NON-LOCAL INTERACTIONS IN RENORMALIZED HAMILTONIANS*

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Renormalization group procedure transforms local Hamiltonian densities of canonical quantum field theories into non-local ones. The nonlocality is illustrated by a generic example of a term with a product of three fields, in which case it can be understood in terms of a wave function of a bound state of two effective particles.

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1. Introduction

Discussing renormalized Hamiltonians at the L Cracow School of Theoretical Physics seems appropriate since the half of a century that the School's history spans is a period of success in the development of quantum field theory and at the same time a period of continuing struggle with its Hamiltonian formulation that could be applied to physics of strong interactions. It is a period of success regarding strong interactions because it included definition of QCD and development of its applications to high-energy interactions of quarks and gluons using asymptotic freedom and the parton model. It also included development of the lattice version of QCD that now enters the period of testing the theory by comparison with data for masses of hadronic ground states and, in some cases, including their excitations, such as for heavy quarkonia. On the other hand, it is a period of continuing struggle because we still do not know how to imagine and mathematically describe a proton in sufficient detail of its quantum structure to make precise predictions about its properties. For example, we still do not know how to calculate its electroweak form factors and structure functions from first principles. This lack is also reflected in the fact that strong interactions between nucleons are not fully understood. For example, in high-energy interactions of nucleons, the parton model enters as a phenomenological tool

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in nearly every consideration based on QCD and it does so at the level of probability instead of a full wave function. Without the parton-model input, high-energy collisions of protons could not be described. On the other hand, strictly speaking, low-energy nuclear potentials remain unrelated to QCD degrees of freedom. These and related puzzles underlie many questions concerning heavy-ion collisions. Theoretical understanding of light hadrons other than nucleons is less satisfactory than for nucleons. In particular, gluon degrees of freedom in hadrons pose challenges since these degrees of freedom are hard to identify experimentally and apparently can interact so strongly among themselves as non-Abelian gauge bosons that our non-relativistic and perturbative intuitions based on analogies with QED are not conceptually sufficient for creating a dynamical quantum picture for them. On a deeper level, basic theory of strong interactions struggles with concepts such as a vacuum condensate, including the gluon condensate. The theory still misses a clear concept of a quantum ground state that could fulfill requirements of relativity in an unambiguous way and hence could apply in a broad range of studies of the visible universe.

The reason for talking about Hamiltonians is that in theory a hadron such as proton could be formally defined as a solution to the eigenvalue problem

$$H|\psi\rangle \;=\; E|\psi\rangle\,, \tag{1}$$

where H is a Hamiltonian for QCD. So far, there is no other way to seek quark and gluon wave functions for a proton in the Minkowski space-time.

2. Interaction Hamiltonian density

In QCD, as in other quantum field theories in their canonical formulation that starts from some local Lagrangian density, Hamiltonians can be written as integrals of a Hamiltonian density over a hyper-plane in space-time,

$$H \;=\; \int d^3x\,\mathcal{H}(x)\,. \tag{2}$$

Characteristic local interaction terms in the density $\mathcal{H}(x)$ in QED, Yukawa theory, and QCD contain a product of three fields, such as

$$\mathcal{H}_I(x) \;=\; g\,:\bar\psi(x)\,\not{A}(x)\,\psi(x):\,, \tag{3}$$

$$\mathcal{H}_I(x) \;=\; g\,:\bar\psi(x)\,\phi(x)\,\psi(x):\,, \tag{4}$$

$$\mathcal{H}_I(x) \;=\; g\,:\mathrm{Tr}\,\partial_\mu A_\nu(x)\,[A^\mu(x),A^\nu(x)]:\,, \tag{5}$$

respectively. All fields in the product have one and the same space-time argument. g is a coupling constant. Given this examples, and to eliminate details of secondary importance in this lecture, it is useful to consider an interaction Hamiltonian density of a generic form

$$\mathcal{H}_I(x) \;=\; g\,:\phi^3(x):\,, \tag{6}$$

where just one Hermitian field ϕ stands for all three fields in physically important theories. In this case, the corresponding non-local renormalized Hamiltonian density has the form (see below)

$$H_\lambda \;=\; \int d^3x_1\, d^3x_2\, d^3x_3\, \mathcal{H}_\lambda(x_1,x_2,x_3)\,, \tag{7}$$

where the density is

$$\mathcal{H}_\lambda(x_1,x_2,x_3) \;=\; g_\lambda\, f_\lambda(x_1,x_2,x_3)\, :\phi_\lambda(x_1)\,\phi_\lambda(x_2)\,\phi_\lambda(x_3): \;. \tag{8}$$

Parameter λ is an arbitrary scale parameter introduced in the renormalization group procedure. Physically, it can be associated with the scale of effective degrees of freedom one employs in order to achieve a simplest possible description of observables of the same scale. The function $f_\lambda(x_1,x_2,x_3)$ is called a vertex form factor in position space. All three points, x_1, x_2, and x_3, lie on the same hyper-plane in space-time.

3. Different forms of Hamiltonian dynamics

Evaluation of H from a Lagrangian density in a field theory requires a choice of the space-time hyper-surface to integrate over in Eqs. (2) and (7). Dirac has classified available choices [1]. One of them is the standard choice in which one integrates over a hyper-plane defined by condition $t = 0$, where t is a time of some inertial observer. Dirac called this choice the instant form (IF) of dynamics. A different form that Dirac distinguished from all other forms by the number of kinematical symmetries (7 instead of typical 6) uses a hyper-plane defined by the condition $x^+ = 0$, where $x^\pm = x^0 \pm x^3$. Such hyper-plane is swept by a wave front of a plane wave of light that moves against the 3rd axis (z-axis, using the standard convention for choosing a frame of reference). The front of a light wave is the reason why the corresponding hyper-plane is named light front (LF). The corresponding form of Hamiltonian dynamics was called by Dirac a front form (FF) and today many people speak of LF dynamics.

From the point of view of strong interaction theory, the LF dynamics is particularly distinguished because it is invariant with respect to boosts along z-axis. This is the additional, 7th kinematical symmetry mentioned above. This special symmetry is potentially very useful because it implies that the LF description of a hadron state $|\psi\rangle$ in Eq. (1) has the same form in the hadron center of mass frame (CMF) of reference and in the infinite momentum frame (IMF) of reference. No other formulation of the strong interaction theory offers such simple relationship between structures of hadrons as seen in these two frames. When one realizes that the parton model is constructed in the IMF while the classification of hadrons in particle data tables refers to models of hadrons in the CMF, it becomes clear that a theory of strong interactions is worth developing in the LF form of Hamiltonian dynamics.

Another difference between the LF and IF of dynamics is that the vacuum state in the LF Fock space, denoted by $|0\rangle$, automatically satisfies the eigenvalue condition $H|0\rangle = 0$ and can be identified with a physical vacuum. A lot of literature exists on the issue if this is a valid identification or if it misses important effects, such as spontaneous symmetry breaking and instantons. An extended discussion and a way out of the trouble is suggested in Ref. [2] using the concept of new terms in the LF Hamiltonian that are not present in a canonical Hamiltonian but do appear in a renormalized effective one. As far as the author knows, there is no evidence found yet in theory that any non-trivial vacuum exists in LF QCD. On the other hand, it is certain that a complete LF QCD Hamiltonian is not a plain canonical one because the canonical one is very singular and requires regularization. All known regularizations require counterterms and the complete structure of the latter continues to be unknown. From this point of view, non-localities of renormalized LF Hamiltonian densities are of major interest as potential locus of conceptual secrets that are hidden in QCD and, more generally, in quantum field theory as we know it [3].

4. How non-local Hamiltonians emerge in renormalization

Non-local interaction terms discussed in this lecture emerge in LF Hamiltonians as a result of application of a renormalization group procedure for effective particles (RGPEP) to a canonical theory[1]. The calculation is based on the assumption that a canonical Hamiltonian is a combination of products of creation and annihilation operators, commonly denoted here by a, with certain coefficients c,

$$H \;=\; \sum_I c_I \prod_{i\in I} a_i \,. \tag{9}$$

The subscript I denotes a set of all relevant kinematical quantum numbers, denoted by i. The operators a are unitarily rotated to new ones that create or annihilate effective particles. The new operators are denoted by a_λ.

[1] For a presentation of RGPEP that is most closely related to this lecture, see Ref. [4] and references therein. In fact, this lecture is to large extent based on Ref. [4].

The RGPEP transformation has the form

$$a_{\lambda i} \;=\; U_\lambda a_i U_\lambda^\dagger \,. \tag{10}$$

The subscript λ indicates the RGPEP momentum scale that characterizes the effective particles. Their spatial size is thought to be of the order of $1/\lambda$ in the sense that they interact significantly only when the energy change associated with an interaction is limited from above by λ. Thus, one can write a Hamiltonian for an effective theory of scale λ as

$$H_\lambda \;=\; \sum_I c_{\lambda I} \prod_{i\in I} a_{\lambda i} \,. \tag{11}$$

The operators a_λ and coefficients c_λ differ from operators a and coefficients c but the whole Hamiltonian as an operator is assumed to stay unchanged, $H_\lambda = H$. At the same time, H provides an initial condition for the RGPEP λ-evolution of a_λ and c_λ. The initial condition in the λ-evolution is set to canonical a and c including counterterms when $\lambda = \infty$. Thus, the canonical H with counterterms is treated as an initial condition in the RGPEP λ-evolution. The origin of difficulty in comprehending the theory is that the unknown counterterms in the initial condition are found only by inspection of solutions to an effective theory of relatively small value of λ and these solutions are not easy to obtain. The advantage of RGPEP method in comparison with all other methods known to the author is that RGPEP is designed to apply in perturbation theory without creating disconnected terms and infrared singularities. These features make the method particularly suitable for application in theories that are asymptotically free, which includes theory of strong interactions.

Instead of using here a lot of new notation (see [4] for examples), we can observe that $H_\lambda(a) = U_\lambda^\dagger H\, U_\lambda$ contains coefficients c_λ and satisfies a differential equation of the form $dH_\lambda(a)/d\lambda = [T_\lambda, H_\lambda(a)]$, where $T_\lambda = -U_\lambda^\dagger dU_\lambda/d\lambda$. The mathematical concept of RGPEP is to introduce candidates for T_λ that lead to desired features of H_λ. The initial condition for λ-evolution of U_λ is $U_\infty = 1$. With these preliminaries, one can now qualitatively understand how the non-local Hamiltonian densities emerge in RGPEP by using a special form for the generator T_λ. The choice used here is adapted from Wegner's flow equation [5] for Hamiltonians he considered in condensed matter physics.

Instead of parameter λ, consider the parameter $s = 1/\lambda^2$ and write the RGPEP equation for $H = H_0 + H_I$ as a function of s in the form

$$\frac{d}{ds} H \;=\; [[H_0, H], H] \,, \tag{12}$$

$$H(s=0) \;=\; H_{\rm can} + {\rm CT} \,, \tag{13}$$

where $H_{\rm can}$ denotes the regulated canonical Hamiltonian and CT denotes corresponding counterterms. Expanding the right-hand side in powers of the interaction, one obtains two terms, a linear one and a quadratic one, so that

$$\frac{d}{ds} H \;=\; -H_0^2 H_I + 2H_0 H_I H_0 - H_I H_0^2 + O\left(H_I^2\right) . \tag{14}$$

Keeping only the linear term in the interaction Hamiltonian, which corresponds to perturbative solution of RGPEP equations in lowest-order perturbation theory, one obtains a simple equation for matrix elements $H_{Imn} = \langle m|H_I|n\rangle$ in the basis of eigenstates of H_0. Namely,

$$\frac{d}{ds} H_{Imn} \sim -(E_m - E_n)^2 H_{Imn} \,, \tag{15}$$

where E_m and E_n are the eigenvalues of H_0 corresponding to the basis states $|m\rangle$ and $|n\rangle$, respectively. This approximate equation is elementary and its solution for the interaction Hamiltonian at $s > 0$ reads

$$H_{Imn}(s>0) \;=\; e^{-s(E_m-E_n)^2}\, H_{Imn}(s=0)\,. \tag{16}$$

This result means that in leading order the effective coefficients $c_{I\lambda}$ in Eq. (11) are related to the canonical coefficients c_I in Eq. (9) by the formula

$$c_{I\lambda} \;=\; e^{-(\Delta E/\lambda)^2}\, c_I \,, \tag{17}$$

where ΔE denotes the change of energy (as measured by H_0) across the interaction term that contains $c_{I\lambda}$. The actual LF calculation proceeds in similar way except that instead of energy one considers invariant mass squared of interacting particles. The reason is that the LF energy is $P^- = (P^{\perp\,2} + \mathcal{M}^2)/P^+$ and, after elimination of $P^\perp$ and P^+ using LF boost symmetry, one deals with $\Delta\mathcal{M}^2$ in ΔP^- instead of ΔE. Thus, one arrives at

$$c_{I\lambda} \;=\; e^{-(\Delta\mathcal{M}^2/\lambda^2)^2}\, c_I \,. \tag{18}$$

Below, the coefficient $f_\lambda = e^{-(\Delta\mathcal{M}/\lambda)^4}$, is called vertex form factor. The vertex form factor leads to a non-local Hamiltonian density.

How the non-local interactions emerge in RGPEP due to vertex form factors can be illustrated in the case of a local Hamiltonian density in Eq. (6). The canonical quantum field ϕ can be decomposed into its Fourier components on the LF with $x^+ = 0$,

$$\phi(x) \;=\; \int [p]\, \left[a_p\, e^{-ip\,x} + \text{h.c.}\right] \,, \tag{19}$$

where $[p]$ denotes a suitable integration measure over momenta p^+ and $p^\perp$ [4]. In terms of the canonical operators a_p with $p^+ > 0$, the bare interaction term of Eq. (6) reads

$$H_I \;=\; g \int [p_1 p_2 p_3]\, \delta_p \left(a^\dagger_{p_1}\, a^\dagger_{p_2}\, a_{p_3} + \text{h.c.} \right) . \tag{20}$$

The symbol δ_p denotes the Dirac δ-function that secures conservation of momentum, including factor $16\pi^3$.

As explained above, RGPEP introduces a vertex counterterm and changes the initial coupling constant g in the canonical Hamiltonian into a function of λ that is denoted by g_λ in the renormalized Hamiltonian at scale λ. The renormalized Hamiltonian also contains form factors in its vertices. Thus, the result of lowest-order RGPEP has the generic form

$$H_{\lambda I} = g_\lambda \int [p_1 p_2 p_3]\, \delta_P\, e^{-[(p_1+p_2)^2 - p_3^2]^2/\lambda^4} \left(a^\dagger_{\lambda p_1}\, a^\dagger_{\lambda p_2}\, a_{\lambda p_3} + \text{h.c.} \right) , \tag{21}$$

in which $a_\lambda = U_\lambda\, a\, U^\dagger_\lambda$ and the invariant mass squared of particles 1 and 2 is written as a square of their total momentum, assuming that the components p_1^- and p_2^- are given by eigenvalues of the free part of the Hamiltonian, H_0, for the corresponding states of free particles.

The next step is to introduce an effective quantum field operator that corresponds to scale λ

$$\phi_\lambda(x) \;=\; \int [p] \; \left[a_{\lambda p}\, e^{-ip\,x} + \text{h.c.} \right] . \tag{22}$$

This operator differs from the canonical quantum field operator in Eq. (19) by replacement of a by a_λ, which means it creates or annihilates effective particles that correspond to complex composites of the canonical field quanta.

Inverting the Fourier transform in Eq. (22), one can express the creation and annihilation operators for effective particles of scale λ in terms of the corresponding effective field operator

$$a_{\lambda p} \;=\; \int [x]\, \phi_\lambda(x)\, e^{+ip\,x} . \tag{23}$$

Note that the sign of p^+ in this expression determines whether one obtains a creation or an annihilation operator. The possibility of distinguishing creation from annihilation of particles by a sign of kinematical momentum is not available in the IF of dynamics because the sign of energy E in dispersion relation $E^2 = m^2 + \vec{p}^{\,2}$ is not determined by the direction of $\vec{p}$. In the LF formulation of theory, the sign of p^+ specifies the sign of p^- and thus distinguishes particles from anti-particles and creation from annihilation [4].

The term displayed explicitly in Eq. (21) can be re-written using the effective field operators and one obtains expressions of the form (details of handling p^+ and normal ordering are available in [4])

$$H_{\lambda I} \;=\; g_\lambda \int [x_1 x_2 x_3] \left[\bar{f}_\lambda(x_1, x_2, x_3)\, \phi_\lambda^\dagger(x_1)\, \phi_\lambda^\dagger(x_2)\, \phi_\lambda(x_3) + \text{h.c.} \right], \quad (24)$$

where the factor $\bar{f}_\lambda(x_1, x_2, x_3)$ is given by a Fourier transform of the vertex form factor f_λ that is introduced in Eq. (21) according to Eq. (18). The notation $\phi^\dagger$ is used here in Eq. (24) to single out the creation part of the field ϕ, while ϕ itself is limited to the annihilation part. After inclusion of all relevant terms and normal-ordering, one obtains the non-local Hamiltonian density in Eq. (8).

5. Non-relativistic intuition for $\bar{f}_\lambda(x_1, x_2, x_3)$

The relativistic lowest-order result of RGPEP for the non-local Hamiltonian density distribution $\bar{f}_\lambda(x_1, x_2, x_3)$ on the LF can be understood by comparing it to a model of a Hamiltonian vertex in the IF of dynamics. This section describes such a model for comparison.

Consider an IF Hamiltonian in a theory with a vertex of the type ϕ^3. Assume a vertex form factor $f_\lambda(p_1, p_2, p_3) = e^{-(\vec{p}_1 - \vec{p}_2)^2/\lambda^2}$ in an IF expression for an effective Hamiltonian at scale λ, built in analogy to the LF Eq. (21) despite that RGPEP in the IF of dynamics does not share all attractive features of the LF dynamics. For the particle 3 at rest, the invariant mass squared of particles 1 and 2 that have the same mass m, is $(p_1 + p_2)^2 = 4(m^2 + \vec{q}^{\,2})$, where $\vec{q} = (\vec{p}_1 - \vec{p}_2)/2$. Thus, the argument $(\vec{p}_1 - \vec{p}_2)^2$ of the model $f_\lambda(p_1, p_2, p_3)$ approximates momentum dependence of the invariant mass squared of particles 1 and 2 for slowly moving particle 3. An effective IF Hamiltonian with this vertex form factor reads

$$H_{\lambda I} \;=\; g_\lambda \int [p_1 p_2 p_3]\, \delta_P\, e^{-4\vec{q}^{\,2}/\lambda^2} \left(a_{\lambda p_1}^\dagger\, a_{\lambda p_2}^\dagger\, a_{\lambda p_3} + \text{h.c.} \right). \quad (25)$$

Strictly speaking, the RGPEP vertex form factor depends on a square of a difference of invariant masses squared of interacting particles before and after interaction, but the difference of squares of masses is a product of a difference and a sum of masses. In the NR approximation, the sum is approximately constant and the difference is a quadratic function of $\vec{q}^{\,2}$. So, the model $f_\lambda(p_1, p_2, p_3) = e^{-4\vec{q}^{\,2}/\lambda^2}$ has some resemblance to the RGPEP result in LF dynamics, except that the exponential includes $1/\lambda^2$ instead of $1/\lambda^4$ because the factor $9m^2/\lambda^2$ is arbitrarily removed from the model (for details, see [4]).

Assuming the IF quantum fields can be defined, one obtains

$$H_{\rm IF} = g_\lambda \int [x_1 x_2 x_3] \left[\bar{f}_\lambda(x_1, x_2, x_3) \, \phi^\dagger_\lambda(x_1) \, \phi^\dagger_\lambda(x_2) \, \phi_\lambda(x_3) + \text{h.c.} \right], \quad (26)$$

where

$$\bar{f}_\lambda(x_1, x_2, x_3) = \int [p_1 p_2 p_3] \, \delta_P \, e^{-4\vec{q}^{\,2}/\lambda^2} \, e^{i(\vec{p}_1 \vec{x}_1 + \vec{p}_2 \vec{x}_2 - \vec{p}_3 \vec{x}_3)} \, . \quad (27)$$

Introducing $\vec{P} = \vec{p}_1 + \vec{p}_2$ and changing integration variables so that

$$\bar{f}_\lambda(x_1, x_2, x_3) \sim \int d^3P \, d^3q \, e^{-4\vec{q}^{\,2}/\lambda^2} \, e^{i\vec{q}(\vec{x}_1 - \vec{x}_2) + i\vec{P}\left(\vec{R} - \vec{x}_3\right)} \quad (28)$$

one arrives at

$$\bar{f}(x_1, x_2, x_3) \sim \delta^3 \left(\vec{x}_3 - \vec{R} \right) \, e^{-(\lambda \vec{r}/4)^2} \, , \quad (29)$$

where

$$\vec{R} = (\vec{x}_1 + \vec{x}_2)/2 \quad (30)$$

denotes co-ordinates of a point in the middle between points of co-ordinates $\vec{x}_1$ and $\vec{x}_2$ on the space-time hyper-plane $t = 0$, and

$$\vec{r} = \vec{x}_1 - \vec{x}_2 \, , \quad (31)$$

denotes co-ordinates of the relative position of points with co-ordinates $\vec{x}_1$ and $\vec{x}_2$ on the same hyper-plane.

The NR intuition for the non-local Hamiltonian density vertex form factor $\bar{f}_\lambda(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ on the hyper-plane $t = 0$ described in this section can be summarized as follows. When two effective particles are created at points $\vec{x}_1$ and $\vec{x}_2$ from particle 3, the particle 3 is annihilated at the point $\vec{R}$ in the middle between points $\vec{x}_1$ and $\vec{x}_2$, and the strength of this interaction quickly decreases when the distance $\vec{r}$ between points $\vec{x}_1$ and $\vec{x}_2$ exceeds $4/\lambda$.

6. Relativistic results for $\bar{f}_\lambda(x_1, x_2, x_3)$

The relativistic RGPEP results for the non-local Hamiltonian density vertex form factor $\bar{f}_\lambda(x_1, x_2, x_3)$ on the LF differ from the intuitive picture described in the previous section. The relativistic results are described below using Figs. 1, 2, and 3.[2]

[2] Figs. 1, 2, and 3 correspond precisely to figures 2, 3, and 5 in [4], respectively, despite that they provide a different representation of the same features.

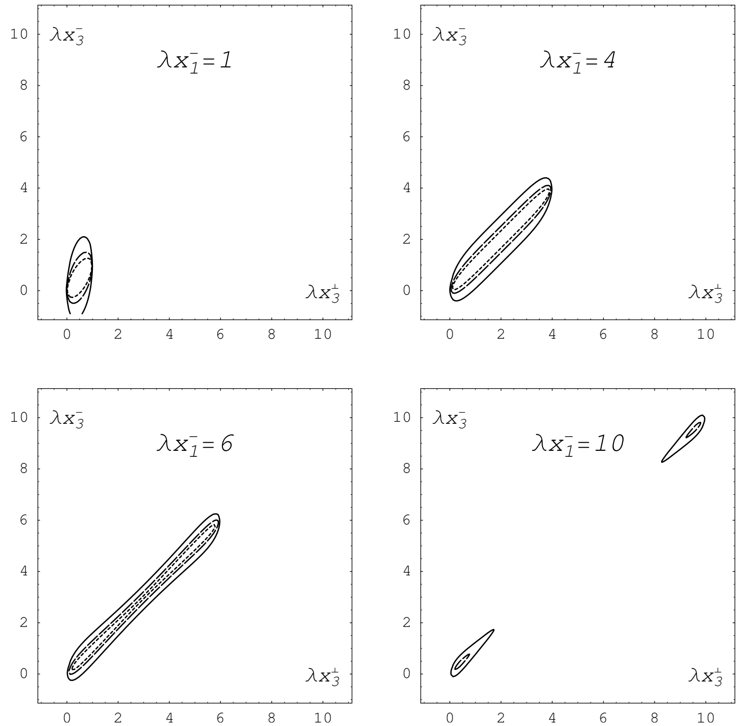

Fig. 1. The non-local Hamiltonian density is drawn on the LF in terms of contours of constant value of the function $|\lambda(x_1 - x_2)^\perp h(x_1, x_2, x_3)|$ in Eq. (32) for $\lambda \gg m$ for 4 different distances between x_1 and x_2. Continuous contour line corresponds to 0.1, dashed line to 0.3, and dotted line to 0.6. x_3 lies on the 2-dimensional space-time plane containing $x_1^\perp - x_2^\perp$. In all four panels $\lambda x_2 = 0$ while $\lambda x_1^- = \lambda |x_1^\perp| = 1$, 4, 6, and 10, as shown. The contours show that point x_3 of annihilation of one effective particle must lie in the vicinity of a line that connects the two points x_1 and x_2 of creation of two effective particles. This figure corresponds to Fig. 2 in Ref. [4]. See the text for more details.

Figs. 1, 2, and 3 show contours of constant value of the vertex form factor $\bar{f}_\lambda(x_1, x_2, x_3)$ on the LF, using conventions that are explained below and in Ref. [4]. All figures display the LF hyper-plane in terms of position co-ordinates of the annihilated particle 3, $x_3 = (x_3^\perp, x_3^-)$, multiplied by the RGPEP scale λ. Thus, the horizontal axes always display dimensionless co-ordinate $\lambda x_3^\perp$ and the vertical axes always display dimensionless co-ordinate λx_3^-.

The vertex form factor $\bar{f}_\lambda(x_1, x_2, x_3)$ on a 3-dimensional hyper-plane, which appears to be a function of 9 position variables, particle masses, and scale λ, can be reduced to a simple function on a plane and its contours

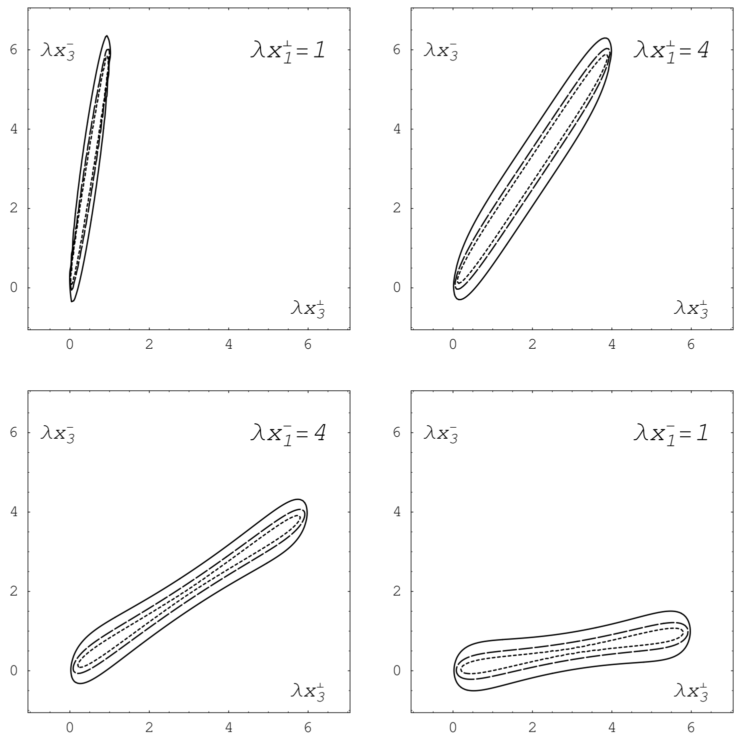

Fig. 2. Changes in the contour pattern drawn in precisely the same case and in the same convention as in Fig. 1 but for different choices of x_1 on the LF: $\lambda(|x_1^\perp|, x^-)$ = (1,6), (4,6), (6,4), and (6,1), as indicated. The contours show the same relativistic pattern of x_3 lying in a vicinity of the line that connects x_1 and x_2 on the LF. This figure corresponds to Fig. 3 in Ref. [4]. See the text for more explanation.

can be drawn on paper, because $\bar{f}_\lambda(x_1, x_2, x_3)$ differs from 0 only when $x_3^\perp$ is parallel to $x_1^\perp - x_2^\perp$. Thus, the three figures present contours drawn on the plane that lies in the LF hyper-plane and contains points x_1 and x_2. In other words, the form factor can be written as

$$\bar{f}_\lambda(x_1, x_2, x_3) \;=\; \frac{3\lambda^6}{16\pi^3}\,\delta\left[\lambda(x_3 - R)^\perp_{\perp r^\perp}\right]\, h_\lambda(x_1, x_2, x_3)\,, \qquad (32)$$

where the function δ is a Dirac δ-function that differs from 0 only when the transverse part of $x_3 - R$ that is transverse to $r^\perp$ is zero. Moreover, the function $h_\lambda(x_1, x_2, x_3)$ is translation invariant and can be drawn as a function of differences of positions alone. Thus, the figures correspond to different choices of $r = x_1 - x_2$. They display contours of constant value of $\lambda|r^\perp||h_\lambda(x_1, x_2, x_3)|$ when one changes x_3 within the plane that contains x_1 and x_2. Finally, $h_\lambda(x_1, x_2, x_3)$ is a function only of co-ordinates multiplied

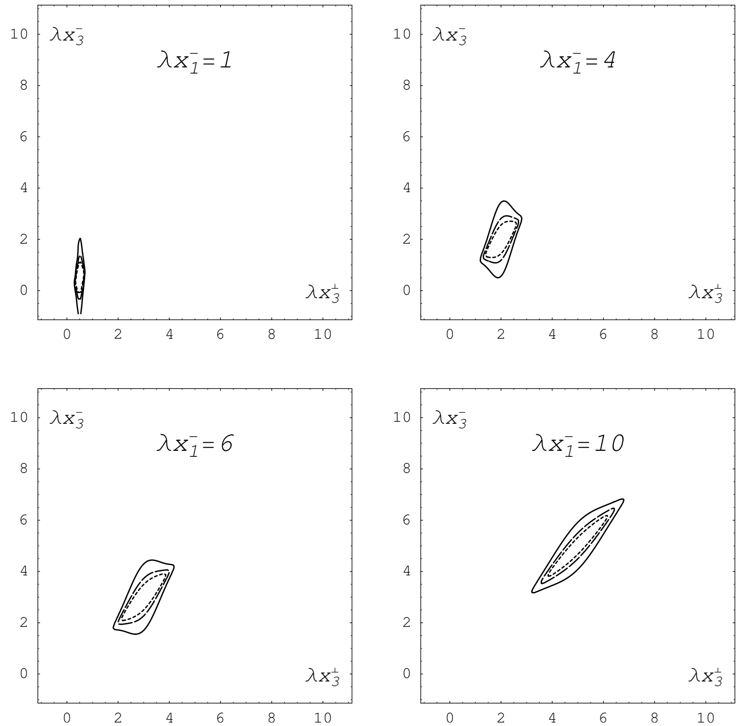

Fig. 3. The same result as in Fig. 1 but for $\lambda = m$ and multiplied in addition by $6e^9$ in order to remove the suppression that the vertex form factor implies for $\lambda \sim m$. This figure corresponds to Fig. 5 in Ref. [4]. See the text for further details.

by the scale λ when the particle mass m is negligible in comparison to λ. When the ratio m/λ becomes significant, the shape of $h_\lambda(x_1, x_2, x_3)$ changes (compare Figs. 3 and 1).

While in Figs. 1 and 2 the function whose contours are displayed is $\lambda |r^\perp| |h_\lambda(x_1, x_2, x_3)|$, in Fig. 3 the same function is displayed multiplied by the factor $6e^9$, which compensates the suppression of interaction strength by f_λ when $\lambda = m$ (see [4] for further explanation of this suppression effect in the case of Fig. 3).

In all figures, the displayed function is actually smoothed out by replacing the singular factor [4]

$$\lambda^{-2} \left[R^- - x_3^- - \frac{r^\perp \left(R^\perp - x_3^\perp \right)}{r^{\perp 2}} \, r^- - i\epsilon \right]^{-2} \tag{33}$$

with $\epsilon \to 0_+$ by the same factor with $\epsilon = 1/5$. This replacement is irrelevant to the discussion of main features of the relativistic non-local Hamiltonian densities that follows.

In all figures, there are three types of contours. The continuous line corresponds to value 0.1, the dashed line to 0.3, and dotted line to 0.6.

Figs. 1 and 2 concern highly relativistic cases where $\lambda \gg m$. Fig. 1 shows what happens when the distance between x_1 and x_2 increases. Fig. 2 shows what happens when the angular orientation of the position x_1 relative to x_2 changes with respect to x^--axis.

Fig. 3 concerns the case when $\lambda \sim m$. It shows what happens in this case when the distance between x_1 and x_2 grows. Comparison with Sec. 5 and Fig. 1, demonstrates that an effective interaction with $\lambda \sim m$ resembles a picture intuitively expected in a NR theory and differs considerably from the interaction in a theory with λ much greater than m.

The angular pattern in the case $\lambda \sim m$ is similar to the case of Fig. 2 in the sense that the contours rotate accordingly to the orientation of x_1 and x_2. There is no need for an extra figure to display this feature.

To summarize, in a relativistic theory, when the RGPEP scale parameter λ is much greater than the particle mass parameter m, the non-locality of renormalized LF Hamiltonians in three-prong interaction terms has the following feature. The interaction strength is distributed along the line that connects points of creation (annihilation) of two particles in the sense that the point of annihilation (creation) of one particle must lie in the vicinity of this line on the LF. In a non-relativistic IF theory, the strength of annihilation (creation) of one particle is concentrated in the middle between points of creation (annihilation) of two particles.

7. Relation to wave functions

One can write a two-body bound-state wave function ψ in terms of a two-body propagator G and a vertex function ϕ as $\psi = G\phi$. When one realizes that the vertex function can be related to a Hamiltonian interaction term, at least in simple models, it becomes clear that the Hamiltonian that changes one particle into two can be interpreted in terms of a two-body wave function: it provides the vertex function ϕ.

Consider a bound state of two effective particles characterized by scale λ. When the bound state has momentum $P = (P^+, P^\perp)$, its vertex function has the form

$$\phi_{\lambda P}(x_1, x_2) \sim \int d^3x_3 \, \bar{f}_\lambda(x_1, x_2, x_3) \, e^{-iPx_3} \, . \tag{34}$$

Using results described in the previous sections, in the case $\lambda \gg m$ one obtains:

$$\phi_{\lambda P}(x_1, x_2) \sim \left(\frac{\lambda}{4\pi}\right)^2 P^+ e^{-iPR} \int_0^1 dz z(1-z) e^{-i(z-1/2)Pr - \frac{1}{4}z(1-z)\lambda^2 r^{\perp 2}} \tag{35}$$

and in the case $\lambda \lesssim m$,

$$\phi_{\lambda P}(x_1, x_2) \;=\; 3g_\lambda \left(\frac{\lambda}{4\pi}\right)^2 P^+ \, e^{-iPR} \, C(\lambda/m) \, e^{-\frac{\lambda^4}{96m^2}\left[\left(\frac{Pr}{2m}\right)^2 + r^{\perp 2}\right]} , \quad (36)$$

where $C(\lambda/m)$ is a number [4].

These LF vertex functions depend on variables R, Pr and $r^\perp$ in a general way that is summarized by writing

$$\phi_{\lambda P} \;\sim\; P^+ \lambda^2 \, e^{-iPR} \, \phi_\lambda(Pr, r) \, . \quad (37)$$

Functions $\phi_\lambda(Pr, r)$ are considerably different in relativistic and non-relativistic cases. Relativistically, when RGPEP scale λ is much greater than the mass parameter m, the LF variables $r^\perp$ and Pr do not mix in a way familiar from the IF of quantum mechanics. Non-relativistically, *i.e.*, in the case when $\lambda \lesssim m$, the same LF variables turn out more familiar: P is approximately $2m$ and for $r^+ = 0$ one has $r- = -2r^z$, which implies that $Pr \sim -2mr^z$. In this case, $\phi_\lambda(Pr, r)$ turns out to be a function of a 3-dimensional position vector $\vec{r}$.

The fact that wave functions depend on variables Pr and $r^\perp$, which are invariant with respect to 7 kinematical transformations in LF formulation of a theory, has an important implication for the program of seeking a solution to LF QCD outlined in Ref. [2]. It is proposed there that a confining potential for two colored particles should depend on $r^\perp$ and $P^+ r^-$. The analysis reported here suggests that the argument $P^+ r^-$ should be replaced by $P^+ r^- - 2P^\perp r^\perp$.

Non-locality of effective three-prong Hamiltonian interaction terms on the LF can be thus understood using bound-state wave functions whose width in position space is of the order of $1/\lambda$. The interaction terms contain integrals over total momentum of such bound-states. One prong corresponds to a bound state, and two remaining prongs correspond to constituents. Characteristic arguments Pr and $r^\perp$ of the corresponding wave functions are identified as the right variables for LF renormalization group studies of Hamiltonians in quantum field theory.

8. Conclusion

Besides questions concerning structure of Hamiltonian interaction terms at scales λ comparable with experimentally accessible momenta, and corresponding space distances of the order of $1/\lambda$, non-local renormalized interactions are of interest also as candidates for defining a theory of particles of finite size, in distinction from the idealization of a local theory for point-like particles. Local theories are singular because locality of interactions leads to singularly violent changes of momenta of the point-like particles when they overlap to interact. One way to avoid the singularities due to locality is to invent a basic non-local theory from scratch, as it is done in string theory. Another way, based on renormalization group approach, is to calculate a finite non-local theory starting from an artificial one that is local, and to subsequently declare that the calculated, scale-dependent effective non-local theories are physically valid only in some range of scales. The latter option leaves the question of structure of a deeper theory not fully answered. The only claim to make would be that the calculated effective theories belong to the same universality class that the desired but unknown deeper theory belongs to. However, the deeper theory may start to differ from the calculated effective theories only at scales so far away from the experimentally accessible scales that no practical need may arise to know any specific features of the deeper theory for a very long time.

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