GENERALIZED UNCERTAINTY PRINCIPLE IN HAWKING RADIATION OF NONCOMMUTATIVE SCHWARZSCHILD BLACK HOLE

R. Fazeli†, S.H. Mehdipour‡

Islamic Azad University, Lahijan Branch, P.O. Box 1616, Lahijan, Iran

S. Sayyadzad§

Islamic Azad University, Noshahr and Chalous Branch
P.O. Box 46615-397, Chalous, Iran

(Received June 14, 2010)

The effects of noncommutativity in the framework of coordinate coherent states for Schwarzschild metric lead to a corrected solution in terms of the noncommutative parameter θ. Using the quasi-classical method, the Hawking radiation of such a noncommutative inspired black hole via the tunneling process is studied. In this situation, utilizing the generalized uncertainty principle, we show that the modification of the de Broglie relation in the quantum tunneling process of the black hole evaporation, provides the non-thermal effects which create the correlations between emitted modes of evaporation. In this setup, at least part of the quantum information becomes encoded in the Hawking radiation, and information can be appeared in the form of the non-thermal GUP correlations merged with the noncommutativity influences.

PACS numbers: 04.70.–s, 04.70.Dy, 04.70.Bw, 11.10.Nx

1. Introduction

Uncovering the black hole evaporation from Hawking hypothesis [1] has led to a long-standing discourse referring to alleged *lost information paradox* (for reviews see [2]) which infers the purely thermal spirit of the spectrum. Possibly, a solution for lost information paradox could prepare a fundamental ingredient in the search for a yet to be created theory of Quantum Gravity.

† elinor.rf@gmail.com
‡ mehdipour@iau-lahijan.ac.ir
§ s.sayyadzad@yahoo.com

In Quantum Mechanics, the information conserves from unitarity. While, in Black Hole Physics, the information is carried by a physical process going down to the singularity hence has never been regained. This would approve a non-unitary quantum evolution which maps a pure state to a mixed state [3]. There have been many proposals to keep the basic principles of Quantum Mechanics neighboring the black holes [2]. A more conventional proposal is that information really flows out from the black hole as non-thermal correlations between different modes of radiation during the evaporation.

The Hawking result can be portrayed as tunneling through the quantum horizon of a black hole. This is accomplished either by using the radial null geodesic method proposed by Parikh and Wilczek [4] or by using the Hamilton–Jacobi method proposed by Shankaranarayanan *et al.* [5] to calculate the probability of tunneling. In this paper, we would like to consider the Parikh–Wilczek method to develop their approach to the extended regime. In Parikh–Wilczek tunneling [4], due to the considering of self-gravitation effects, the evaporation is made by a lessening in the black hole horizon just by the emitted particle itself which leads to non-thermal corrections to the black hole radiation spectrum. Nevertheless, due to its absence of correlations between the tunneling rates of different modes in the black hole radiation spectrum the form of the corrections is not sufficient by itself to conserve the information [6, 7].

Recently, Smailagic and Spallucci [8] proposed an interesting method of noncommutativity relating to Coordinate Coherent States (CCS) which supports Lorentz invariance, Unitarity and UV-finiteness of Quantum Field theory. Possibly, space-time noncommutativity, namely an intrinsic property of the manifold by itself, can open a way to find an interpretation to the black hole information paradox. Nicolini, Smailagic and Spallucci (NSS) [9] (see also [10]) by applying this method have exhibited a physically inspired model of noncommutativity corrections to black hole solutions. In this methodology, the point-like structure of mass M, instead of being entirely localized at a point, is described by a smeared structure throughout a region of linear size $\sqrt{\theta}$. Utilizing the CCS approach, it has been demonstrated that the modified metric does not allow the black hole to decay lower than Planck size remnant thus the evaporation process finishes when the size of the black hole reaches to a Planckian relic; as a result, no curvature singularity at the origin. In view of the fact that space-time noncommutativity can improve some kinds of divergences which emerge in General Relativity, we hope to perfect a step further and modify the tunneling perspective by applying the CCS methodology.

Furthermore, it is widely believed that the so-called Heisenberg Uncertainty Principle (HUP), should be reformulated due to the noncommutative nature of space-time at Planck scale. In other words, it has been demonstrated that in Quantum Gravity there exists a minimum observable length [11], namely the order of the Planck length, which is a quick outcome of Generalized Uncertainty Principle (GUP). Therefore, the measurements in Quantum Gravity scenarios should be done by the GUP. There has been much discussion in the literature (*e.g.* [12]) about the applying of the GUP in black hole thermodynamics which leads to the significant modifications to the emission procedure, specifically at the ultimate phase of evaporation. Recently, Nozari and Mehdipour have generalized the Parikh–Wilczek tunneling by using the GUP effects [13, 14]. They have shown that the self-gravitation effects including the Planck-scale modifications cannot be relinquished once the black hole mass becomes corresponding to the Planck mass.

Here, in the study of black hole evaporation, we first briefly generalize the Schwarzschild metric to the noncommutative model of CCS. Next, to find the possible non-thermal correlations within the Hawking radiation, we investigate the tunneling methodology by the radial null geodesic approach in the background of CCS-noncommutativity including the Planck-scale corrections from the GUP origin. It might has been desired that, when the effects of gravitational back-reaction including the CCS-noncommutativity are incorporated with the Planck-scale corrections via the GUP, one would perceive the occurrence of correlations between the tunneling probability of different modes in the black hole radiation spectrum. The appearance of these correlations can shed more light on information loss problem.

The organization of this paper is the following. In Sec. 2, the incorporation of noncommutativity effects in the framework of CCS for Schwarzschild metric is briefly investigated. In Sec. 3, we study a feasible role of the Quantum Gravity corrections as the GUP effects. This trend continues to present the Parikh–Wilczek tunneling including the GUP outcomes for such a noncommutative inspired Schwarzschild solution. The tunneling amplitude at which massless particles tunnel across the event horizon is computed and then its result as a non-thermal spectrum for escaping the information via the Hawking radiation is studied. Finally, in Sec. 4, the paper is concluded with summary.

2. Noncommutative inspired Schwarzschild black hole

In this section, we will briefly show the appearance of black hole remnant in the context of CCS-noncommutativity. In accord with the NSS results [9, 10], the simple concept of point-like structure is not physically relevant and should be substituted for a minimum width Gaussian distribution of mass/energy which satisfies the principles of Quantum Mechanics. The procedure we prefer here is to accomplish an analysis which provides a solution in the case of a non-static, spherically symmetric, asymptotically flat, minimal width, Gaussian distribution of mass/energy whose noncommutative size is characterized by the parameter $\sqrt{\theta}$. For this purpose, the mass/energy distribution should be exhibited by a smeared delta function

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{\frac{3}{2}}} e^{-\frac{r^2}{4\theta}} \,. \tag{1}$$

The noncommutative inspired Schwarzschild metric in the presence of smeared mass source, by solving Einstein equations with (1) as a matter source, can be found as

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\Omega^2 \,, \tag{2}$$

with

$$F(r) = 1 - \frac{2M_\theta(r)}{r} \,, \tag{3}$$

where the Gaussian-smeared mass distribution immediately reads

$$M_\theta(r) = \int\limits_0^r \rho_\theta(r) 4\pi r^2 dr = M\left[\mathcal{E}\left(\frac{r}{2\sqrt{\theta}}\right) - \frac{r}{\sqrt{\pi\theta}} e^{-\frac{r^2}{4\theta}}\right] \,, \tag{4}$$

where $\mathcal{E}(x)$ shows the *Gauss error function* defined as $\mathcal{E}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$. In the commutative limit, $\theta \to 0$, the Gauss error function tends to one and the other term will exponentially be reduced to zero. Thus one retrieves the conventional source, $M_\theta \to M$, and the noncommutative inspired Schwarzschild black hole reduces to the usual Schwarzschild black hole, $F(r) = 1 - \frac{2M}{r}$.

It is evident that the metric (2) has a coordinate singularity at the event horizon as[1]

$$r_{\rm H} = 2M_\theta(r_{\rm H}) \,. \tag{5}$$

Analytical solution of Eq. (5) for $r_{\rm H}$ in a closed form is unfeasible, however, it is feasible to solve (5) to find M, which prepares the mass as a function of the horizon radius $r_{\rm H}$ and can be solved to yield

$$M = \frac{\sqrt{\pi\theta}\, r_{\rm H}}{2}\left[\sqrt{\pi\theta}\,\mathcal{E}\left(\frac{r_{\rm H}}{2\sqrt{\theta}}\right) - r_{\rm H} e^{-\frac{r_{\rm H}^2}{4\theta}}\right]^{-1} \,. \tag{6}$$

[1] Note that since there is no analytical solution for $r_{\rm H}$ *versus* M, then one can approximately compute the noncommutative horizon radius *versus* the mass by setting $r_{\rm H} = 2M$ into the function of Gaussian-smeared mass distribution $M_\theta(r_{\rm H})$, namely $r_{\rm H} = 2M\left[\mathcal{E}\left(\frac{M}{\sqrt{\theta}}\right) - \frac{2M}{\sqrt{\pi\theta}}\, e^{-\frac{M^2}{\theta}}\right]$.

The numerical result of mass *versus* the horizon radius is depicted in Fig. 1. As expected, from the mass equation (see Fig. 1), one obtains that noncommutativity discloses a *minimal non-zero mass* M_0, namely the black hole remnant, in order to have an event horizon. So, the black hole in the noncommutative case does not allow to decay lower than the *minimal mass* M_0, *i.e.* black hole remnant, and for $M < M_0$ there is no event horizon [9,10,15].

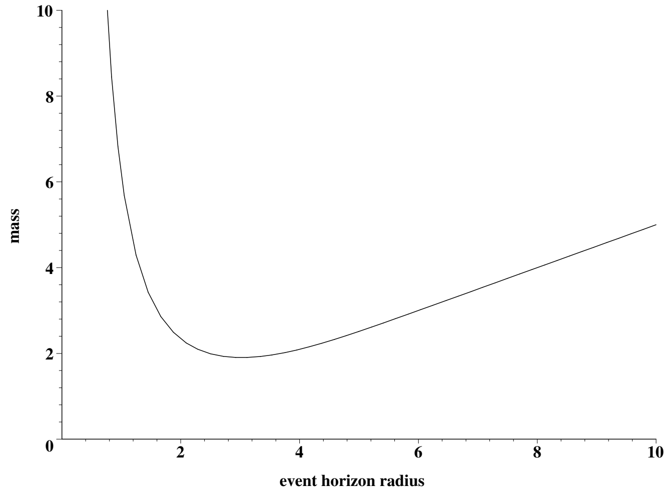

Fig. 1. The mass, $\frac{M}{\sqrt{\theta}}$, *versus* the event horizon radius, $\frac{r_{\rm H}}{\sqrt{\theta}}$. As can be seen from the figure, the existence of a minimal non-zero mass is clear, *i.e.* $M_0 \approx 1.9\sqrt{\theta}$.

3. Parikh–Wilczek tunneling including the GUP corrections

Now, we would like to investigate the radiating behavior of such a modified Schwarzschild solution by using the tunneling procedure proposed by Parikh and Wilczek [4]. To describe the tunneling methodology where a particle moves in dynamical geometry and travels through the horizon without singularity on the passage, we should use a coordinate system which is regular at the horizon. Painlevé coordinates [16] which are applied to remove coordinate singularity are specifically useful options in this investigation. Under the Painlevé time coordinate transformation

$$dt \to dt - \frac{\sqrt{1-F(r)}}{F(r)}dr\,, \tag{7}$$

the noncommutative Painlevé metric now is given by

$$ds^2 = -F(r)dt^2 + 2\sqrt{1-F(r)}dtdr + dr^2 + r^2 d\Omega^2\,. \tag{8}$$

The line element is now stationary, and there is no coordinate singularity at the horizon. The outgoing radial null geodesics, $ds^2 = d\Omega^2 = 0$, takes the following form

$$\dot{r} = 1 - \sqrt{1 - F(r)} = 1 - \sqrt{\frac{2M_\theta}{r}}\,, \tag{9}$$

where overdot abbreviates d/dt. According to the original work proposed by Parikh and Wilczek [4], the WKB approximation is valid near the horizon. Thus, the tunneling probability for the classically forbidden region as a function of imaginary part of the action for a particle in a tunneling process takes the form[2]

$$\Gamma \sim e^{-2\mathrm{Im}\, I}. \tag{10}$$

Now, we consider a spherical positive energy shell comprising the components of massless particles each of which journeys on a radial null geodesic like an s-wave outgoing particle which crosses the horizon in the outward direction from r_{in} to r_{out}, on the condition that $r_{\mathrm{in}} > r_{\mathrm{out}}$. Hence, the imaginary part of the action is given by

$$\mathrm{Im}\, I = \mathrm{Im} \int\limits_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} p_r dr = \mathrm{Im} \int\limits_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \int\limits_{0}^{p_r} dp'_r dr\,. \tag{11}$$

If we consider the particle's self-gravitation effect, in accordance with the original work proposed by Kraus and Wilczek [21], then both the noncommutative Painlevé metric and the geodesic equation should be modified by the response of the background geometry.

As mentioned briefly in the introduction, the resulting expectation of various models of Quantum Gravity is the appearance of a minimal observable distance of the order of the Planck length that cannot be probed [11], *e.g.* in String Theory there exists a restraint to probe the distances smaller than the string length. Hence the HUP is generalized to include this restrained resolution of the space-time structure. The result of this generalization is the so-called GUP which actually has the origin on the quantum fluctuations of the space-time at Planck scale.

---

[2] It should be noted that there exists another outlook on utilizing the relation (10). There has been a problem here known as "factor 2 problem" [17]. Recently, some authors have stated that relation (10) is not invariant under canonical transformations, however, the same formula with a factor of 1/2 in the exponent is canonically invariant (see [18, 19] and references therein). This method yields a temperature which is higher than the Hawking temperature by a factor of 2. In Ref. [20], a solution to this problem was prepared concerning the overlooked temporal contribution to the tunneling amplitude. When one comprises this temporal contribution one obtains exact the correct temperature and precisely when one utilizes the canonically invariant tunneling amplitude.

In this section, we want to apply the GUP to find a method of recovering the lost information in the black hole evaporation process. This method is related to the correlations between different modes of evaporation. A first effort in this direction was accomplished, in Ref. [7], by using the quantum corrected entropy with an additional logarithmic correction term. However, such a formulation does not give a non-zero statistical correlation function. In our method, we follow a different way to recover the information. According to the recent papers [13, 14], we would like to formulate the modifications of the Hawking radiation via the tunneling process by using the GUP-corrected de Broglie wave length and then a GUP-corrected energy (see for instance [22] and references therein)

$$\lambda \simeq \frac{1}{p}\left(1+\alpha L_{\mathrm{Pl}}^2 p^2\right) , \tag{12}$$

$$\mathtt{E} \simeq E\left(1+\alpha L_{\mathrm{Pl}}^2 E^2\right) . \tag{13}$$

Now, in the tunneling process, it is necessary to take into account the reaction of the background geometry to an emitted GUP-corrected energy $\mathtt{E}$. We hold the total ADM mass (M) of the space-time fixed, and allow the hole mass to fluctuate. In other words, a massless particle as a shell travels on the geodesics of a space-time with M replaced by $M-\mathtt{E}$. Next, we should first substitute M by $M-\mathtt{E}$ in Eq. (9) and then apply the deformed Hamilton's equation of motion [13, 14]

$$\dot{r} \simeq \left(1+\alpha L_{\mathrm{Pl}}^2 \mathtt{E}^2\right) \left.\frac{dH}{dp_r}\right|_r , \tag{14}$$

to alter the integral variable of the imaginary action (11) from momentum to energy. So, we have

$$\mathrm{Im}\, I = \mathrm{Im} \int\limits_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \int\limits_{M}^{M-\mathtt{E}} \frac{1+\alpha L_{\mathrm{Pl}}^2 \mathtt{E}'^2}{\dot{r}} dH dr , \tag{15}$$

where the Hamiltonian is $H = M - \mathtt{E}'$. We evaluate the integral (15) by writing the explicit form for the radial null geodesic which includes the back-reaction effects, namely

$$\dot{r} = 1 - \sqrt{\frac{2M_\theta\left(M-\mathtt{E}\right)}{r}} , \tag{16}$$

where

$$M_\theta\left(M-\mathtt{E}\right) = \left(M-\mathtt{E}\right)\left[\mathcal{E}\left(\frac{M-\mathtt{E}}{\sqrt{\theta}}\right) - \frac{2\left(M-\mathtt{E}\right)}{\sqrt{\pi\theta}}\, e^{-\frac{(M-\mathtt{E})^2}{\theta}}\right] . \tag{17}$$

Thus, we find

$$\mathrm{Im}\, I = \mathrm{Im} \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \int_{0}^{\mathtt{E}} \frac{1+\alpha L_{\mathrm{Pl}}^2 \mathtt{E}'^2}{1-\sqrt{\frac{2M_\theta\left(M-\mathtt{E}'\right)}{r}}} (-d\mathtt{E}')dr\,. \tag{18}$$

The r integral of Eq. (18) can be done first by deforming the contour for lower half $\mathtt{E}'$ plane, due to escape from the pole at the horizon, as follows

$$\mathrm{Im}\, I = \mathrm{Im} \int_{0}^{\mathtt{E}} 4\pi i M_\theta (M-\mathtt{E}') \left(1+\alpha L_{\mathrm{Pl}}^2 \mathtt{E}'^2\right) d\mathtt{E}'\,. \tag{19}$$

Finally, the imaginary action leads to the following form

$$\begin{aligned}\mathrm{Im}\, I =\ & 3\pi\theta(\mathcal{E}_2-\mathcal{E}_1) + 6\sqrt{\pi\theta}\ e^{-\frac{M^2+E^2+2\alpha L_{\mathrm{Pl}}^2 E^4}{\theta}} \Bigg[M e^{\frac{E^2+2\alpha L_{\mathrm{Pl}}^2 E^4}{\theta}} \\ & - e^{\frac{2M\left(E+\alpha L_{\mathrm{Pl}}^2 E^3\right)}{\theta}} (M-E) \Bigg] + \pi\alpha L_{\mathrm{Pl}}^2 \left[\frac{M^4}{3}(\mathcal{E}_1-\mathcal{E}_2) + \mathcal{E}_2 E^3 \left(\frac{16}{3}M - 5E\right)\right] \\ & + 2\pi\left[M^2(\mathcal{E}_1-\mathcal{E}_2) + \mathcal{E}_2 E(2M-E)\right] + O(\alpha^2 L_{\mathrm{Pl}}^4)\,, \end{aligned} \tag{20}$$

where

$$\begin{cases} \mathcal{E}_1 \equiv \mathcal{E}\left(\frac{M}{\sqrt{\theta}}\right), \\ \mathcal{E}_2 \equiv \mathcal{E}\left(\frac{M-E-\alpha L_{\mathrm{Pl}}^2 E^3}{\sqrt{\theta}}\right). \end{cases}$$

We consider the leading-order correction to be just proportional to $(\alpha L_{\mathrm{Pl}}^2)$. These new corrections cannot be ignored when the black hole mass is adjacent to the Planck mass. However, the corrections are fundamentally trifling, one could respect this as a consequence of quantum inspection at the level of semi-classical Quantum Gravity. Note that, for simpleness and without loss of generality, we have eliminated the terms proportional to $(\alpha L_{\mathrm{Pl}}^2\sqrt{\theta})$ and also $(\alpha L_{\mathrm{Pl}}^2\theta)$ due to the fact that these corrections are extremely small to be substantial. From the Eq. (20), it is obviously observed that the corresponding tunneling rate differs from the purely thermal essence of the spectrum.

The tunneling rate in the high energy depends on the final and initial number of microstates available for the system [23–25]. Thus, we have

$$\varGamma \sim \frac{e^{S_{\mathrm{final}}}}{e^{S_{\mathrm{initial}}}} = e^{\Delta S}, \tag{21}$$

where S is the black hole entropy. In this viewpoint the tunneling rate is proportional to the difference in black hole entropies before and after emission which means that the emission spectrum cannot be accurately thermal at higher energies.

The question which arises here is the possible dependencies between different modes of radiation during the evaporation. In this situation, as confirmed in [13, 14], we want to show that there are correlations between emitted particles due to the including of GUP corrections. This means it can be demonstrated that the probability of tunneling of two particles of energy E_1 and E_2 is not equal to the probability of tunneling of one particle with their compound energies, $E = E_1 + E_2$, *i.e.*

$$\chi(E_1 + E_2; E_1, E_2) = \Delta S_{(E_1+E_2)} - \Delta S_{E_1} - \Delta S_{E_2} \neq 0\,, \tag{22}$$

where

$$\begin{cases} \Delta S_{E_1} \equiv \Delta S(M, E = E_1)\,, \\ \Delta S_{E_2} \equiv \Delta S(M = M - E_1, E = E_2)\,, \\ \Delta S_{(E_1+E_2)} \equiv \Delta S(M, E = E_1 + E_2)\,. \end{cases}$$

One can clearly verify that the additional or combined terms of the expression (20) depending on the parameters GUP and noncommutativity, *i.e.* α and θ respectively, lead to a non-zero statistical correlation function between rates of tunneling of two particles with different energies which reads

$$\begin{aligned}
&\chi(E_1+E_2; E_1, E_2) = 10\pi\mathcal{E}_2\Big(M, E=E_1+E_2\Big)\Bigg[\alpha L_{\rm Pl}^2\Bigg((E_1+E_2)^4 \\
&-\frac{16}{15}M\left(E_1^3+E_2E_1^2+E_1E_2^2+E_2^3-\frac{M^3}{16}\right)\Bigg)+\frac{2}{5}(M-E_1-E_2)^2-\frac{3}{5}\theta\Bigg] \\
&+12\sqrt{\pi\theta}(M-E_1-E_2)e^{-\frac{\left(M-E_1-E_2-\alpha L_{\rm Pl}^2(E_1+E_2)^3\right)^2}{\theta}}-\frac{2}{3}\pi\mathcal{E}_2\Big(M=M-E_1, E=E_2\Big) \\
&\times\left[\alpha L_{\rm Pl}^2\left((M-E_1)^4-16E_2^3\left(M-E_1-\frac{15}{16}E_2\right)\right)+6(M-E_1-E_2)^2-9\theta\right] \\
&-10\pi\mathcal{E}_2\Big(M, E=E_1\Big)\left[\alpha L_{\rm Pl}^2\left(E_1^4+\frac{M}{15}\left(M^3-16E_1^3\right)\right)+\frac{2}{5}(M-E_1)^2-\frac{3}{5}\theta\right] \\
&-12\sqrt{\pi\theta}(M-E_1-E_2)\,e^{-\frac{\left(M-E_1-E_2-\alpha L_{\rm Pl}^2E_2^3\right)^2}{\theta}} \\
&-12\sqrt{\pi\theta}(M-E_1)\,e^{-\frac{\left(M-E_1-\alpha L_{\rm Pl}^2E_1^3\right)^2}{\theta}} \\
&+12\sqrt{\pi\theta}(M-E_1)\,e^{-\frac{(M-E_1)^2}{\theta}} \\
&+\frac{2}{3}\pi\mathcal{E}_1\Big(M-E_1\Big)\left[\alpha L_{\rm Pl}^2(M-E_1)^4+6(M-E_1)^2-9\theta\right].
\end{aligned} \tag{23}$$

Therefore, the tunneling probabilities for the different modes of radiation throughout the evaporation are reciprocally connected from a statistical viewpoint. It is useful to examine this result in some limits. In the limit $\theta \rightarrow 0$ and $\alpha \neq 0$, we recover the same results to Ref. [13]. But in the HUP limit, $\alpha = 0$, one regains the zero statistical correlation function even with non-vanishing θ (see [15]). This means that the appearance of correlations is just dependent to the emerging of GUP parameter. In fact, the inclusion of the effects of Quantum Gravity as the GUP expression yields the generation of correlations between the different modes of radiation. These features cause the information flows out from the black hole as the non-thermal GUP correlations within the Hawking radiation and this can shed more light on lost information paradox in the black hole evaporation process.

4. Summary

In the framework of noncommutative model of CCS, we have presented the Schwarzschild metric. We have studied the Hawking radiation of such a noncommutative inspired solution via the tunneling procedure. By using the Parikh–Wilczek method, the tunneling probability including the GUP corrections has been achieved in terms of the noncommutative parameter θ. Finally the issue of possible correlations between emitted particles has been discussed. According to Refs. [13, 14], we have introduced the GUP effects to provide a way to observe the correlations between emitted modes. We have shown that the incorporation of Quantum Gravity effects such as GUP merged with the noncommutativity effects generates the correlations between emitted particles which means at least a part of information coming out of the black hole can be preserved in these correlations.

We would like to thank K. Nozari for a long-time collaboration on this area.

REFERENCES

[1] S.W. Hawking, *Commun. Math. Phys.* **43**, 199 (1974).

[2] J. Preskill, [arXiv:hep-th/9209058]; D.N. Page, *Phys. Rev. Lett.* **71**, 3743 (1993); C.R. Stephens, G. 't Hooft, B.F. Whiting, *Class. Quantum Grav.* **11**, 621 (1994); A. Strominger, [arXiv:hep-th/9501071]; T. Banks, *Nucl. Phys. (Proc. Suppl.)* **41**, 21 (1995); J.G. Russo, [arXiv:hep-th/0501132] and references therein.

[3] S.W. Hawking, *Phys. Rev.* **D14**, 2460 (1976).

[4] M.K. Parikh, F. Wilczek, *Phys. Rev. Lett.* **85**, 5042 (2000).

[5] S. Shankaranarayanan, T. Padmanabhan, K. Srinivasan, *Class. Quantum Grav.* **19**, 2671 (2002); *Mod. Phys. Lett.* **A16**, 571 (2001).

[6] M.K. Parikh, `arXiv:hep-th/0402166`.

[7] M. Arzano, A. Medved, E. Vagenas, *J. High Energy Phys.* **09**, 037 (2005); A.J.M. Medved, E. Vagenas, *Mod. Phys. Lett.* **A20**, 1723 (2005); *Mod. Phys. Lett.* **A20**, 2449 (2005).

[8] A. Smailagic, E. Spallucci, *Phys. Rev.* **D65**, 107701 (2002); *J. Phys. A: Math. Gen.* **35**, L363 (2002); *J. Phys. A: Math. Gen.* **36**, L467 (2003); *J. Phys. A: Math. Gen.* **36**, L517 (2003); *J. Phys. A: Math. Gen.* **37**, 7169 (2004).

[9] P. Nicolini, A. Smailagic, E. Spallucci, *Phys. Lett.* **B632**, 547 (2006); S. Ansoldi, P. Nicolini, A. Smailagic, E. Spallucci, *Phys. Lett.* **B645**, 261 (2007); P. Nicolini, *Int. J. Mod. Phys.* **A24**, 1229 (2009).

[10] T.G. Rizzo, *J. High Energy Phys.* **09**, 021 (2006); R. Casadio, P. Nicolini, *J. High Energy Phys.* **11**, 072 (2008); E. Spallucci, A. Smailagic, P. Nicolini, *Phys. Lett.* **B670**, 449 (2009); K. Nozari, S.H. Mehdipour, *Commun. Theor. Phys.* **53**, 503 (2010).

[11] D. Amati, M. Ciafaloni, G. Veneziano, *Phys. Lett.* **B216**, 41 (1989); K. Konishi, G. Paffuti, P. Provero, *Phys. Lett.* **B234**, 276 (1990); M. Maggiore, *Phys. Rev.* **D49**, 5182 (1994); *Phys. Lett.* **B304**, 65 (1993); *Phys. Lett.* **B319**, 83 (1993); L.J. Garay, *Int. J. Mod. Phys.* **A10**, 145 (1995); A. Kempf, G. Mangano, R.B. Mann, *Phys. Rev.* **D52**, 1108 (1995); H. Hinrichsen, A. Kempf, *J. Math. Phys.* **37**, 2121 (1996); C. Castro, *Found. Phys. Lett.* **10**, 273 (1997); F. Scardigli, *Phys. Lett.* **B452**, 39 (1999); S. Capozziello, G. Lambiase, G. Scarpetta, *Int. J. Theor. Phys.* **39**, 15 (2000).

[12] There is a large literature on this subject see for example: R.J. Adler, P. Chen, D.I. Santiago, *Gen. Relativ. Gravitation* **33**, 2101 (2001); A.J.M. Medved, E.C. Vagenas, *Phys. Rev.* **D70**, 124021 (2004); M. Cavaglia, S. Das, *Class. Quantum Grav.* **21**, 4511 (2004); B. Bolen, M. Cavaglia, *Gen. Relativ. Gravitation* **37**, 1255 (2005); K. Nozari, S.H. Mehdipour, *Mod. Phys. Lett.* **A20**, 2937 (2005); K. Nozari, S.H. Mehdipour, *Int. J. Mod. Phys.* **A21**, 4979 (2006); *Elec. J. Theor. Phys.* **3**, 151 (2006); K. Nozari, A.S. Sefiedgar, *Phys. Lett.* **B635**, 156 (2006).

[13] K. Nozari, S.H. Mehdipour, *Europhys. Lett.* **84**, 20008 (2008); S.H. Mehdipour, *Int. J. Mod. Phys.* **A24**, 5669 (2009).

[14] K. Nozari, S.H. Mehdipour, *J. High Energy Phys.* **03**, 061 (2009).

[15] K. Nozari, S.H. Mehdipour, *Class. Quantum Grav.* **25**, 175015 (2008); S.H. Mehdipour, *Commun. Theor. Phys.* **52**, 865 (2009).

[16] P. Painlevé, *Compt. Rend. Acad. Sci. (Paris)* **173**, 677 (1921).

[17] E.T. Akhmedov, V. Akhmedova, D. Singleton, *Phys. Lett.* **B642**, 124 (2006); E.T. Akhmedov, V. Akhmedova, T. Pilling, D. Singleton, *Int. J. Mod. Phys.* **A22**, 1705 (2007).

[18] B.D. Chowdhury, *Pramana* **70**, 593 (2008).

[19] T. Pilling, *Phys. Lett.* **B660**, 402 (2008).

[20] V. Akhmedova, T. Pilling, A. de Gill, D. Singleton, *Phys. Lett.* **B666**, 269 (2008); E.T. Akhmedov, T. Pilling, D. Singleton, *Int. J. Mod. Phys.* **D17**, 2453 (2008).

[21] P. Kraus, F. Wilczek, *Nucl. Phys.* **B433**, 403 (1995); *Mod. Phys. Lett.* **A9**, 3713 (1994).

[22] G. Amelino-Camelia, M. Arzano, Y. Ling, G. Mandanici, *Class. Quantum Grav.* **23**, 2585 (2006).

[23] E. Keski-Vakkuri, P. Kraus, *Nucl. Phys.* **B491**, 249 (1997).

[24] R. Banerjee, B.R. Majhi, S. Samanta, *Phys. Rev.* **D77**, 124035 (2008); R. Banerjee, B.R. Majhi, *Phys. Lett.* **B662**, 62 (2008); *J. High Energy Phys.* **06**, 095 (2008).

[25] S. Massar, R. Parentani, *Nucl. Phys.* **B575**, 333 (2000).