A BPS SKYRME MODEL — MATHEMATICAL PROPERTIES AND PHYSICAL APPLICATIONS*

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(Received November 10, 2010)

Using the framework of generalized integrability, the BPS chiral model is constructed. This model is integrable in the sense of the existence of infinitely many conservation laws, solvable as it leads to exact soliton solutions carrying arbitrary topological charge and, by construction, very topological in nature. Moreover, solutions are of the Bogomolny type and saturate the corresponding topological bound, which immediately guarantees their stability. When applied to nuclear physics, the model seems to cure several serious problems appearing in the standard Skyrme model as well as in its typical generalizations. At the classical level, it qualitatively reproduces the main features of the liquid drop model of nuclei, providing proper relations between masses and radii of nuclei and the baryon number. In spite of its rather unusual form, the model also allows for the semiclassical quantization.

PACS numbers: 12.39.Fe, 12.39.Dc

1. Introduction

The Skyrme proposal for the formulation of the low energy sector of QCD by means of an effective Lagrangian built out of meson matrix fields U [1] is still one of the most accepted and successful attempts for the description of the static properties of baryons and atomic nuclei as well as their

* Lecture presented at the L Cracow School of Theoretical Physics "Particle Physics at the Dawn of the LHC", Zakopane, Poland, June 9–19, 2010.

interactions. This old idea became even more attractive when it was shown that in the large N_c limit QCD is equivalent to an unknown effective theory of mesons [2, 3]. Although in the real world the number of colors is finite, it is believed that $N_c = 3$ is sufficiently large to support the description of the non-perturbative QCD in terms of mesonic degrees of freedom. The next important feature of the Skyrme models is their close connection to topology. In fact, baryons appear as solitonic, *i.e.*, collective excitations of the original mesonic field with an identification between the baryon number and the pertinent topological charge. In the simplest case, U takes values in the SU(2) Lie group. Then, static solutions are maps from the physical $\mathbb{R}^3$ space into the $\mathrm{SU}(2) \cong \mathbb{S}^3$ target space

$$U: \;\; \mathbb{R}^3 \cup \{\infty\} \cong \mathbb{S}^3 \ni \vec{x} \;\; \longrightarrow \;\; U(\vec{x}\,) \in \mathbb{S}^3 \,,$$

where the compactification of the base space to $\mathbb{S}^3$ is due to the finite energy boundary condition $U(\vec{x}\,) \rightarrow U_0 = \text{const.}$ as $\vec{x} \rightarrow \infty$. Such maps can be divided into disconnected homotopy classes and characterized by the corresponding topological index $B \in \pi_3(\mathbb{S}^3)$,

$$B = \frac{1}{24\pi^2} \int d^3x \, \epsilon^{ijk} \mathrm{Tr}\left(L_i L_j L_k\right) \,, \tag{1}$$

where $L_i = U^\dagger \partial_i U$.

In its basic form, the Skyrme Lagrangian is given by three terms

$$\mathcal{L}_{\mathrm{Skr}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_0 \,, \tag{2}$$

where apart from the standard sigma-model term

$$\mathcal{L}_2 = -\frac{f_\pi^2}{4} \mathrm{Tr}\left(L_\mu L^\mu\right) \,, \tag{3}$$

one adds a four derivative part, the so-called Skyrme term,

$$\mathcal{L}_4 = -\frac{1}{32e^3} \mathrm{Tr}\left([L_\mu \,, L_\nu]^2 \right) \tag{4}$$

and a potential

$$\mathcal{L}_0 = -\mu^2 V\left(\mathrm{Tr}\, U\right) \,. \tag{5}$$

The inclusion of these terms seems to be quite natural. The sigma model part corresponds to the standard kinetic term for the mesonic field. The quartic term is, on the other hand, needed from the stability point of view. Namely, it allows to circumvent the Derrick's arguments for the nonexistence of static soliton solutions. Moreover, the Skyrme term $\mathcal{L}_4$ is a rather special one as it leads to equations of motion which are of the second order in time and allows for the semiclassical quantization procedure [4]. The potential term provides masses for the perturbative pseudoscalar pions.

In spite of its apparent success, the Skyrme model as well as its further generalizations have several serious problems.

Derivation of the model. The typical way to derive the Skyrme type models is to simultaneously perform the large N_c and small derivative expansion. In practice, one calculates one-fermion-loop contributions to the low energy effective action and truncates it at a given power of derivatives. For the fourth order Lagrangian this procedure gives two new terms which contribute at the same level as the standard Skyrme term [5,6]

$$2\mathrm{Tr}\left(\partial_\mu^2 U^\dagger \partial_\nu^2 U\right) \quad \text{and} \quad -\mathrm{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right)^2 .$$

However, these terms are rather problematic as they lead to second order time derivatives in the action (or four power of the first order). Usually, for models with Lagrangians containing second order derivatives one has to extend the variational principle to get equations of motion. Such equations are of the fourth order and therefore require extended, rather unphysical Cauchy data. It also results in a very non-standard time dynamics and serious difficulties with the semiclassical quantization. Moreover, the second term tends to destabilize solitons. In the case of sixth order Lagrangians, even more terms appear without any grading between them [7].

There is an additional conceptual obstacle which makes the small derivative expansion disputable. Baryons, being solitons in a pure mesonic model, are configurations with significant values of the spatial derivatives. Because of that, higher derivative terms seem to be as important as quadratic or quartic [8–14] ones. In other words, one should consider the complete derivative *Gasser–Leutwyler* expansion of an unknown correct effective theory which finally would provide us with a model containing infinitely many terms. As higher power derivative terms give rise to higher n-body interactions, such a Lagrangian describes a strongly correlated system for which many body interactions are not small perturbations to the two-body interaction. Undoubtedly, such a picture is plausible in the context of the low energy QCD which, indeed, is believed to be an example of a strongly correlated system. Of course, we do not know the complete expansion of the effective theory and, even if we knew it, it would be very difficult to treat this model in an analytical way.

Large binding energies. It has been proved that stable solitonic solutions of the Skyrme model (massless [15] as well massive [16–18], which are interpreted as nuclei with a certain baryon number B, give non-physically large binding energies. They are significantly bigger than the experimental energies which do not exceed 1% of the masses of the nuclei. The reason for that is a certain mathematical property of the Skyrme model *i.e.*, the fact that it is not an exact BPS theory. As a consequence, the corresponding energy-topological charge bound cannot be saturated by soliton solutions resulting in the appearance of nonlinear energy-baryon charge relations and binding energies. From the point of view of the large N_c expansion this indicates that the binding energies scale like $N_c \Lambda_{\rm QCD}$, instead of $\Lambda_{\rm QCD}/N_c$ as expected for the weakly bound nuclear matter [19].

Shell-like baryons. Another feature of the Skyrme model is that shell-like structures are preferred rather than core or ball configurations. In fact, in the massless Skyrme model all stable solitons possess fulleren-like structure with empty regions inside (almost zero energy density). Further, the size of the solutions scales wrongly with the topological charge $R \sim \sqrt{B}$ [15]. In the case of the massive Skyrme model, the situation is improved and one asymptotically gets the correct size-baryon number relation $R \sim \sqrt[3]{B}$. However, for physically acceptable values of the model parameters, the first few skyrmions are of the shell type again [16–18].

Crystal state of matter. Numerical results, especially in the limit of the large baryon charge, show that the matter described by the Skyrme model behaves like a crystal [15]. This is obviously in contradiction with the standard, liquid picture of nuclear matter. This result is further supported by analytical calculations for $N_c \to \infty$ [20].

Strong forces. Because of the enhancement of the pion coupling constant in the Skyrme model $g_{\pi NN} \sim N_c^{3/2}$, the axial coupling constant grows linearly with N_c. This results in strong spin–isospin forces at distances larger than the size of the nucleus, and, obviously, is in contradiction to experimental as well as lattice data [21].

Quantitative results. The quantitative agreement between observables derived from the Skyrme model (masses of light baryons and nuclei, charge radii, magnetic moments, coupling constants *etc.*) and experimental data is on the level of approximately 10–20%. Nonetheless, for some observables as for example the axial coupling constant g_A, one observes much bigger disagreement.

Taking into account all the problems indicated above, one may wonder whether the Skyrme proposal in general, and the Skyrme model in particular, can have a chance to be the right framework for the description of the low energy QCD. Indeed, Skyrme's main idea, *i.e.*, the existence of the pure mesonic effective model has been criticized, and some generalizations or totally new theories have been proposed. Basically, they consist in the inclusion of quarks to the low energy effective action even at the lowest order.

On the other hand, the Skyrme model is able to reproduce a vast number of experimental values with reasonably good precision. It allows also for an extremely precise calculation of the excitation spectra of several light nuclei [18]. In fact, it has been established that many properties of Skyrme type models are model-independent, while others are only weakly modified by the inclusion of some additional terms. So, why the model works at all?

A possible answer, supported by the model-independent results, is that, as proven by large N_c calculations, the bosonic matrix field is indeed a proper degree of freedom, at least for $N_c \to \infty$. Thus, baryons are still topological solitons *i.e.*, collective excitations of the primary mesonic field. The main problem is that we do not know the right effective action and apparently the small derivative expansion does not lead to a satisfactory approximation. Therefore, we believe that one does not have to abandon Skyrme's idea but rather should significantly modify the action.

Of course, if we doubt the relevance of the small derivation expansion, we cannot build the action starting with the sigma model term. We need a different principle to propose the new model. The unique tool, which may help us, is again the topological concept of Skyrme. Namely, we want to consider *the most topological* action for the matrix field, which seems to be quite natural as we would like to describe the strongly correlated system. In other words, we apply the Skyrme proposal to the extreme and try to encode as many properties of baryons as possible into topological aspects of the model.

To summarize, our strategy is the following:

(i) we use the standard mesonic d.o.f. $U \in SU(2)$,

(ii) baryons are still topological solitons,

(iii) we build the action using the "maximal topology" principle.

Quite surprisingly, a model constructed using this procedure cures all qualitative problems indicated above and still gives reasonable numerical values for masses, radii as well as magnetic momenta of some baryons.

2. BPS Skyrme model

The most topological model means that its action is built out of topological quantities. Of course, it cannot be an example of a topological model, but a metric tensor must be nontrivially included into the construction. A topological quantity relevant for the Skyrme field is the baryon current

$$\mathbb{B}^\mu = \frac{1}{24\pi^2}\mathrm{Tr}\,\left(\epsilon^{\mu\nu\rho\sigma}U^\dagger\partial_\nu U\, U^\dagger\partial_\rho U\, U^\dagger\partial_\sigma U\right)\,. \tag{6}$$

Then, the Lorentz invariant metric dependent term reads

$$\mathcal{L}_6 = \lambda^2\pi^4\mathbb{B}_\mu\mathbb{B}^\mu\,, \tag{7}$$

where index 6 refers to the number of derivatives. Such a term is included in many generalizations of the standard Skyrme action by many authors and it usually improves phenomenological results. Effectively, this term is induced by a massive vector meson ω_μ coupled to the baryon density $\omega^\mu\mathbb{B}_\mu$. Due to that, treating (7) as the main ingredient for our model realizes, in some sense, the old concept of the vector meson dominance in QCD.

One may show using the Derrick theorem that a model containing only the $\mathcal{L}_6$ term does not allow for stable static soliton configurations. One can for example consider a scaling transformation for the spatial coordinates $\vec{x}\to a\vec{x}$, where a is a parameter of the transformation. Then, the static energy functional transforms as

$$E\left[U(a\vec{x}\,)\right] = \frac{1}{a^3}E\left[U(\vec{x}\,)\right]\,.$$

Therefore, the energy can be lowered to arbitrarily small values taking sufficiently large a, which means that solitons are unstable with respect to expansion. We have to stabilize solitons by adding a new term which would scale in opposite direction. The simplest term one may use is a potential $\mathcal{L}_0$, *i.e.*, a non-derivative part of the model. Hence, the model we want to propose is given by the expression [22, 23]

$$\mathcal{L}_{06} = \mathcal{L}_6 + \mathcal{L}_0\,. \tag{8}$$

We call this model the BPS Skyrme model, for reasons explained below. Let us also underline that although the model contains a term of sixth order in derivatives it contains maximally time derivatives squared. Because of that, it gives a second order Euler–Lagrange equation, for which the standard Cauchy data are enough to uniquely describe the time evolution. Moreover, such a model possesses a Hamiltonian formulation and, therefore, can be quantized by the semiclassical method.

Below we present some mathematical properties of the BPS Skyrme model, which play an important role in its phenomenological applications.

2.1. Symmetries

First of all, it is convenient to use the following parametrization of the SU(2) chiral field

$$U = e^{i\xi\vec{n}\cdot\vec{\sigma}} = \cos\xi + i\sin\xi\vec{n}\cdot\vec{\sigma}\,, \qquad \vec{n}^2 = 1\,,$$

where $\vec{\tau}$ are the Pauli matrices, ξ is a real field and $\vec{n}$ is an unit three component vector field, which is further related to a complex field u by means of the stereographic projection

$$\vec{n} = \frac{1}{1+|u|^2}\left(u+\bar{u}, -i(u-\bar{u}), 1-|u|^2\right)\,.$$

Then, the BPS model takes the form

$$L_{06} = -\frac{\lambda^2\sin^4\xi}{(1+|u|^2)^4}\,\left(\epsilon^{\mu\nu\rho\sigma}\xi_\nu u_\rho \bar{u}_\sigma\right)^2 - \mu^2 V(\xi)\,, \tag{9}$$

where we make the additional assumption that the potential depends only on $\mathrm{Tr}\,U$.

In addition to the standard Poincare symmetries, the BPS model possesses an infinite number of target space symmetries. The origin of these symmetries is a very special geometric property of the sextic term $\mathcal{L}_6$. It is the square of the pullback of the volume form on the target space $\mathbb{S}^3$, where the target space volume element is

$$dV = -i\frac{\sin^2\xi}{(1+|u|^2)^2}d\xi du d\bar{u} \tag{10}$$

with the exterior product of the differentials. Therefore, all transformations which preserve the volume form, *i.e.*, the volume preserving diffeomorphisms (VPD) on $\mathbb{S}^3$, are also symmetries of the sextic term. The potential part, in general, breaks all or some of these symmetries depending on its particular form. In our case, $\mathcal{L}_0 = V(\xi)$, it respects a certain subgroup of the volume preserving diffeomorphisms. Concretely, it is invariant under those VPD which act nontrivially only on $u, \bar{u}$ leaving ξ unchanged

$$\xi \to \xi\,, \quad u \to \tilde{u}(u,\bar{u},\xi)\,, \quad \left(1+|\tilde{u}|^2\right)^{-2} d\xi d\tilde{u} d\bar{\tilde{u}} = \left(1+|u|^2\right)^{-2} d\xi d\tilde{u} d\bar{u}\,. \tag{11}$$

They form a one parameter family of the groups of the area preserving diffeomorphisms on $\mathbb{S}^2$ [24]. Such an infinite dimensional family is a symmetry of the full action, so it is a Noether symmetry with the corresponding infinitely many conserved currents.

There are also additional symmetries which are symmetries of the static energy functional but not the action (for the baby Skyrme model see [25]). Thus, there are no Noether currents corresponding to them but nevertheless they are very important for physical applications of the model. Consider the static energy functional

$$E = \int d^3x \left(\frac{\lambda^2 \sin^4 \xi}{(1+|u|^2)^4} \left(\epsilon^{mnl} i \xi_m u_n \bar{u}_l \right)^2 + \mu^2 V \right) . \tag{12}$$

Both the base space volume element d^3x and $\epsilon^{ijk}\xi_i u_j \bar{u}_k$ are invariant under coordinate transformations of the base space which leave the volume form d^3x invariant. In other words, the static energy functional has the volume preserving diffeomorphisms on base space as symmetries. However, these symmetries are exactly the symmetries of an incompressible ideal fluid (see also [26]).

2.2. Integrability

The existence of infinitely many Noether transformations and related with them infinitely many conserved quantities usually indicates the integrability of a system. Indeed, it is exactly the case in the BPS Skyrme model, which is integrable in the sense of the *generalized integrability* [27]. From a mathematical point of view the BPS Skyrme model belongs to a large family of models built out of the pullback of the volume form on the target space squared [28–32] together with a stabilizing potential term, and a procedure for deriving conservation laws for these models is very well known [33–35]. Following previous works, we may present these currents in an exact form

$$J_\mu = \frac{\delta G}{\delta \bar{u}} \bar{\mathcal{K}}_\mu - \frac{\delta G}{\delta u} \mathcal{K}_\mu \, , \tag{13}$$

where

$$\mathcal{K}^\mu = \frac{K^\mu}{(1+|u|^2)^2} \, , \qquad K_\mu = \frac{\partial}{\partial \bar{u}^\mu} \left(\epsilon^{\alpha\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma \right)^2 \tag{14}$$

and $G = G(u, \bar{u}, \xi)$ is an arbitrary function of its arguments. Now, we have to calculate the four-divergence

$$\begin{aligned} \partial^\mu J_\mu \;=\; & G_{\bar{u}\bar{u}} \bar{u}_\mu \bar{\mathcal{K}}^\mu + G_{\bar{u}u} u_\mu \bar{\mathcal{K}}^\mu + G_{\bar{u}} \partial_\mu \bar{\mathcal{K}}^\mu - G_{u\bar{u}} \bar{u}_\mu \mathcal{K}^\mu \\ & - G_{uu} u_\mu \mathcal{K}^\mu - G_u \partial_\mu \mathcal{K}^\mu + G_{\bar{u}\xi} \xi_\mu \bar{\mathcal{K}}^\mu - G_{u\xi} \xi_\mu \mathcal{K}^\mu \, . \end{aligned}$$

Using the identities satisfied by $\mathcal{K}_\mu$,

$$u_\mu \mathcal{K}^\mu = \xi_\mu \mathcal{K}^\mu = \bar{u}_\mu \mathcal{K}^\mu = u_\mu \bar{\mathcal{K}}^\mu = 0$$

and the equations of motion

$$\partial_\mu \mathcal{K}^\mu = 0\,, \tag{15}$$

we arrive at $\partial^\mu J_\mu = 0$. The existence of infinitely many conservation laws is a very special property and occurs rather rarely for higher dimensional solitonic theories. However, their appearance may indicate the solvability of the model, that is, the possibility to find the energy minimum (or minima) in every topological sector in an exact form. As we show below, this connection between integrability and solvability holds for the BPS Skyrme model. The standard Skyrme model, as well as its generalizations, is an example of a non-integrable theory, which makes analytical computations very difficult. Fortunately, for some cases (the Skyrme model without potential) one may apply the rational map ansatz method [36] allowing for an analytical treatment of the model and for a derivation of approximate solutions. However, in general one has to perform refined numerical calculations[1]. Therefore, the possibility for an analytical treatment of the BPS Skyrme model is another advantage of this theory.

2.3. Bogomolny bound

Now, we show that our solitons are of the BPS type and saturate a Bogomolny bound. The energy functional reads

$$\begin{aligned} E &= \int d^3x \left(\frac{\lambda^2 \sin^4 \xi}{(1+|u|^2)^4} (\epsilon^{mnl} i\xi_m u_n \bar{u}_l)^2 + \mu^2 V \right) \\ &= \int d^3x \left(\frac{\lambda \sin^2 \xi}{(1+|u|^2)^2} \epsilon^{mnl} i\xi_m u_n \bar{u}_l \pm\ \mu\sqrt{V} \right)^2 \\ &\quad \mp \int d^3x \frac{2\mu\lambda \sin^2 \xi \sqrt{V}}{(1+|u|^2)^2} \epsilon^{mnl} i\xi_m u_n \bar{u}_l \\ &\geq \mp \int d^3x \frac{2\mu\lambda \sin^2 \xi \sqrt{V}}{(1+|u|^2)^2} \epsilon^{mnl} i\xi_m u_n \bar{u}_l \\ &= \pm \left(2\lambda\mu\pi^2\right) \left[\frac{-i}{\pi^2} \int d^3x \frac{\sin^2 \xi \sqrt{V}}{(1+|u|^2)^2} \epsilon^{mnl} \xi_m u_n \bar{u}_l \right] \equiv 2\lambda\mu\pi^2 C[V]|B|, \end{aligned} \tag{16}$$

[1] There is one exceptional case for charge one sector, where the hedgehog configuration allows for the reduction of the full system of PDEs to an ODE. A plausible explanation may be again formulated in the language of the generalized integrability. The Skyrme model possesses integrable sectors if one imposes some additional constraints, so-called integrability conditions. The hedgehog ansatz obeys these constraints for an arbitrary profile function, which remains to be determined by the reduced equation of motion.

where B is the baryon number (topological charge) and the sign has to be chosen appropriately (upper sign for $B > 0$). The constant $C[V]$ depends on the potential but not on a particular solution. If we replace $\sqrt{V}$ by one, then the result (*i.e.*, the last equality in (16)) follows immediately (and the constant $C[V] = 1$). Indeed, for $V = 1$ the expression in brackets is just the topological charge, see *e.g.* [37], chapter 1.4. An equivalent derivation, which shall be useful below, starts with the observation that this expression is just the base space integral of the pullback of the volume form on the target space $\mathbb{S}^3$, normalized to one. Further, while the target space $\mathbb{S}^3$ is covered once, the base space $\mathbb{S}^3$ is covered B times, which implies the result. The same argument continues to hold with the factor $\sqrt{V}$ present (remember that $V = V(\xi)$), up to a constant $C[V]$. Indeed, we just have to introduce a new target space coordinate $\bar{\xi}$ such that

$$\sin^2 \xi \sqrt{V(\xi)}\, d\xi = C[V] \sin^2 \bar{\xi}\, d\bar{\xi}\,. \tag{17}$$

The constant $C[V]$ and a second constant C_2, which is provided by the integration of Eq. (17), are needed to impose the two conditions $\bar{\xi}(\xi = 0) = 0$ and $\bar{\xi}(\xi = \pi) = \pi$, which have to hold if $\bar{\xi}$ is a good coordinate on the target space $\mathbb{S}^3$. Obviously, $C[V]$ depends on the potential $V(\xi)$. Specifically, for the standard Skyrme potential $V = 1 - \cos \xi$, $C[V]$ is

$$C[V] = \frac{32\sqrt{2}}{15\pi}$$

as may be checked easily by an elementary integration. We remark that an analogous Bogomolny bound in one lower dimension has been derived in [38] for the baby Skyrme model.

The Bogomolny inequality is saturated by configurations obeying the first order Bogomolny equation

$$\frac{\lambda \sin^2 \xi}{(1+|u|^2)^2} \epsilon^{mnl} i\xi_m u_n \bar{u}_l = \mp \mu \sqrt{V}\,.$$

The saturation of the energy-charge inequality by our solutions proves their stability. It is not possible to find configurations with lesser energy in a sector with a fixed value of the baryon charge.

2.4. Axially symmetric ansatz

The field equations for the BPS model read

$$\frac{\lambda^2 \sin^2 \xi}{(1+|u|^2)^4} \partial_\mu \left(\sin^2 \xi\ H^\mu\right) + \mu^2 V'_\xi = 0\,,$$

$$H_\mu = \frac{\partial}{\partial \xi^\mu} \left(\epsilon^{\alpha\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma\right)^2\,, \tag{18}$$

$$\partial_\mu \left(\frac{K^\mu}{(1+|u|^2)^2}\right) = 0\,. \tag{19}$$

As we are interested in static topologically nontrivial solutions, u must cover the whole complex plane ($\vec{n}$ covers at least once $\mathbb{S}^2$) and $\xi \in [0, \pi]$. Then the natural ansatz is an axially symmetric generalization of the hedgehog configuration

$$\xi = \xi(r)\,, \qquad u(\theta, \phi) = g(\theta) e^{in\phi}\,, \tag{20}$$

where (r, θ, ϕ) are spherical coordinates. Then, the field equation for u reads

$$\frac{1}{\sin\theta}\partial_\theta \left(\frac{g^2 g_\theta}{(1+g^2)^2 \sin\theta} \right) - \frac{g g_\theta^2}{(1+g^2)^2 \sin^2\theta} = 0\,,$$

and the solution with the right boundary conditions is

$$g(\theta) = \tan\frac{\theta}{2}\,. \tag{21}$$

Observe that this solution holds for all values of n. The equation for the real scalar field is

$$\frac{n^2\lambda^2 \sin^2\xi}{2r^2} \partial_r \left(\frac{\sin^2\xi\ \xi_r}{r^2} \right) - \mu^2 V_\xi = 0\,.$$

This equation can be simplified by introducing the new variable

$$z = \frac{\sqrt{2}\mu r^3}{3|n|\lambda}\,. \tag{22}$$

It reads

$$\sin^2\xi\ \partial_z \left(\sin^2\xi\ \xi_z \right) - V_\xi = 0 \tag{23}$$

and may be integrated to

$$\tfrac{1}{2} \sin^4\xi\ \xi_z^2 = V\,, \tag{24}$$

where we chose a vanishing integration constant to get finite energy solutions. We remark that this first integration of the field equation is equivalent to a Bogomolny equation and, thus, to a Bogomolny bound for the dimensionally reduced, effectively one-dimensional problem. It is easy to see that this equation can be derived from the full Bogomolny equation without any symmetry reduction, once the ansatz is inserted.

2.5. Qualitative results

In terms of the variable r the integrated (Bogomolny) equation reads

$$\frac{n^2\lambda^2}{4\mu^2 r^4} \sin^4\xi \xi_r^2 = V \tag{25}$$

and it is this form which will be useful for the discussion of the energy to be performed next. Indeed, the energy is

$$E = \int d^3x \left(-\frac{\lambda^2 \sin^4 \xi}{(1+|u|^2)^4} (\nabla_r \xi)^2 (\nabla_\theta u \nabla_\phi \bar{u} - \nabla_\phi u \nabla_\theta \bar{u})^2 + \mu^2 V \right) , \quad (26)$$

or, after inserting the hedgehog ansatz with the solution (21) for u,

$$E = 4\pi \int r^2 dr \left(\frac{\lambda^2 n^2 \sin^4 \xi}{4r^4} \xi_r^2 + \mu^2 V \right) . \quad (27)$$

It follows from the Bogomolny equation for r, (25), that the sextic term and the potential contribute the same amount to the energy density for arbitrary values of r. Therefore, we may further simplify the expression for the energy like

$$E = 4\pi \, 2\mu^2 \int r^2 dr V(\xi(r)) = 4\sqrt{2}\pi\mu\lambda|n| \int dz V(\xi(z)) . \quad (28)$$

Further, we may already draw some qualitative conclusions about the behavior of the energy density profiles for different types of potentials. Finiteness of the energy requires that the fields take values in the vacuum manifold of the potential V in the limit $r \to \infty$. For the class of potentials $V = V(\xi)$ we consider this just means that $\lim_{r\to\infty} V(\xi(r)) = 0$. Further, the topology of skyrmion fields requires that the matrix field U takes a constant, direction-independent value in the limit $r \to \infty$. Within the hedgehog ansatz this implies that the field ξ must take one of its two boundary values $\xi = 0, \pi$ in this limit. For skyrmions with finite energy, therefore, at least one of these two boundary values must belong to the vacuum manifold of the potential. Without loss of generality, let us assume that ξ takes the value $\xi = 0$ in the limit $r \to \infty$. For a wide class of potentials this implies that ξ must take the opposite boundary value $\xi = \pi$ at $r = 0$, because it follows easily from (25) that ξ is a monotonous function of r in the region, where $V \neq 0$. These observations lead to the following conclusions. For one-vacuum potentials with the only vacuum at $\xi = 0$, the energy density cannot be zero inside the skyrmion. If, in addition, the potential is a monotonous function of ξ in the range of ξ, then the energy density is a monotonous function of r and takes its maximum value at $r = 0$, *i.e.*, the soliton is of the core type. If the potential has the two vacua $\xi = 0, \pi$, then the energy density is zero also at $r = 0$, and the soliton is of the shell type. For more complicated vacuum manifolds of V, more complicated soliton structures emerge, but they may still be found by a variant of the simple qualitative reasoning applied in this paragraph. We remark that a qualitatively similar relation between the vacuum manifold of the potential and the skyrmion structure also is observed in the original Skyrme model with a potential. The difference is that in the latter case this relation is the result of complicated, three-dimensional numerical integrations, whereas in our case it follows from some simple, analytical arguments.

2.6. The Skyrme potential

The first obvious possibility is to consider the standard Skyrme potential

$$V = \tfrac{1}{2}\mathrm{Tr}(1-U) \quad \rightarrow \quad V(\xi) = 1-\cos\xi\,. \tag{29}$$

Imposing the boundary conditions for topologically non-trivial solutions we get

$$\xi = \begin{cases} 2\arccos\sqrt[3]{\frac{3z}{4}} & z\in\left[0,\frac{4}{3}\right]\,, \\ 0 & z\geq \frac{4}{3}\,. \end{cases} \tag{30}$$

The corresponding energy is

$$E \;=\; 8\sqrt{2}\pi\mu\lambda|n|\int\limits_0^{4/3}\left(1-\left(\frac{3z}{4}\right)^{\frac{2}{3}}\right)dz = \frac{64\sqrt{2}\pi}{15}\mu\lambda|n|\,. \tag{31}$$

The solution is of the compacton type, *i.e.*, it has a finite support [39] (compact solutions of a similar type in different versions of the baby Skyrme models have been found in [31, 40]). The function ξ is continuous but its first derivative is not. The jump of the derivative is, in fact, infinite at the compacton boundary $z=4/3$, as the left derivative at this point tends to minus infinity. Nevertheless, the energy density and the topological charge density (baryon number density) are continuous functions at the compacton boundary, and the field equation (23) is well-defined there. The reason is that ξ_z always appears in the combination $\sin^2\xi\,\xi_z$, and this expression is finite (in fact, zero) at the compacton boundary. We could make the discontinuity disappear altogether by introducing a new variable $\tilde{\xi}$ instead of ξ which satisfies

$$\tilde{\xi}_z = \sin^2\xi\,\xi_z\,.$$

We prefer to work with ξ just because this is the standard variable in the Skyrme model.

In order to extract the energy density it is useful to rewrite the energy with the help of the rescaled radial coordinate

$$\tilde{r} = \left(\frac{\sqrt{2}\mu}{4\lambda}\right)^{\frac{1}{3}} r \equiv \frac{r}{R_0} = \left(\frac{3|n|z}{4}\right)^{\frac{1}{3}} \tag{32}$$

(here R_0 is the compacton radius) like

$$E = 8\sqrt{2}\mu\lambda \left(4\pi \int_0^{|n|^{\frac{1}{3}}} d\tilde{r}\tilde{r}^2 (1 - |n|^{-\frac{2}{3}}\tilde{r}^2) \right)$$

such that the energy density per unit volume (with the unit of length set by $\tilde{r}$) is

$$\begin{aligned} \mathcal{E} &= 8\sqrt{2}\mu\lambda \left(1 - |n|^{-\frac{2}{3}}\tilde{r}^2\right) \qquad \text{for} \qquad 0 \leq \tilde{r} \leq |n|^{\frac{1}{3}} \\ &= 0 \qquad \text{for} \qquad \tilde{r} > |n|^{\frac{1}{3}} \,. \end{aligned} \tag{33}$$

$\tilde{r}$ does not depend on the topological charge $B = n$, so the dependence of $\mathcal{E}$ on n is explicit.

In the same fashion we get for the topological charge (baryon number), see *e.g.* Chapter 1.4 of [37]

$$\begin{aligned} B &= -\frac{1}{\pi^2} \int d^3x \frac{\sin^2 \xi}{(1+|u|^2)^2} i\epsilon^{mnl} \xi_m u_n \bar{u}_l = -\frac{2n}{\pi} \int dr \sin^2 \xi \, \xi_r \\ &= -\frac{2n}{\pi} \int dz \sin^2 \xi \, \xi_z = \frac{4n}{\pi} \int_0^{\frac{4}{3}} dz \left(1 - \left(\frac{3}{4}\right)^{\frac{2}{3}} z^{\frac{2}{3}}\right)^{\frac{1}{2}} \\ &= \operatorname{sign}(n) \frac{4}{\pi^2} \left(4\pi \int_0^{|n|^{\frac{1}{3}}} d\tilde{r}\tilde{r}^2 \left(1 - |n|^{-\frac{2}{3}}\tilde{r}^2\right)^{\frac{1}{2}} \right) = n \end{aligned} \tag{34}$$

and for the topological charge density per unit volume

$$\begin{aligned} \mathcal{B} &= \operatorname{sign}(n) \frac{4}{\pi^2} \left(1 - |n|^{-\frac{2}{3}}\tilde{r}^2\right)^{\frac{1}{2}} \qquad \text{for} \qquad 0 \leq \tilde{r} \leq |n|^{\frac{1}{3}} \\ &= 0 \qquad \text{for} \qquad \tilde{r} > |n|^{\frac{1}{3}} \,. \end{aligned} \tag{35}$$

Both densities are zero outside the compacton radius $\tilde{r} = |n|^{\frac{1}{3}}$. We remark that the values of the densities at the center $\tilde{r} = 0$ are independent of the topological charge $B = n$, whereas the radii grow like $n^{\frac{1}{3}}$. For $n = 1$, we plot the two densities in Fig. 1, where we normalize both densities (*i.e.*, multiply them by a constant) such that their value at the center is one.

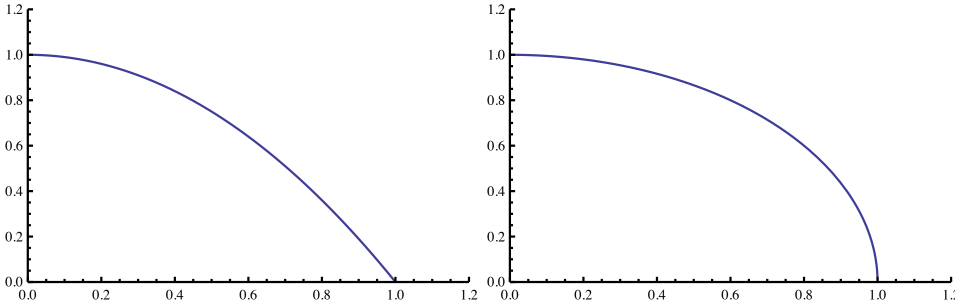

Fig. 1. Normalized energy density (left figure) and topological charge density (right figure) as a function of the rescaled radius $\tilde{r}$, for topological charge $n = 1$. For $|n| > 1$, the height of the densities remains the same, whereas their radius grows like $|n|^{\frac{1}{3}}$.

3. Classical aspects

After having discussed the properties of the Lagrangian and its classical solutions in the preceding sections, let us now try to apply it to the description of some properties of nuclei. After all, this possibility is one of the rationales for the original Skyrme model and its generalizations. We are, of course, far from considering this model as the correct effective model of QCD, but one may wonder whether solitons of this integrable model can be related in one way or another to some properties of real baryons. Here we shall first focus on the classical theory and solutions, and we will find that at this level the model already reproduces quite well some properties of the nuclear drop model. In a next step, we perform the semi-classical quantization of the (iso)rotational degrees of freedom of the $B = 1$ soliton, *i.e.*, the nucleon. Further, we choose the standard Skyrme potential $V = 1 - \cos\xi$ for simplicity throughout this and the next section.

We find immediately that the classical solutions of the BPS Skyrme model seem to describe surprisingly well some static properties of nuclei. As was discussed already in [22], it provides an alternative starting point for an effective soliton model of baryons, which by construction is much more topological in nature.

3.1. Mass spectrum and linear energy-charge relation

As a consequence of the BPS nature of the classical solutions, the energy of the solitons is proportional to the topological (baryon) charge

$$E = E_0|B|\,,$$

where $E_0 = 64\sqrt{2}\pi\mu\lambda/15$. Such a linear dependence is a well established fact in nuclear physics. For the moment (*i.e.*, in the context of the purely classical reasoning), let us fix the energy scale by assuming that $E_0 = 931.75$ MeV. This is equivalent to the assumption that the mass of the solution with $B = 4$ is equal to the mass of He^4. One usually assumes this value because the ground state of He^4 has zero spin and isospin [17]. Therefore, possible corrections to the mass from spin–isospin interactions are absent. In Table I we compare energies of the solitons in the BPS model with experimental values and energies obtained in the vector-Skyrme [13] and standard massive Skyrme model [16]. (We use the numerical data, if accessible, or calculate them from fitted functions [16]. The energy scale is set by the same prescription.) It is interesting to note that instead of the approximate 7% accuracy typical for the soliton energies of standard Skyrme theories we get maximally only a 0.7% discrepancy. Besides, the masses of the BPS Skyrme model solitons are slightly smaller than the experimental masses in *all* cases (except for the He^4 used for the fit, of course). This goes into the right direction, because the (iso)rotational excitation energies should be added to the classical soliton masses (except for the He^4, of course) for a more reliable comparison with physical masses of nuclei.

TABLE I

Energies of the solutions in the BPS Skyrme model, compared with masses for the vector-Skyrme and Skyrme model, as well as with the experimental date. All numbers are in MeV.

B	E_{BPS}	$E_{\mathrm{vec-Skyrme}}$	E_{Skyrme}	$E_{\mathrm{experiment}}$
1	931.75	996	1024	939
2	1863.5	1999	1937	1876
3	2795.25	2913	2836	2809
4	3727	3727	3727	3727
6	5590.5	—	5520	5601
8	7454	—	7327	7455
10	9317.5	—	9113	9327

3.2. No binding energy

It follows from the BPS nature of the model that the binding energy is zero. This is different from the standard Skyrme model, where binding energies are rather big. For example, the energy of the baryon number two skyrmion exceeds the topological energy bound by 23%. Of course, such binding energies are significantly larger than experimentally observed, which usually do not reach 1%. Therefore, as pointed out by Sutcliffe [18], a BPS Skyrme theory seems to be a better starting point to get realistic binding energies. Small (non-zero) binding energies could be produced by small perturbations around a BPS theory [41].

3.3. Size of nuclei and compactons

Due to the compact nature of the solitons, their radius is well defined and can be easily computed

$$R_B = R_0 \sqrt[3]{B}\,, \qquad R_0 = \left(\frac{2\sqrt{2}\lambda}{\mu}\right)^{\frac{1}{3}}\,,$$

which again reproduces the well-known experimental relation. The numerical value which best reproduces the known radii of nuclei is approximately $R_0 = 1.25$ fm.

Further, the compact nature of our solutions probably can be viewed as an advantage of the model rather than a defect. In fact, the absence of interactions (or, more precisely, of *finite range interactions*) corresponds quite well with the very short range of forces between nuclei.

3.4. Core type energy density

For the ansatz of Section 2.4, which provides spherically symmetric energy densities for all baryon numbers, the resulting energy density takes its maximum value at the origin. It is of some interest to compare this result with the densities in the standard massless or massive Skyrme models. For the massless Skyrme model, solitons are geometrically complicated shell-like structures with empty space regions inside [15]. In addition, the size of the shell-skyrmions grows like $\sqrt{B}$, which is in contradiction to the experimental data. In the case of the massive Skyrme model, the situation is slightly more subtle [16–18]. The proper size-charge relation has been reported [16]. Moreover, depending on the mass of the pion field and baryon number, squeezed clustered solutions, instead of shell ones, begin to be preferred. Precisely speaking, for a fixed value of the mass parameter, the first few skyrmions possess a shell-like structure, whereas for higher baryon charge a clustered solution is the true minimum. The critical charge, for which shell-skyrmion occurs seems to be smaller if the mass is increased [17]. However, even for the physically acceptable value $m = 1$ (which is more or less twice the bare pion mass), skyrmions with $B \leq 9$ are shells. In the modern interpretation this problem can be cured by treating the massive parameter as a renormalized pion mass which should be adjusted to best reproduce observed data [17, 42]. Then, increasing m one gets rid of unwanted shell solutions, leaving only clustering ones.

Let us also notice that there is a reminiscence of this clustering phenomenon in the BPS Skyrme model, even though it is quite trivial. Namely, due to the compact and BPS nature of the solutions of the BPS Skyrme model, it is possible to construct a collection of separated components provided they are sufficiently separated (they do not touch each other). Such a clustered configuration has a total baryon number equal to the sum of the components. In the BPS Skyrme model, none of these clustered (multi-center) solutions is energetically preferred, which again is a simple outcome of the BPS origin of the solitons.

Finally, the values of the energy and charge densities of the solutions of Section 2.6 at the center do not depend on the baryon number, which, again, is a property which holds reasonably well for physical nuclei.

3.5. The liquid drop property

The energy functional for static field configurations has the volume-preserving diffeomorphisms on the three-dimensional base space as symmetries. In physical terms, all deformations of solitons which correspond to these volume-preserving diffeomorphisms may be performed without any cost in energy. But these deformations are exactly the allowed deformations for an ideal, incompressible droplet of liquid when surface contributions to the energy are neglected. These symmetries are not symmetries of a physical nucleus. A physical nucleus has a definite shape, and deformations which change this shape cost energy. Nevertheless, deformations which respect the local volume conservation (*i.e.*, deformations of an ideal incompressible liquid) cost much less energy than volume-changing deformations, as an immediate consequence of the liquid drop model of nuclear matter. This last observation also further explains the nature of the approximation our model provides for physical nuclei. It reproduces some of the classical features of the nuclear liquid drop model at least on a qualitative level, and the huge amount of symmetries of the model is crucial for this fact. Its soliton energies, *e.g.*, correspond to the bulk (volume) contribution of the liquid drop model, with the additional feature that the energies are quantized in terms of a topological charge.

This should be contrasted with the expected behavior for large baryon number for the standard Skyrme model. In the standard Skyrme model, there remain some long range forces between different Skyrmions, whose attractive or repulsive character depends on the relative orientation of the Skyrmions. As a consequence, it is expected that for large baryon number the energy-minimizing configurations are Skyrmion crystals, where all the Skyrmions are brought into the right positions and orientations to minimize the total energy. For physical nuclear matter, there is no sign of this crystal type behavior. Instead, nuclear matter seems to be in a liquid state, which is well described by our BPS Skyrme model.

3.6. Absence of pion fluctuations

In the model, both the quadratic and the quartic kinetic terms are absent. As a consequence, neither propagating pions nor the two-body interaction between pions can be described in the model. Nevertheless, already at the classical level the model seems to describe some nuclear properties reasonably well, which seems to indicate that in certain circumstances the sextic term could be more important than the terms $\mathcal{L}_2$ and $\mathcal{L}_4$. The quadratic term is kinetic in nature, whereas the quartic term provides, as a leading behavior, two-body interactions. On the other hand, the sextic term is essentially topological in nature, being the square of the topological current (baryon current). So in circumstances where our model is successful this seems to indicate that a *collective* (topological) contribution to the nucleus is more important than kinetic or two-body interaction contributions. This behavior is, in fact, not so surprising for a system at strong coupling (or for a strongly non-linear system).

Let us notice that, although the kinetic term for the pion fields is not included, we do have pions in the model. They enter in the definition of the meson field U and are partially taken into account in the potential term of the model. Therefore, one may think about the BPS Skyrme model as a very nontrivial realization of the concept of the *pionless* theory of baryons with *hidden* pions.

A first consequence of the absence of dynamical pions is the compact nature of the solutions, *i.e.*, the absence of the exponentially decaying pion cloud. A second consequence is the absence of linear pion radiation, and one may wonder whether there exists classical radiation at all in this model. The answer is probably yes, although the study of radiation is inherently nonlinear in compacton-supporting models of this type (the field equations remain nonlinear in the weak-field limit). The simplest way to find some indications of radiation is the study of rotating solitons. In the standard Skyrme model (with a nonzero pion mass), it is found that rotating solutions exist for not too large angular velocities but cease to exist if the angular velocity exceeds a certain limit. The reason for this behavior may be understood easily from the linearized weak-field analysis. If the corresponding angular frequency is too large (essentially larger than the pion mass), then the formal solution is oscillatory instead of exponentially decaying, and so has infinite energy. Physically this is interpreted as the onset of pion radiation at that frequency. So one may wonder what happens for rotating solitons in the BPS Skyrme model. Unfortunately, the field equations in this case can no longer be reduced to an ordinary differential equation. There exists, however, a baby Skyrme version of the BPS Skyrme model in one dimension lower, where the dimensional reduction of a rotating baby Skyrmion ansatz to an ODE is possible and has been performed in [31]. The result is as follows: the rotating baby Skyrmion solution exists and can be found exactly if the angular velocity remains below a certain critical value. It remains compact, and its radius even shrinks with the angular velocity (although the moment of inertia increases, as one would expect). For frequencies above the critical value, on the other hand, a solution does not exist. This may be viewed as an indication that radiation will set in also for a sufficiently fast rotating BPS Skyrmion, although we repeat that radiation for compactons is an inherently nonlinear and, therefore, complicated problem.

Summary. As announced previously, we found that the classical model already describes rather well some features of the liquid drop model of nuclei. These classical results are probably more trustworthy for not too small nuclei, because:

(i) The contribution of the pion cloud (which is absent in our model) to the size of the nucleus is of lesser significance for larger nuclei. We remind that in addition to the core of a nucleus (with a size which grows essentially with the third root of the baryon number) a surface term is known to exist for physical nuclei whose thickness is essentially independent of the baryon number.

(ii) The description of a nucleus as a liquid drop of nuclear matter is more appropriate for larger baryon number.

(iii) The contribution of (iso)rotational quantum excitations to the total mass of a nucleus is smaller for larger nuclei, essentially because of the larger moments of inertia of larger nuclei.

We will find further indication for this behavior in the Section 4.1, where a rigid rotor quantization of the (iso)rotational degrees of freedom is performed for the $B = 1$ nucleon. Indeed, as we shall see, both the corresponding (iso)rotational excitations and the (missing) pion cloud will be of some importance in this case.

4. Quantum aspects

4.1. $B = 1$ sector

Let us now discuss the issue of quantization of the BPS Skyrme model. As the model is rather unusual, not containing the quadratic, sigma model kinetic part, one might doubt whether the quantization procedure can be performed. However, the sextic derivative term used in the construction, the square of the pullback of the volume on the target space, is a very special one. It is the unique term with sextic derivatives which leads to a Lagrangian of second order in time derivatives. Therefore, we deal with a Hamiltonian of second order in time derivatives and the system can be quantized in the standard manner.

We want to perform the semiclassical quantization about a soliton solution in the same way it is performed for the standard Skyrme model. Let us recall that for the nonzero or vibrational modes, the semiclassical quantization consists in a quantization of the quadratic oscillations about the classical solution. These oscillations presumably just amount to renormalizations of the couplings of the theory and therefore may be taken into account implicitly by fitting the model parameters to their physical values. The zero mode fluctuations related to the symmetries, on the other hand, cannot be approximated by quadratic fluctuations and have to be treated by the method of collective coordinates. In principle, one collective coordinate has to be introduced for each symmetry transformation of the model which does not leave invariant the soliton about which the quantization is performed. Here, nevertheless, we only shall consider the collective coordinate quantization of the rotational and isorotational degrees of freedom. The physical reason for this restriction is, of course, the fact that the excitational spectra of nuclei are classified exactly by the corresponding quantum numbers of spin and isospin. A more formal justification of this restriction could be, for instance, that the additional collective coordinates do not provide discrete spectra of excitations but, instead, just renormalize the coupling constants, like the vibrational modes do. A definite answer to this question would require a more detailed investigation of the full moduli space of the theory, where all the infinitely many symmetries are taken into account. This is probably a very difficult problem which is beyond the scope of the present paper. A second justification consists in the assumption that, in any case, the given model is just an approximation, whereas a more detailed application to the properties of nuclei requires the inclusion of additional terms in the Lagrangian which, although being small in some sense, have the effect of breaking the symmetries down to the ones of the standard Skyrme model.

We start from the classical, static field configuration U_0 found in Section 2.6. For simplicity, we only consider the hedgehog configuration with baryon number $B = 1$. This configuration is invariant under a combined rotation in base and isotopic space, therefore, it is enough to introduce the collective coordinates of one of the two. Allowed excitational states will always have the corresponding quantum numbers of spin and isospin equal, as a consequence of the symmetries of the hedgehog. Following the standard treatment, we introduce the collective coordinates of the isospin by including a time-dependent iso-rotation of the classical soliton configuration

$$U(x) = A(t)U_0(x)A^\dagger(t)\,, \tag{36}$$

where $A(t) = a_0 + ia_i\tau_i \in \mathrm{SU}(2)$ and $a_0^2 + \vec{a}^2 = 1$. Inserting this expression into the Lagrangian, we get

$$L = -E_0 + \mathcal{I}\,\mathrm{Tr}\,\left[\partial_0 A^\dagger(t)\partial_0 A(t)\right]\,, \tag{37}$$

where the energy (mass) of the classical solution is

$$E_0 = \frac{64\sqrt{2}\pi}{15}\mu\lambda \tag{38}$$

and the moment of inertia is

$$\mathcal{I} = \frac{4\pi}{3}\lambda^2 \int\limits_0^\infty dr \left(\sin^4 \xi \xi_r'^2\right) = \frac{4\sqrt{2}\pi}{3}\sqrt{\lambda\mu}\left(\frac{3\lambda}{\sqrt{2}\mu}\right)^{2/3}\gamma\,, \tag{39}$$

where

$$\gamma = \int\limits_0^\infty dz \left(z^{2/3}\sin^4 \xi \xi_z'^2\right) = 4\int\limits_0^{4/3} z^{2/3}\left(1-\left(\frac{3z}{4}\right)^{2/3}\right) dz = \frac{32}{35}\left(\frac{4}{3}\right)^{2/3}.$$

Then, finally

$$\mathcal{I} = \frac{2^8\sqrt{2}\pi}{15\times 7}\lambda\mu\left(\frac{\lambda}{\mu}\right)^{2/3}. \tag{40}$$

Introducing the conjugate momenta π_n to the coordinates a_n on $\mathrm{SU}(2) \simeq \mathbb{S}^3$ we get the Hamiltonian

$$H = E_0 - \frac{1}{8\mathcal{I}}\sum_{n=0}^{3}\pi_n^2 = E_0 - \frac{\hbar^2}{8\mathcal{I}}\sum_{n=0}^{3}\frac{\partial^2}{\partial a_n^2}\,,$$

where the usual canonical quantization prescription $\pi_n \to -i\hbar\partial/\partial a_n$ has been performed. Finally we get

$$H = E_0 + \frac{\hbar^2 I^2}{2\mathcal{I}} = E_0 + \frac{\hbar^2 S^2}{2\mathcal{I}}\,,$$

where I^2 is the isospin operator (the spherical Laplacian on $\mathbb{S}^3$). We introduced $\hbar$ explicitly because later on we want to use units where $\hbar$ is different from one. Further, S^2 is the spin operator, and we took into account the equality of spin and isospin for the hedgehog. It is interesting to note that the isospin operator automatically allows for wave functions on $\mathbb{S}^3$ both for integer isospin (homogeneous polynomials of even degree) and half-odd integer isospin (homogeneous polynomials of odd degree).

The soliton with baryon number one is quantized as a fermion. Concretely, the nucleon has spin and isospin $1/2$, whereas the Δ resonance has spin and isospin $3/2$, so we find for their masses

$$M_N = E_0 + \frac{3\hbar^2}{8\mathcal{I}}\,, \qquad M_\Delta = E_0 + \frac{15\hbar^2}{8\mathcal{I}} \;\Rightarrow\; M_\Delta - M_N = \frac{3\hbar^2}{2\mathcal{I}} \tag{41}$$

which is exactly like the nucleon and delta mass splitting formula of the standard Skyrme model. The difference comes only from particular expressions for E_0 and I. These expressions may now be fitted to the physical masses of the nucleon ($M_N = 938.9$ MeV) and the Δ resonance ($M_\Delta = 1232$ MeV) which determines fitted values for the coupling constants. Concretely we get

$$\lambda\mu = 45.66 \text{ MeV}\,, \qquad \frac{\lambda}{\mu} = 0.2556 \text{ fm}^3\,,$$

where we used $\hbar = 197.3$ MeV fm. These may now be used to "predict" further physical quantities like, *e.g.* the charge radii of the nucleons. For this purpose, we need the linear (*i.e.*, per unit radius) isoscalar and isovector charge densities. These expressions have already been determined for a generalized Skyrme model including the sextic term in [12], so we just use their results in the appropriate limit.

For the isoscalar (baryon) charge density per unit r we find

$$\rho_0 = 4\pi r^2 B^0 = -\frac{2}{\pi}\sin^2\xi\xi_r' \tag{42}$$

and for the isovector charge density per unit r

$$\rho_1 = \frac{1}{\mathcal{I}}\frac{4\pi}{3}\lambda^2\sin^4\xi\xi_r'^2\,. \tag{43}$$

Then the electric charge densities for proton and neutron, $\rho_{E,p(n)} = \frac{1}{2}(\rho_0 \pm \rho_1)$ read

$$\rho_{e,p(n)} = \frac{2\sqrt{2}}{\pi}\frac{\mu}{\lambda}r^2\sqrt{1-\left(\frac{\mu}{2\sqrt{2}\lambda}\right)^{2/3}r^2}\left(1 \pm \frac{4\sqrt{2}\pi^2\lambda\mu r^2}{3\mathcal{I}}\sqrt{1-\left(\frac{\mu}{2\sqrt{2}\lambda}\right)^{2/3}r^2}\right). \tag{44}$$

The corresponding isoscalar and isovector mean square electric radii are

$$\left\langle r^2\right\rangle_{e,0} = \int drr^2\rho_0 = \left(\frac{\lambda}{\mu}\right)^{2/3}, \tag{45}$$

$$\left\langle r^2\right\rangle_{e,1} = \int drr^2\rho_1 = \frac{10}{9}\left(\frac{\lambda}{\mu}\right)^{2/3}. \tag{46}$$

Further, the isoscalar magnetic radius is defined as the ratio

$$\left\langle r^2\right\rangle_{m,0} = \frac{\int drr^4\rho_0}{\int drr^2\rho_0} = \frac{5}{4}\left(\frac{\lambda}{\mu}\right)^{2/3}. \tag{47}$$

With the numerically determined values of the coupling constants we find the values for the radii displayed in Table II.

TABLE II

Compacton radius and some charge radii and their ratios for the nucleon. The numbers for the massive Skyrme model are from Ref. [8]. All radii are in fm.

Radius	BPS Skyrme	Massive Skyrme	Experiment
Compacton	0.897	—	—
$r_{e,0}$	0.635	0.68	0.72
$r_{e,1}$	0.669	1.04	0.88
$r_{m,0}$	0.710	0.95	0.81
$r_{e,1}/r_{e,0}$	1.054	1.529	1.222
$r_{m,0}/r_{e,0}$	1.118	1.397	1.125
$r_{e,1}/r_{m,0}$	0.943	1.095	1.086

In relation to Table II, some comments are appropriate. Firstly, observe that in the BPS Skyrme model all radii are bound by the compacton radius $R_0 = \sqrt{2}(\lambda/\mu)^{(1/3)}$. This bound holds because all radii can be expressed as moments of densities $(\int drr^n \rho_i)^{(1/n)}$, where ρ_i is a density normalized to one. Secondly, all radii in the BPS Skyrme model are smaller than their physical values, as well as significantly smaller than the values predicted in the standard massive Skyrme model. This, however, has to be expected, because we know already that the pion cloud is absent in the BPS Skyrme model, and the densities strictly go to zero at the compacton radius. We also display the ratios of some radii for the following reason. If the deviations of the BPS Skyrme model radii from their physical values are mainly due to the same systematic error (the absence of the pion cloud in the model), then we expect that this systematic error should partly cancel in the ratios. This is precisely what happens. The errors in the radii themselves are of the order of 30%, whereas the errors in the ratios never exceed 15%, providing us with a nice consistency check for our interpretation of the model. The understanding of the origin of errors in the BPS Skyrme model is undoubtedly an advantage of this model, if compared with the standard Skyrme models, where quantitative errors seem to be distributed randomly without any deeper explanation.

Finally, let us display the numerical results for the magnetic moments of the proton and neutron. The corresponding expressions are

$$\mu_{p(n)} = 2M_N \left(\frac{1}{12\mathcal{I}} \left\langle r^2 \right\rangle_{e,0} + (-) \frac{\mathcal{I}}{6\hbar^2} \right) , \tag{48}$$

and the resulting numerical values are given in Table III.

TABLE III

Proton and neutron magnetic moments. The numbers for the massive Skyrme model are from Ref. [8].

	BPS Skyrme	Massive Skyrme	Experiment
μ_p	1.918	1.97	2.79
μ_n	-1.285	-1.24	-1.91
$\lvert\mu_p/\mu_e\rvert$	1.493	1.59	1.46

The quality of the values is comparable to the case of the standard massive Skyrme model, so the absence of the pion cloud apparently does not have such a strong effect on the magnetic moments.

4.2. $B > 1$ sector

For the standard Skyrme theory and its generalizations, higher Skyrmions have rather complicated discrete symmetries and are known only in numerical form, so their quantization is a rather complicated procedure. Nevertheless, recently the rotational and isorotational excitations of the rigid rotor quantization of the solitons of the standard massive Skyrme model have been applied quite successfully to the corresponding spectra of excitations of light nuclei [18]. As the solutions in the standard Skyrme model are sometimes quite different from ours, one might think that this fact casts some serious doubts on the applicability of our model to the phenomenology of nuclei. Here we just want to point out that this does not have to be the case. In fact, the information which is most important for the spectra of excitations consists in the *symmetries* of the solitons, and not in the full dynamical contents of the soliton solutions. These symmetries determine the Finkelstein–Rubinstein constraints on the allowed excited states and, therefore, the spectra of excitations for each baryon number. Further, the solutions in our model typically have higher symmetry due to the special properties of this model.

As a consequence, the following picture is quite plausible. Our model as it stands already describes quite well some bulk properties of nuclei like masses and charge and energy densities. A more detailed description does require the addition of further terms, but these will be small in some sense (*e.g.* their contribution to the total mass is small). On the other hand, these additional terms will break the symmetries of the resulting soliton solutions, and the resulting solutions probably have the symmetries of the standard Skyrme model, and, consequently, their spectra of excitations. If this symmetry breaking is small, then the spectral lines should still show some approximate degeneracy, that is, some spectral lines should be spaced more narrowly than others. A detailed investigation of this issue is beyond the scope of the present paper and will be presented in future publications. Of course, in the simplest baryon number one case (the hedgehog), the symmetries and the excitational spectra coincide.

The first step in the derivation of spectra of nuclei with higher baryon numbers has been recently done by Bonenfant and Marleau [43]. The authors investigate a version of the BPS Skyrme model with a different potential. They first calculate the exact static soliton solutions plus the (iso)rotational energies in the rigid rotor quantization for the general baryon charge $B = n$ for the spherical symmetric ansatz and observe the appearance of a non-zero binding energy. It may be explained by the fact that the semiclassical quantization takes into account time dependence of the static solutions and therefore breaks the volume preserving diffeomorphism symmetry of the static energy functional. The obtained binding energies agree to some extent with the experimental data: one gets smaller values for light nuclei with a saturation around $B = 7$. For higher nuclei the binding energies per nucleon are almost the same forming a plateau. A significantly better agreement is achieved if one allows for small contributions to the total energy from the quadratic and quartic Skyrme terms treated perturbatively (explicitly broken VPD symmetry). Then, they fit the resulting binding energies to the experimental binding energies of the most abundant isotopes of higher nuclei, assuming, as is usually done, that these correspond to the states with the lowest value of the isospin. The resulting agreement between calculated and experimentally determined masses and binding energies is impressive, lending further support to the viability of the BPS Skyrme model as the leading contribution to an effective theory for the properties of nuclear matter.

5. Issue of $N_c \to \infty$

First, let us emphasize again the possible relevance of the BPS Skyrme model in the limit of a large number of colors N_c of the underlying QCD-type theory. Indeed, as was pointed out, *e.g.*, in [19], some problems of the standard Skyrme model when applied to QCD-like theories become more severe in the large N_c limit. For instance, in the Skyrme model rather strong forces of the order of N_c are generated between nuclei, and the ground state of sufficiently high baryon number tends to be a Skyrmion crystal with binding energies again of the order of N_c. Both of these findings are in conflict with lattice simulations and with known properties of physical nuclei, respectively. Let us, therefore, discuss properties of the BPS model from a large N_c perspective. First of all, three qualitative results are N_c independent, *i.e.*, they are not sensitive to how N_c enters the parameters of the model and are observed for any finite as well as infinite value of N_c. Namely, *(i)* BPS nature of solitons and linear energy-baryon number relation, which leads to zero classical binding energies,

(ii) compact baryons and contact interaction, *i.e.*, no medium or long range forces,

(iii) incompressible liquid property.

Thus, both above mentioned problems are absent in the BPS Skyrme model (there are no long-range forces and no binding energies) although in a rather radical way. Here, two comments are appropriate.

Firstly, in the large N_c expansion the meson–meson couplings are of the order of $1/N_c$. Hence, mesons become free and noninteracting at $N_c = \infty$. From this perspective, the BPS Skyrme model (at $N_c = \infty$) provides an acceptable, although again radical, result. That is to say, it not only holds that mesons do not interact; they disappear completely from the particle spectrum, their only remnants being some collective (solitonic) excitations, and the chiral symmetry breaking aspects of pion dynamics taken into account by the potential. This fact is crucial to cure the unwanted strong forces at intermediate range in the Skyrme model at large N_c.

Secondly, in QCD at $N_c = \infty$ the instanton liquid becomes incompressible, as well. Whether this appearance of an incompressible liquid at large N_c, both in the BPS Skyrme model and in the instanton liquid, is more than a mere coincidence remains to be seen.

Further, assuming that both terms in the BPS model depend linearly on N_c, *i.e.*, $\mu^2, \lambda^2 \sim N_c$ (which is a natural assumption for Skyrme-type models), some additional remarks can be made. Classical energies of solutions are proportional to the number of colors, $E \sim N_c$, while their radii are N_c independent, $R \sim N_c^0$. Moreover, this does not change after taking into account the semiclassical quantization corrections, that is, $\langle r^2 \rangle \sim N_c^0$. As the moment of inertia scales as $I \sim N_c$, we get the following relations for the nucleon-Delta mass splitting d and binding energies δE

$$d = M_N - M_\Delta \sim N_c^{-1} \qquad \text{and} \qquad \delta E \sim N_c^{-1} \,.$$

Thus, one observes a degeneracy in the $B = 1$ spectrum and vanishing quantum binding energies at $N_c = \infty$.

So one might speculate that the BPS Skyrme model provides more accurate results as an effective field theory in the large N_c limit. Unfortunately, however, there is no obvious large N_c limit which would produce just the BPS Skyrme model as its leading order, so the rather good large N_c properties of the model must be due to some more subtle mechanism. A better theoretical understanding of the conditions under which the BPS Skyrme theory provides a reasonable limit as an effective field theory for large N_c

QCD-like theories would be highly desirable. For instance, the two terms might be enhanced by two different physical mechanisms, where the sextic term is related to some collective or topological excitations, whereas the potential is related, *e.g.*, to the chiral quark condensate of QCD.

6. Perspectives

We believe that we have identified and solved an important submodel in the space of Skyrme-type effective field theories, which is singled out both by its capacity to reproduce qualitative properties of the liquid drop approximation of nuclei and by its unique mathematical structure. The model directly relates the nuclear mass to the topological charge, and it naturally provides a finite size of the nuclei, as well as the liquid drop behavior, which probably is not easy to get from an effective field theory. So our model solves a conceptual problem by explicitly deriving said properties from a (simple and solvable) effective field theory and may be a good starting point for an effective field theory description of nuclei. For a further development of this application of the model, however, the following problems have to be resolved or further investigated.

(i) Symmetry breaking: the infinitely many symmetries of the model are not shared by physical nuclei. In addition, it is not clear how to select the correct soliton from the infinitely many ones related by symmetries or how to quantize these infinitely many symmetries. Therefore, a realistic phenomenological application will require the breaking of these symmetries. The challenge will be to identify a breaking mechanism which effectively breaks the unwanted symmetries without perturbing too much the good properties of the model (like weak binding energies, weak internuclear forces, *etc.*).

(ii) Quantization of higher nuclei: the semiclassical quantization of higher solitons should be performed and applied to higher nuclei, along the lines of what was done for the standard Skyrme model. Investigations of the semiclassical quantization with the inclusion of higher excitations of nuclei and electrostatic effects, among others, is the next necessary step.

(iii) Motivation from QCD: it would be very interesting to see whether the rather good phenomenological properties of the model can be justified in a more rigorous manner from the fundamental theory of strong interactions, *i.e.*, QDC. We want to emphasize that the two terms of the BPS Skyrme model are rather specific. The sextic term is essentially topological in nature and is, therefore, naturally related to collective

excitations of the underlying microscopic degrees of freedom. The potential, on the other hand, provides the chiral symmetry breaking and might therefore be related to collective degrees of freedom of the quarks, like the quark condensate. A related issue is the behavior for a large number of colors N_c. Indeed, the good phenomenological properties of the BPS Skyrme model seem to further improve in the limit of large N_c, but a deeper understanding of this fact is still missing.

C.A. and J.S.-G. thank the Ministry of Science and Investigation, Spain (grant FPA2008-01177), and the Xunta de Galicia (grant INCITE09.296.035 PR and Conselleria de Educacion) for financial support. Further, A.W. thanks M.A. Nowak and L. McLerran for interesting discussions.

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