phi_3 \\ +\hat{a}_{m+1}\hat{b}_m\hat{b}_{m-1}\Phi_1 + i\hat{a}_{m+1}\hat{b}_m\left(\epsilon\Phi_0 + k\Phi_2\right) - M\sigma\hat{a}_{m+1}\hat{b}_m C &= 0\,. \quad (54) \end{aligned}$$

It is better to introduce new field variables

$$\begin{aligned} F(r) &= k\Phi_0 + \epsilon\Phi_2\,, \qquad G(r) = \epsilon\Phi_0 + k\Phi_2\,, \\ \hat{b}_{m-1}\Phi_1 &= Z_1\,, \qquad \hat{a}_{m+1}\Phi_3 = Z_3\,, \end{aligned} \quad (55)$$

then the system (53)–(54) reads

$$\begin{aligned} & \left(-\hat{b}_{m-1}\hat{a}_m + \epsilon^2 - k^2 - M^2\right)Z_1 \\ +\hat{b}_{m-1}\hat{a}_m Z_3 + i\hat{b}_{m-1}\hat{a}_m G - M\sigma\hat{b}_{m-1}\hat{a}_m C &= 0\,, \quad (56) \\ & \left(-\hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2 - M^2\right)Z_3 \\ +\hat{a}_{m+1}\hat{b}_m Z_1 + i\hat{a}_{m+1}\hat{b}_m G - M\sigma\hat{a}_{m+1}\hat{b}_m C &= 0\,. \quad (57) \end{aligned}$$

Again, it is convenient to define new variables $f(r)$, $g(r)$

$$Z_1 = \frac{f+g}{2}\,, \qquad Z_3 = \frac{f-g}{2}\,, \qquad Z_1 + Z_3 = f\,, \qquad Z_1 - Z_3 = g\,, \quad (58)$$

then Eqs. (56) and (57) give

$$\begin{aligned} & \left(-\hat{b}_{m-1}\hat{a}_m + \epsilon^2 - k^2 - M^2\right)\frac{f+g}{2} \\ +\hat{b}_{m-1}\hat{a}_m\frac{f-g}{2} + i\hat{b}_{m-1}\hat{a}_m G - M\sigma\hat{b}_{m-1}\hat{a}_m C &= 0\,, \quad (59) \\ & \left(-\hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2 - M^2\right)\frac{f-g}{2} \\ +\hat{a}_{m+1}\hat{b}_m\frac{f+g}{2} + i\hat{a}_{m+1}\hat{b}_m G - M\sigma\hat{a}_{m+1}\hat{b}_m C &= 0\,. \quad (60) \end{aligned}$$

After simple manipulation, from two last equations it follows that

$$\begin{aligned} & \left[-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2 - M^2\right]g \\ + & \left(-\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m\right)\left(-iG + M\sigma C\right) = 0\,, \end{aligned}$$

$$\left(-\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m\right) g + \left(\epsilon^2 - k^2 - M^2\right) f$$
$$+ \left(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m\right) (-i\, G + M\sigma C) \; = \; 0\,. \qquad (61)$$

With the use of identities

$$-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m = \varDelta\,, \qquad -\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m = 2B$$

Eqs. (61) can be written as

$$\left[\varDelta + \epsilon^2 - k^2 - M^2\right] g + 2B\,(-iG + M\sigma C) \; = \; 0\,,$$
$$2B\, g + \left(\epsilon^2 - k^2 - M^2\right) f + \varDelta\,(-iG + M\sigma C) \; = \; 0\,. \qquad (62)$$

In turn, Eqs. (47), (48) will read (in the new variables)

$$\left(\varDelta - k^2 - M^2 + \epsilon^2\right) F = 0\,,$$
$$\left(\varDelta - M^2\right) G = -i\left(\epsilon^2 - k^2\right) f + i\sigma M\left(\epsilon^2 - k^2\right) C\,. \qquad (63)$$

Let us collect results together

$$\left(\varDelta - k^2 - M^2 + \epsilon^2\right) F = 0\,, \qquad (64)$$
$$\left(\varDelta - M^2\right) G = -i\left(\epsilon^2 - k^2\right) f + i\sigma M\left(\epsilon^2 - k^2\right) C\,, \qquad (65)$$
$$\left(\varDelta + \epsilon^2 - k^2 - M^2\right) g + 2B\,(-iG + M\sigma C) = 0\,, \qquad (66)$$
$$2B\, g + \left(\epsilon^2 - k^2 - M^2\right) f + \varDelta\,(-iG + M\sigma C) = 0\,. \qquad (67)$$

It is possible to eliminate the function $C(r)$ in the above equation. To show how it can be done, let us turn to a couple of equations in (39), containing the terms $M\,\varPhi_1$, $M\,\varPhi_3$, and find the combination

$$\hat{b}_{m-1}M\varPhi_1 + \hat{a}_{m+1}\varPhi_3$$
$$= i\epsilon\hat{b}_{m-1}E_1 - \sigma\hat{b}_{m-1}\hat{a}_m C + i\hat{b}_{m-1}\hat{a}_m H_2 - k\hat{b}_{m-1}H_1$$
$$+ i\epsilon\hat{a}_{m+1}E_3 - \sigma\hat{a}_{m+1}\hat{b}_m C - i\hat{a}_{m+1}\hat{b}_m H_2 + k\hat{a}_{m+1}H_3$$
$$= i\epsilon\left(\hat{b}_{m-1}E_1 + \hat{a}_{m+1}E_3\right) - \sigma\left(\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m\right) C$$
$$+ i\left(\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m\right) H_2 - k\left(\hat{b}_{m-1}H_1 - \hat{a}_{m+1}H_3\right)$$

from whence, with the help of the first and third equations in (39) in the form

$$\left(\hat{b}_{m-1}E_1 + \hat{a}_{m+1}E_3\right) \; = \; -i\epsilon\sigma C - ik\, E_2 - M\varPhi_0\,,$$
$$\left(\hat{b}_{m-1}H_1 - \hat{a}_{m+1}H_3\right) \; = \; \epsilon E_2 + k\sigma C + iM\varPhi_2\,,$$

we obtain

$$\begin{aligned}
&\hat{b}_{m-1} M\Phi_1 + \hat{a}_{m+1}\Phi_3 \\
&= i\epsilon\left(-i\epsilon\sigma C - ikE_2 - M\Phi_0\right) - \sigma\left(\hat{b}_{m-1}\hat{a}_m + \hat{a}_{m+1}\hat{b}_m\right) C \\
&+ i\left(\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m\right) H_2 - k\left(\epsilon E_2 + k\sigma C + iM\Phi_2\right) .
\end{aligned}$$

From this, after evident calculation, we arrive at

$$\begin{aligned}
&\hat{b}_{m-1} M\Phi_1 + \hat{a}_{m+1}\Phi_3 = -iM\left(\epsilon\Phi_0 + k\Phi_2\right) \\
&+\sigma\left(-\hat{b}_{m-1}\hat{a}_m - \hat{a}_{m+1}\hat{b}_m + \epsilon^2 - k^2\right) C - 2iB\, H_2 \,.
\end{aligned} \tag{68}$$

Comparing (42) and (68), we conclude that there exists a linear relation

$$2iB\, H_2(r) = M^2 C(r)\,. \tag{69}$$

Due to Eq. (40), it holds

$$i\hat{b}_{m-1}\Phi_1 - i\hat{a}_{m+1}\Phi_3 = M\, H_2 \qquad \Longrightarrow \qquad ig = M\, H_2\,, \tag{70}$$

therefore, the function $C(r)$ is expressed through $g(r)$

$$C(r) = -\frac{2B}{M^3} g(r)\,. \tag{71}$$

The system (64)–(67), after excluding $C(r)$, takes the form

$$\left(\Delta - k^2 - M^2 + \epsilon^2\right) F = 0\,, \tag{72}$$

$$\left(\Delta - M^2\right) G = -i\left(\epsilon^2 - k^2\right) f - i\sigma\left(\epsilon^2 - k^2\right)\frac{2B}{M^2} g\,, \tag{73}$$

$$\left[\Delta + \epsilon^2 - k^2 - M^2 - \sigma\left(\frac{2B}{M}\right)^2\right] g - 2iB\, G = 0\,, \tag{74}$$

$$2B\, g + \left(\epsilon^2 - k^2 - M^2\right) f + \Delta\left(-iG - \sigma\frac{2B}{M^2} g\right) = 0\,. \tag{75}$$

The structure of the system allows to separate evident, linearly independent solution as follows

$$\begin{aligned}
f(r) &= 0\,, \qquad g(r) = 0\,, \qquad H(r) = 0\,, \\
F(r) &\neq 0\,, \qquad \left(\Delta - k^2 - M^2 + \epsilon^2\right) F = 0\,,
\end{aligned} \tag{76}$$

corresponding functions and the energy spectrum are known (also see below).

We are to solve the system of three last equations in (73)–(75). With the help of (74)

$$\Delta G \;=\; M^2 G - i\left(\epsilon^2 - k^2\right) f - i\sigma\left(\epsilon^2 - k^2\right)\frac{2B}{M^2}g\,,$$
$$\Delta g \;=\; -\left[\epsilon^2 - k^2 - M^2 - \sigma(\frac{2B}{M})^2\right] g - 2iB\,G\,,$$

Eq. (75) takes the form of the linear relation

$$M^2 f = i\left(-M^2 + \sigma\frac{4B^2}{M^2}\right)G + 2B\left(1 - \sigma - \frac{\sigma^2}{M^2}\frac{4B^2}{M^2}\right)g\,. \qquad (77)$$

Now, returning to Eqs. (73)–(74), after excluding the function f and using the notation

$$\gamma = \frac{\epsilon^2 - k^2}{M^2}\,, \qquad \beta = \sigma\frac{4B^2}{M^2}\,, \qquad \alpha = \gamma\rho\,, \qquad \rho = 1 - \frac{4B^2\sigma^2}{M^4}\,,$$

we arrive at two equations

$$\left(\varDelta + \epsilon^2 - k^2 - M^2\right) g \;=\; \beta g(r) + 2iB\,G(r)\,,$$
$$\left(\varDelta + \epsilon^2 - k^2 - M^2\right) G \;=\; -2iB\alpha g(r) + \beta\gamma G(r)\,. \qquad (78)$$

In the matrix form they read

$$\left(\varDelta + \epsilon^2 - M^2 - k^2\right)\begin{vmatrix} g(r) \\ G(r) \end{vmatrix} = \begin{vmatrix} \beta & 2iB \\ -2iB\alpha & \beta\gamma \end{vmatrix}\begin{vmatrix} g(r) \\ G(r) \end{vmatrix}\,. \qquad (79)$$

Let us construct the transformation changing the matrix on the right to a diagonal form

$$\left(\varDelta + \epsilon^2 - M^2 - k^2\right)\begin{vmatrix} g' \\ G' \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix}\begin{vmatrix} g' \\ G' \end{vmatrix}\,,$$
$$\begin{vmatrix} g' \\ G' \end{vmatrix} = S\begin{vmatrix} g \\ G \end{vmatrix}\,, \qquad S = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}\,; \qquad (80)$$

the problem is reduced to a couple of linear systems

$$\left\{\begin{array}{l} (\beta - \lambda_1)\,s_{11} - 2iB\alpha s_{12} = 0\,, \\ 2iBs_{11} + (\beta\gamma - \lambda_1)_{12} = 0\,, \end{array}\right. \qquad \left\{\begin{array}{l} (\beta - \lambda_2)\,s_{21} - 2iB\alpha s_{22} = 0\,, \\ 2iBs_{21} + (\beta\gamma - \lambda_2)s_{22} = 0\,. \end{array}\right.$$

The eigenvalues λ_1, λ_2 are

$$\lambda_1 = \frac{\beta(1+\gamma) + \sqrt{\beta^2(1-\gamma)^2 + 16B^2\rho\gamma}}{2}\,,$$
$$\lambda_2 = \frac{\beta(1+\gamma) - \sqrt{\beta^2(1-\gamma)^2 + 16B^2\rho\gamma}}{2}\,, \qquad (81)$$

let it be

$$s_{12} = 1\,, \qquad s_{22} = 1\,, \qquad s_{11} = \frac{\lambda_1' - \beta\gamma}{2iB}\,, \qquad s_{21} = \frac{\lambda_2 - \beta\gamma}{2iB}\,,$$

$$g' = \frac{\lambda_1 - \beta\gamma}{2iB} g + G\,, \qquad G' = \frac{\lambda_2 - \beta\gamma}{2iB} g + G\,. \tag{82}$$

In the new (primed) basis, Eqs. (80) take the form of two separated differential equations

$$\begin{aligned} \left(\Delta + \epsilon^2 - k^2 - M^2 - \lambda_1\right) g' &= 0\,, \\ \left(\Delta + \epsilon^2 - k^2 - M^2 - \lambda_2'\right) G' &= 0\,. \end{aligned} \tag{83}$$

Recalling the meaning of Δ, let us specify the second order equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+Br^2)^2}{r^2} + \lambda^2\right)\varphi(r) = 0\,. \tag{84}$$

This equation was examined above. We obtain two possibilities for the energy spectrum:

$$\begin{aligned} g' &\neq 0\,, \qquad \epsilon^2 - k^2 = \lambda^2 + M^2 + \lambda_1\,, \\ G' &\neq 0\,, \qquad \epsilon^2 - k^2 = \lambda^2 + M^2 + \lambda_2'\,. \end{aligned} \tag{85}$$

Both Eqs. (85) can be written as

$$M^2\gamma = \lambda^2 + M^2 + \frac{\beta(1+\gamma) \pm \sqrt{\beta^2(1-\gamma)^2 + 16B^2\rho\gamma}}{2}\,. \tag{86}$$

It is convenient to introduce new variable $x = \gamma - 1$, and also with the help of

$$\beta = \sigma\frac{4B^2}{M^2}\,, \qquad \rho = 1 - \frac{4B^2\sigma^2}{M^4} = 1 - \frac{\beta^2}{4B^2}\,, \qquad 16\rho B^2 = 16B^2 - 4\beta^2$$

to eliminate the parameter ρ

$$\left(2M^2 - \beta\right)x - 2\left(\lambda^2 + \beta\right) = \pm\sqrt{\beta^2x^2 + \left(16B^2 - 4\beta^2\right)(x+1)}\,. \tag{87}$$

Thus, we get the second order equation

$$\begin{aligned} &M^2\left(M^2 - \beta\right)x^2 - \left[\left(\lambda^2+\beta\right)\left(2M^2-\beta\right) + \left(4B^2 - \beta^2\right)\right]x \\ &+ \left(\lambda^2+\beta\right)^2 - \left(4B^2-\beta^2\right) = 0\,, \end{aligned} \tag{88}$$

its solutions read

$$\epsilon^2 - M^2 - k^2 = \frac{1}{2(M^2-\beta)}\Big\{ \left[\left(\lambda^2+\beta\right)\left(2M^2-\beta\right)+\left(4B^2-\beta^2\right)\right] \\ \pm\Big[\left[\left(\lambda^2+\beta\right)\left(2M^2-\beta\right)+\left(4B^2-\beta^2\right)\right]^2 \\ -4M^2\left(M^2-\beta\right)\left(\lambda^2+\beta\right)^2-\left(4B^2-\beta^2\right)\Big]\Big\}. \qquad (89)$$

Note, that the case (25) gives the following spectrum $\epsilon^2 = M^2 + k^2 + \lambda^2$, so for these solutions the polarizability does not manifest itself in a magnetic field.

Thus, on the base of general covariant formalism in the vector particle with polarizability, the exact solutions for such a particle are constructed in the presence of an external homogeneous magnetic field. There are separated three types of linearly independent solutions, and corresponding energy spectra are found.

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