PSEUDOSPIN SYMMETRY IN DEFORMED NUCLEI WITH TRIAXIAL-SYMMETRIC HARMONIC OSCILLATOR POTENTIAL

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Pseudospin symmetry is a perfectly valid concept which can reliably be used in calculations of any deformed heavy nuclei. Therefore, we examined this symmetry in deformed nuclei with triaxial-symmetry and eigenfunctions. Energy equation were obtained for the case of triaxial-symmetric harmonic oscillator potential in the Dirac equation.

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1. Introduction

Pseudospin symmetry is based on experimental observation of the quasi-degeneracy between two single-particle orbitals with quantum numbers $(n_{\rm r}, l, j = l+1/2)$ and $(n_{\rm r}-1, l+2, j' = j+1 = l+3/2)$, where $n_{\rm r}$, l and j are single-nucleon radials, orbital and total angular momentum quantum numbers, respectively [1,2]. The structure of the doublet is expressed in terms of a "pseudo" orbital angular momentum $\tilde{l} = l + 1$ and "pseudo" spin $\tilde{s} = 1/2$. For example, $(n_{\rm r}s_{1/2}, (n_{\rm r} - 1)d_{3/2})$ we have $\tilde{l} = 1$, and $(n_{\rm r}p_{3/2}, (n_{\rm r} - 1)f_{3/2})$ we have $\tilde{l} = 2$, *etc.* These doublets are almost degenerate in proportion to pseudospin $\tilde{s} = 1/2$, since $j = \tilde{l} + \tilde{s}$, for the two states in the doublet. This is the concept of pseudospin symmetry in the spherical nuclei.

Recently, it has been shown that the pseudospin symmetry is a good approximation in deformed nuclei, including axially and triaxially deformed nuclei, and the implementation based on non-relativistic Nilsson model have been reported [3–6]. In Ref. [4] the validity of the pseudospin concept for heavy triaxially deformed nuclei was explored using correlation coefficient measure between a generalized Nilsson Model Hamiltonian and pseudospin–orbit interaction. Analysis of the correlation coefficient measures for the generalized Nilsson Hamiltonian and pseudospin–orbit operator showed that

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the goodness of pseudospin symmetry remains virtually unchanged from axial to triaxial deformations. This points out the importance of using pseudospin symmetry based shell-model schemes for heavy nuclei at any reasonable deformation including, in particular, triaxial shapes [4].

A study of the goodness of pseudospin dynamical symmetry in triaxial nuclei has been done in Ref. [5]. In this reference, an explicit form for the extended pseudospin transformation for arbitrary deformations has been suggested and applied to some modifications of triaxial Nilsson Hamiltonian. Nilsson-type model could be viewed as a further clarification on triaxial model which was intended to correctly reproduce the structure of the basic states in both spherical and cylindrical limits and extend this to arbitrary triaxial shapes [6].

In presence of deformation, doublets with quantum numbers

$$[N, n_z, \Lambda]\, \Omega = \Lambda + \tfrac{1}{2} \qquad \text{and} \qquad [N, n_z, \Lambda' = \Lambda + 2]\, \Omega' = \Lambda + \tfrac{3}{2}$$

can be expressed in terms of pseudo-orbital and total angular momentum projections $\tilde{\Lambda} = \Lambda + 1$, $\Omega = \tilde{\Lambda} \pm \frac{1}{2}$ [7]. The pseudospin symmetry has been used to explain features of deformed nuclei [8–10].

In the past decades, Relativistic Mean-Field Theory (RMF) achieved great success in description of nuclear properties, especially in pseudospin symmetry [11] and spin symmetry in anti-nucleon spectra [20]. In the Dirac equation of nucleon, when scalar potential $S(r)$ and vector potential $V(r)$ are equal in amplitudes but opposite in sign, *i.e.*, $S(r) + V(r) = 0$, or more generally, $d[S(r) + V(r)]/dr = 0$, there is an exact pseudospin symmetry in single-particle spectra [11–14]. These conditions imply some special relations between four components of Dirac wavefunctions which have been used to test the pseudospin symmetry in spherical and axially deformed nuclei [15–19].

In the nucleus, the charge-conjugation transformation relates spin symmetry of antinucleons to pseudospin symmetry of the nucleons [20]. This has also been discussed in Ref. [21], analyzing harmonic oscillator for antinucleons with spin symmetry ($S(r) = V(r)$). Castro *et al.* [22] have solved generalized relativistic harmonic oscillator in $1 + 1$ dimensions, *i.e.*, including a linear pseudoscalar potential and quadratic scalar and vector potentials which have equal or opposite signs.

They considered positive and negative quadratic potentials and discussed in detail their bound state solutions for fermions and antifermions. Some authors studied relativistic harmonic oscillator for spin $\frac{1}{2}$ particles, and obtained bound state solutions for Dirac equation whit spin and pseudospin symmetry conditions [23, 24]. In this paper, some part of Ref. [19] will be reviewed. Then, Dirac equation for triaxial-symmetric harmonic oscillator will be solved.

At first, eigenfunctions and energy equation for Dirac equation with triaxial-symmetric harmonic oscillator under pseudospin symmetry condition were obtained. Then, the results were compared with previous results for this problem in spherical nuclei, *i.e.* the nuclei with spherical harmonic oscillator which was done in [23, 24]. In the previous work, this procedure for axial-symmetric harmonic oscillator was done [25]. Finally, our results were compared with those of Ginocchio's work through chiral and charge-conjugation transformations.

2. Dirac Hamiltonian and pseudospin symmetry

In this section, Ref. [19] is reviewed briefly.

Hamiltonian of a Dirac particle of mass M in an external scalar, $S(\vec{r})$, and vector, $V(\vec{r})$, potentials is given by

$$H = \vec{\hat{\alpha}} \cdot \vec{p} + \hat{\beta}(M + S(\vec{r})) + V(\vec{r})\,, \tag{1}$$

where α and β are Dirac matrices. Dirac equation can be written as ($\hbar = c = 1$)

$$\left\{\vec{\hat{\alpha}} \cdot \vec{p} + \hat{\beta}[M + S(\vec{r})] + V(\vec{r})\right\} \Phi(\vec{r}) = E\Phi(\vec{r})\,. \tag{2}$$

Dirac Hamiltonian with spherically symmetric scalar and vector potentials are invariant under a SU(2) algebra for two limits: $S(\vec{r}) = V(\vec{r}) + C_{\rm s}$ and $S(\vec{r}) = -V(\vec{r}) + C_{\rm ps}$, where $C_{\rm s}$, $C_{\rm ps}$ are constants [26]. When the former limit occurs, we have spin symmetry [27]. The latter limit leads to pseudospin symmetry [9]. Generators of pseudospin are given by

$$\vec{\tilde{S}}_i = \begin{pmatrix} \vec{\tilde{s}}_i & 0 \\ 0 & \vec{s}_i \end{pmatrix} = \begin{pmatrix} U_p \vec{s}_i U_p & 0 \\ 0 & \vec{s}_i \end{pmatrix}, \tag{3}$$

where $s_i = \frac{\sigma_i}{2}, \sigma_i$ are Pauli matrices and $U_p = \frac{\sigma \cdot p}{p}$ is unitary matrix operator [28]. This generators commute with Dirac Hamiltonian for the limit of $S(\vec{r}) = -V(\vec{r}) + C_{\rm ps}, [H_{\rm ps}, S_i] = 0$. Thus, Dirac Hamiltonian and pseudospin symmetry have simultaneous eigenfunction

$$H_{\rm ps}\Phi^{\rm ps}_{k,\tilde{\mu}}(\vec{r}) = E_k \Phi^{\rm ps}_{k,\tilde{\mu}}(\vec{r})\,, \tag{4}$$

where k is just a label for the remaining quantum numbers besides $\tilde{\mu}$ [19], and $\tilde{\mu} = \pm\frac{1}{2}$ is eigenvalue of $\tilde{S}_z$

$$\tilde{S}_z \Phi^{\rm ps}_{k,\tilde{\mu}}(\vec{r}) = \tilde{\mu}\Phi^{\rm ps}_{k,\tilde{\mu}}(\vec{r})\,. \tag{5}$$

Eigenstate in pseudospin doublet will be connected by $\tilde{S}_\pm$ generators

$$\tilde{S}_\pm \Phi^{\rm ps}_{k,\tilde{\mu}}(\vec{r}) = \sqrt{\left(\tfrac{1}{2} \mp \tilde{\mu}\right)\left(\tfrac{3}{2} \pm \tilde{\mu}\right)}\, \Phi^{\rm ps}_{k,\tilde{\mu}}(\vec{r})\,. \tag{6}$$

Dirac four-component wavefunction, $\Phi^{\rm ps}_{k,\tilde{\mu}}(\vec{r})$, is given by

$$\Phi^{\rm ps}_{k,\tilde{\mu}}(\vec{r}) = \begin{pmatrix} g^+_{k,\tilde{\mu}}(\vec{r}) \\ g^-_{k,\tilde{\mu}}(\vec{r}) \\ i f^+_{k,\tilde{\mu}}(\vec{r}) \\ i f^-_{k,\tilde{\mu}}(\vec{r}) \end{pmatrix}, \tag{7}$$

where $g^\pm_{k,\tilde{\mu}}(\vec{r})$ are upper Dirac components, here $+$ indicates spin up and $-$ spin down and, $f^\pm_{k,\tilde{\mu}}(\vec{r})$ are lower Dirac components, where $+$ indicates spin up and $-$ spin down. Pseudospin symmetry create some relations between these components which are derived from Eqs. (5) and (6) [18]

$$f^+_{k,-\frac{1}{2}}(\vec{r}) = f^-_{k,\frac{1}{2}}(\vec{r}) = 0\,, \tag{8}$$

$$f^+_{k,+\frac{1}{2}}(\vec{r}) = f^-_{k,-\frac{1}{2}}(\vec{r}) \equiv f_k(\vec{r})\,, \tag{9}$$

$$g^+_{k,\frac{1}{2}}(\vec{r}) = -g^-_{k,-\frac{1}{2}}(\vec{r}) \equiv g_k(\vec{r})\,, \tag{10}$$

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) g^-_{k,\frac{1}{2}}(\vec{r}) = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) g^+_{k,-\frac{1}{2}}(\vec{r})\,, \tag{11}$$

$$\frac{\partial}{\partial z} g^\pm_{k,\mp\frac{1}{2}}(\vec{r}) = \pm\left(\frac{\partial}{\partial x} \mp i\frac{\partial}{\partial y}\right) g^\pm_{k,\pm\frac{1}{2}}(\vec{r})\,. \tag{12}$$

Therefore, Dirac eigenfunction in pseudospin doublet are given by

$$\Phi^{\rm ps}_{k,\frac{1}{2}}(\vec{r}) = \begin{pmatrix} g_k(\vec{r}) \\ g^-_{k,\frac{1}{2}}(\vec{r}) \\ i f_k(\vec{r}) \\ 0 \end{pmatrix}, \tag{13}$$

$$\Phi^{\rm ps}_{k,-\frac{1}{2}}(\vec{r}) = \begin{pmatrix} g^+_{k,-\frac{1}{2}}(\vec{r}) \\ -g_k(\vec{r}) \\ 0 \\ i f_k(\vec{r}) \end{pmatrix}. \tag{14}$$

3. Pseudospin symmetry for triaxially deformed nuclei

In Ref. [4] it has been shown that although near level degeneracy of pseudospin–orbit partners in axial systems is lost for triaxial geometries, pseudospin symmetry breaking induced by triaxiality is small and comparable to that found in axial cases, and therefore pseudospin symmetry remains an important physical concept.

We replace Eqs. (13) and (14) in Eq. (2) and obtain the following relations for upper and lower components of Dirac wavefunctions, respectively

$$g^{\pm}_{k,\mp\frac{1}{2}}(\vec{r}) = \frac{-1}{M-E+\Sigma}\left(\frac{\partial}{\partial x} \mp i\frac{\partial}{\partial y}\right) f_k(\vec{r})\,, \tag{15}$$

$$g_k(\vec{r}) = \frac{-1}{M-E+\Sigma}\frac{\partial}{\partial z} f_k(\vec{r})\,, \tag{16}$$

$$f_k(\vec{r}) = \frac{-1}{M+E-\Delta}\left(\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) g^{+}_{k,-\frac{1}{2}}(\vec{r}) + \frac{\partial}{\partial z}\right) g_k(\vec{r})\,, \tag{17}$$

where $\Sigma = S(\vec{r}) + V(\vec{r})$ and $\Delta = V(\vec{r}) - S(\vec{r})$.

By substituting $g^{+}_{k,-\frac{1}{2}}(r)$ and $g_k(\vec{r})$ from Eqs. (15) and (16) in Eq. (17) a second order deferential equation for lower component, $f_k(\vec{r})$, is obtained

$$(M+E-\Delta)(M-E+\Sigma)f_k(\vec{r}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{M-E+\Sigma}\right.$$
$$\left.\left(\left[\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\Sigma\right]\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) + \frac{\partial \Sigma}{\partial z}\frac{\partial}{\partial z}\right)\right) f_k(\vec{r})\,. \tag{18}$$

By applying pseudospin symmetry condition, *i.e.* $\Sigma = 0$, or $\frac{\partial \Sigma}{\partial x} = 0$, $\frac{\partial \Sigma}{\partial y} = 0$, $\frac{\partial \Sigma}{\partial z} = 0$ and replacing $\Delta = 2V$, this equation is reduced to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + E^2 - M^2 + 2V(M-E)\right) f_k(\vec{r}) = 0\,. \tag{19}$$

For triaxial deformed nuclei, Eq. (19) with the following vector and scalar potentials

$$V(\vec{r}) = -S(\vec{r}) = \tfrac{1}{2}M\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right) \tag{20}$$

can be written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - (E-M)M\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right) + E^2 - M^2\right) f_k(\vec{r})\,. \tag{21}$$

This equation can be written as three separate equations for x_1, y_1 and z_1

$$\left(\frac{\partial^2}{\partial x_1^2} - x_1^2 + 2\tilde{n}_x + 1\right) f_{\tilde{n}_x}(\vec{r}) \;=\; 0\,, \tag{22}$$

$$\left(\frac{\partial^2}{\partial y_1^2} - y_1^2 + 2\tilde{n}_y + 1\right) f_{\tilde{n}_y}(\vec{r}) \;=\; 0\,, \tag{23}$$

$$\left(\frac{\partial^2}{\partial z_1^2} - z_1^2 + 2\tilde{n}_z + 1\right) f_{\tilde{n}_z}(\vec{r}) \;=\; 0\,, \tag{24}$$

where

$$x_1 \;=\; \left(M\omega_x^2(E-M)\right)^{\frac{1}{4}} x = \alpha_x x\,, \tag{25}$$

$$y_1 \;=\; \left(M\omega_y^2(E-M)\right)^{\frac{1}{4}} y = \alpha_y y\,, \tag{26}$$

and

$$z_1 = \left(M\omega_z^2(E-M)\right)^{\frac{1}{4}} z = \alpha_z z\,. \tag{27}$$

Here we consider

$$f_k(\vec{r}) = N f_{\tilde{n}_x}(x) f_{\tilde{n}_y}(y) f_{\tilde{n}_z}(z)\,, \tag{28}$$

where N is normalization constant which is determined by

$$\int dx \int dy \int dz \left|\Phi^{\rm ps}_{k,\pm\frac{1}{2}}(\vec{r})\right|^2 = 1\,. \tag{29}$$

Energy equation becomes

$$E^2_{\tilde{n}_x,\tilde{n}_y,\tilde{n}_z} - M^2 = \alpha_x^2(2\tilde{n}_x+1) + \alpha_y^2(2\tilde{n}_y+1) + \alpha_z^2(2\tilde{n}_z+1)\,, \tag{30}$$

where $\tilde{n}_x, \tilde{n}_y$ and $\tilde{n}_z$ are quantum numbers of oscillator of lower component in x, y and z directions, respectively.

In the spherical harmonic oscillator each level has a $(N+2)(N+1)$ degeneracy because of pseudospin symmetry and because allowed pseudo-orbital angular momenta are $\tilde{l} = N, N-2, \ldots 0$ or 1 and allowed pseudo-orbital angular momentum projections are $m = \tilde{l}, \tilde{l}-1, \ldots, \tilde{l}$.

Each group of N in deformed harmonic oscillator contains the levels for $\tilde{n}_z = 0, 1, \ldots, N$ with excitation energy increasing with decreasing $\tilde{n}_z$. Each level has a $2(N-\tilde{n}_z)+1$ degeneracy for $(N-\tilde{n}_z)$ even and a $2(N-\tilde{n}_z+1)$ degeneracy for $(N-\tilde{n}_z)$ odd due to spin symmetry and because the allowed orbital angular momentum projections are $\tilde{\Lambda} = (N-\tilde{n}_z)$, $(N-\tilde{n}_z-2)$, 1 or 0. The splitting of levels within each N appears to be approximately linear with $\tilde{n}_z$ (see Fig. 2 from Ref. [21]).

Eqs. (22)–(24) are Hermite equations and solution of these equations can be written in terms of Hermite functions, thus we can obtain lower component wavefunction, $f_k(\vec{r})$, from Eq. (28)

$$f_k(\vec{r}) = e^{-\frac{\alpha_x^2 x^2+\alpha_y^2 y^2+\alpha_z^2 z^2}{2}} H_{\tilde{n}_x}(\alpha_x x) H_{\tilde{n}_y}(\alpha_y y) H_{\tilde{n}_z}(\alpha_z z)\,. \tag{31}$$

Now, we are going to obtain energy spectrum in the spherical symmetry case.

For spherical harmonic oscillator potential we have $\omega_x = \omega_y = \omega_z = \omega_1$ then, by using Eqs. (25)–(27), one can obtain $\alpha_x = \alpha_y = \alpha_z = \alpha_1$. Therefore, the energy equation of Eq. (30) can be rewritten as

$$E^2_{\tilde{n}_x,\tilde{n}_y,\tilde{n}_z} - M^2 = 2\alpha_1^2\left(\tilde{n}_x + \tilde{n}_y + \tilde{n}_z + \tfrac{3}{2}\right). \tag{32}$$

On the other hand, energy spectrum of spherical harmonic oscillator corresponding to Eq. (62) of Ref. [23], becomes as following

$$E^2 - M^2 \;=\; 2\left(M\omega_1^2(E-M)\right)^{\frac{1}{2}}\left(2\tilde{n} + \tilde{l} + \tfrac{3}{2}\right) = 2\alpha_1^2\left(2\tilde{n} + \tilde{l} + \tfrac{3}{2}\right), \tag{33}$$

where the potential, used in our paper, is twice the potential introduced in Ref. [23]. The $\tilde{l}$ and $\tilde{n}$ are the orbital angular momentum and the number of nodes of lower component of radial wavefunction.

By using the Eqs. (32) and (33), we have

$$\tilde{n}_x + \tilde{n}_y + \tilde{n}_z = 2\tilde{n} + \tilde{l}\,. \tag{34}$$

Then the energy equation of triaxial-symmetric harmonic oscillator for $\omega_x = \omega_y = \omega_z = \omega_1$ is equal to that of spherical harmonic oscillator.

Now, we derive upper components from Eqs. (15) and (16)

$$g_k(\vec{r}) \;=\; \frac{\alpha_z}{M-E}\frac{1}{H_{\tilde{n}_z}(\alpha_z z)}\left(\frac{H_{\tilde{n}_z+1}(\alpha_z z)}{2} - \tilde{n}_z H_{\tilde{n}_z-1}(\alpha_z z)\right) f_k(\vec{r})\,, \tag{35}$$

$$\begin{aligned} g^{\pm}_{k,\mp\frac{1}{2}}(\vec{r}) \;=\; \frac{1}{M-E}\Bigg(&\frac{\alpha_x}{H_{\tilde{n}_x}(\alpha_x x)}\left(\frac{H_{\tilde{n}_x+1}(\alpha_x x)}{2} - \tilde{n}_x H_{\tilde{n}_x-1}(\alpha_x x)\right) \\ &\mp \frac{i\alpha_y}{H_{\tilde{n}_y}(\alpha_y y)}\left(\frac{H_{\tilde{n}_y+1}(\alpha_y y)}{2} - \tilde{n}_y H_{\tilde{n}_y-1}(\alpha_y y)\right)\Bigg) f_k(\vec{r})\,. \end{aligned} \tag{36}$$

In an interesting paper Castro *et al.* [22] found that solutions for zero pseudoscalar potential are related to spin and pseudospin symmetry of Dirac equation in $3+1$ dimensions. They showed how charge conjugation and chiral transformations are related to spectra of spin and pseudospin symmetries. They also found that there is the same spectrum, but different spinor solutions for massless particles of the spin and pseudospin symmetries.

And now, our results are compared with those of Ginocchio's work [21] through chiral and charge-conjugation transformations. At first, we introduce charge-conjugation operation:

The charge-conjugation operation changes the sign of the vector potential in (1). This is performed by the transformation [29]

$$\Phi \rightarrow \Phi_{\rm c} = \alpha\,\Phi^{*}\,. \tag{37}$$

After applying this charge-conjugation operation to Dirac equation (2), time independent Dirac equation becomes

$$H_{\rm c}\tilde{\Phi}_{\rm c} = -E\,\tilde{\Phi}_{\rm c}\,, \tag{38}$$

where $\tilde{\Phi}_{\rm c} = \alpha\,\tilde{\Phi}^{*}$, $\tilde{\Phi}(x) = e^{\frac{i}{\hbar}Et}\phi\,(x,t)$ and $H_{\rm c}$ is given by

$$H_{\rm c} = \vec{\hat{\alpha}}\cdot\vec{p} + \hat{\beta}(M + S(\vec{r})) - V(\vec{r})\,. \tag{39}$$

In terms of the potentials Δ and Σ, this Hamiltonian becomes

$$H_{\rm c} = \vec{\hat{\alpha}}\cdot\vec{p} + \hat{\beta}Mc^2 - \frac{I+\hat{\beta}}{2}\Delta - \frac{I-\hat{\beta}}{2}\Sigma\,. \tag{40}$$

We can see that charge-conjugation operation changes the sign of the energy and of the potential $V(r)$. This means that Σ turns into $-\Delta$ and Δ into $-\Sigma$. Therefore, to be invariant under charge conjugation, the Hamiltonian must contain only a scalar potential [22]. Now, we introduce chiral transformation: Chiral operator for a Dirac spinor is the matrix γ^5. Transformed Dirac spinor under chiral transformation is given by $\Phi_\chi = \gamma^5\Phi$ and transformed Dirac Hamiltonian $H_\chi = \gamma^5 H\gamma^5$. Chiral transformed Dirac equation is

$$H_\chi\tilde{\Phi}_\chi = E\,\tilde{\Phi}_\chi\,, \tag{41}$$

where H_χ is given by

$$H_\chi = \vec{\hat{\alpha}}\cdot\vec{p} - \hat{\beta}\left(Mc^2 + S(\vec{r})\right) + V(\vec{r})\,, \tag{42}$$

in terms of Σ and Δ, this Hamiltonian becomes

$$H_\chi = \vec{\hat{\alpha}}\cdot\vec{p} - \hat{\beta}Mc^2 + \frac{I+\hat{\beta}}{2}\Delta + \frac{I-\hat{\beta}}{2}\Sigma\,. \tag{43}$$

This means that the chiral transformation changes the sign of the mass and that of scalar potential, thus turning Σ into Δ and *vice versa*. A chiral-invariant Hamiltonian needs to have zero mass and $S(r)$ zero everywhere [22].

With respect to this fact that charge-conjugation transformation performs the changes of $\Delta \to -\Sigma$, $\Sigma \to -\Delta$ and $E \to -E$, we can conclude that charge-conjugation transformation relates the spin symmetry of the negative bound-state solutions (antinucleons) to pseudospin symmetry of positive bound-state solutions (nucleons) [20]. Therefore, under charge-conjugation transformation, the nucleons energy spectrum, Eq. (30), becomes

$$E^2_{n_x,n_y,n_z} - M^2 = \alpha_x^2(2n_x+1) + \alpha_y^2(2n_y+1) + \alpha_z^2(2n_z+1)\,, \tag{44}$$

where

$$\alpha_x = \left(M\omega_x^2(E+M)\right)^{\frac{1}{4}}\ , \tag{45}$$

$$\alpha_y = \left(M\omega_y^2(E+M)\right)^{\frac{1}{4}}\ , \tag{46}$$

and

$$\alpha_z = \left(M\omega_z^2(E+M)\right)^{\frac{1}{4}}\ . \tag{47}$$

Here, n_x, n_y and n_z are the number of nodes for upper component in x, y and z directions, respectively.

Eq. (44) is the same energy equation (Eq. (19), Ref. [21]) which is applied for the study of antinucleons. Ginocchio got this equation for triaxial harmonic oscillator for the case $\Delta = 0$, $\Sigma = M(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)$, *i.e.*, spin symmetry.

Moreover, since under chiral transformation we have $\Sigma \to \Delta$, $\Delta \to \Sigma$ and $M \to -M$ changes, so we can obtain the conclusions related to $\Delta = 0$, $\Sigma = M(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)$, by changing M sign in relevant parameters. Therefore, under chiral transformation, nucleons energy spectrum, Eq. (30), becomes

$$E^2_{n_x,n_y,n_z} - M^2 = \alpha_x^2(2n_x+1) + \alpha_y^2(2n_y+1) + \alpha_z^2(2n_z+1)\,. \tag{48}$$

Eq. (48) is the same as Eq. (44) but for nucleons with spin symmetry. By comparing Eqs. (44) and (48), one may note that the spectra of nucleons and antinucleons are the same for spin symmetry. It means that, for Dirac harmonic oscillator which we considered, spectra of nucleon $\Delta = 0$ states are degenerate with antinucleon $\Delta = 0$ states.

The number of nodes of the function $g_k(\vec{r})$, in the z direction is different from that of lower component, $f_k(\vec{r})$, by one unit, but they are the same in x and y directions. Although, the number of nodes of $g^{\pm}_{k,\mp\frac{1}{2}}(\vec{r})$ functions is the same as that of lower component in z direction, they are different in x and y direction (by one unit).

In fact, this shows the structure of nodes in the pseudospin doublet of deformed nuclei with triaxial-symmetric harmonic oscillator potential.

The structure of nuclear rotational bands has been investigated in Ref. [30]. Szymanski considered single-particle motion of a nucleon in a highly deformed and fast rotating mean field by the rather well known procedure of the cranking model. Pseudospin picture which will be used extensively throughout this work is mainly connected to cranking treatment of the rotation. The cranking Hamiltonian [30]

$$H^\omega = H - \omega j_1 \tag{49}$$

will, therefore, be employed here together with the pseudospin picture. Here, the angular momentum j_1 is the sum of the pseudo-orbital angular momentum $\tilde{l}_1$ and pseudospin $\tilde{s}_1$. In Ref. [30], the whole dynamics has been described in terms of a simple picture of a Rotating Harmonic Oscillator (RHO) in the coordinate space. An exact solution to the cranking Hamiltonian H^ω exists [31, 32] and may be employed to investigate explicitly the single-particle Routhians. The solution has the form of three independent normal modes of the Harmonic Oscillator (HO) type and obtained the one-nucleon Routhians as [30]

$$e^\omega_\nu = \left(n_1 + \tfrac{1}{2}\right)\omega_1 + \left(n_2 + \tfrac{1}{2}\right)\Omega_2 + \left(n_3 + \tfrac{1}{2}\right)\Omega_3\,. \tag{50}$$

Here, $\omega_1, \omega_2, \omega_1$ are the three original harmonic oscillator frequencies. The two modified (normal) frequencies Ω_2 and Ω_3 are simple functions of ω_2 and ω_3, and rotational frequency ω [31, 32]. Integers n_1, n_2 and n_3 are the three quantum numbers of the RHO. It seems to be a remarkable result of such a model that whenever the condition

$$n_2 = n_3 \tag{51}$$

is fulfilled, the orbit (n_1, n_2, n_3) becomes almost a flat line in the $e^\omega_{n_1,n_2,n_3} = f(\omega)$ representation in a rather large interval of ω (see Fig. 1 of Ref. [30]).

The angular momentum operator j_1 from Eq. (48) couples all the states $|\tilde{N} n_3 \tilde{\Lambda} \Omega\rangle$ in the pseudospin picture so that expansion of any RHO state (n_1, n_2, n_3) into the states $|\tilde{N} n_3 \tilde{\Lambda} \Omega\rangle$ is infinite. Nevertheless, for slow rotation a certain correspondence between the two representations can be established approximately (see Table II from Fig. [30]). The states (n_1, n_2, n_3) are related to asymptotic HO representation in pseudospin picture. The approximate correspondence between the states in the two representations shows that the state (n_1, n_2, n_3) is related to the state $|\tilde{N} n_3 \tilde{\Lambda} \Omega\rangle$ as $\tilde{N} = n_1 + n_2 + n_3$ and $\tilde{\Lambda} = (\tilde{N} - n_3), (\tilde{N} - n_3) - 2, \ldots$ while the quantum number n_3 remains the same in both representations.

In order to transform pseudospin deformed HO representation into a usual deformed HO representation $|N n_3 \Lambda \Omega\rangle$, the corresponding procedure is well known as follows [1, 3]

$$\left|\tilde{N} n_3 \tilde{\Lambda} \Omega\right\rangle \longrightarrow |N + 1 n_3 \Lambda \Omega\rangle\,, \tag{52}$$

where $\Lambda = \tilde{\Lambda} \pm 1$ for $\Omega = \tilde{\Lambda} \pm (\frac{1}{2})$.

4. Conclusions

A study of goodness of pseudospin symmetry in triaxial nuclei was the subject matter of this paper. An analysis was carried out within the framework of the harmonic-oscillator shell model. In this paper, energy equation was given by Eq. (30). If $\omega_x = \omega_y = \omega_z = \omega_1$, then $\alpha_x = \alpha_y = \alpha_z = \alpha_1$, in this case, the energy equation would be converted into Eq. (32), which is the same corresponding energy equation for spherical symmetric nuclei. So the consistent condition of our result with previous findings [23] is $\tilde{n}_x + \tilde{n}_y + \tilde{n}_z = 2\tilde{n} + \tilde{l}$. The wavefunctions for triaxial nuclei and also for spherical nuclei are 4-component Dirac spinors, but in the former case one has four distinct spatial wavefunctions (and therefore four differential equations to solve) while in the latter it has only 2 distinct radial functions [23]. In this work, four-component Dirac wavefunctions were obtained as given by Eq. (31) and Eqs. (35), (36). We showed how the charge-conjugation and chiral transformations connect different spectra related to spin and pseudospin symmetries. In fact, we mapped our solution to Ginocchio's by appropriate changes of signs in the relevant parameters.

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