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PHASE SYNCHRONISATION IN MUTUALLY COUPLED CHAOTIC JOSEPHSON JUNCTIONS: EFFECT OF ASYMMETRY AND INCOMMENSURATE FREQUENCIES

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In this paper, synchronising two coupled ratchet Josephson junctions subjected to a quasiperiodic field is achieved. In the limit of weak perturbation of irrational frequencies equal to the square root of the transcendental number π and for small damping parameters, phase locking occurs as the coupling between both junctions is increased. It turns out that the transition from non-synchronous to synchronous chaotic state does not involve attractors appearing and disappearing. The undertaken symmetry analysis of the system demonstrates the suppression of the massive phase fluctuations as the coupling rises, allowing chaos synchronisation between both junctions to take place. The calculations also reveal the persistence of the synchronous state for high coupling strengths, taking into consideration the symmetry particularity of the external drive and potential.

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1. Introduction

Since the seminal work of Pecora and Carroll [1], which was based originally on earlier publication [2] by Fujisaka and Yamada in 1983, investigating the synchronisation of chaotic systems evolving from dissimilar initial conditions has been rising. Chaos synchronisation has enticed researchers due to its potential applications in the field of communications and cryptography [3–5]. The phenomenon occurs between two distinct chaotic oscillators once they are coupled, that both continuously preserve their temporal oscillations in step with each other. In spite of the sensitivity to initial conditions, and complexity of the attractors, synchronisation can be achieved in such systems, though with different methods of control and stability. Achieving a stable synchronous state between nonidentical chaotic systems has proven more arduous than between identical ones. Amongst the numerous procedures developed to fulfil a controlled and optimum synchronisation in both cases are the active control [6], sliding mode control [7], backstepping design [8], and non-linear control [9], to name a few. For most of the practical applications like the encryption of digital messages, synchronising non-identical chaotic systems has become of utmost demand. This issue has been inadequately handled, particularly from the parameter mismatch and incompatible dynamical structures perspectives [10]. Amid the few techniques that have been used to synchronise different chaotic systems is the *master–slave* approach, for which one particular system is the *master*, while the other is the *slave* [1, 7]. A nonlinear controller receiving signals from the master and slave systems is included, in order to achieve synchronisation between both via the one-way coupling method. On the other hand, studies on achieving synchronisation in the so-called ratchets have been lately growing. The significance lies in extracting a usable work from unbiased thermal fluctuations, which is dubbed ratchet [11, 12]. An increasing interest of such effect in nonlinear systems has been intensifying for technological applications, particularly at the micro and nano scales [13]. In late publications we have examined the behaviour of chaotic Josephson junctions in a ratchet potential [14,15], and investigated the effect of the incommensurable quasiperiodicity and asymmetry on the transport properties. A bifurcation from periodicity to chaos and adversely in the dynamical and directed transport characteristics has been reported, whenever the junction was acted upon by an external time-dependent force of dual frequency and zero average. In a recent paper, Vincent *et al.* [16] showed the existence of phase synchronisation in bidirectionally coupled deterministic ratchets, via simulating a particle driven by an external force of single frequency under the influence of an asymmetric potential. They illustrated the possibility of a transition from a regime of which the phases rotate with different velocities to a synchronous state, where the phase difference does not increase with time as the coupling strength is raised.

In the present paper, we extend Vincent *et al.* [16] work and embark on the two coupled rotators model proposed by Osipov *et al.* [17] to explore the phase synchronisation in underdamped ratchet Josephson junctions, driven by a quasiperiodic excitation of dual frequency in the absence of noise. We demonstrate the existence of phase synchronisation between the mutually coupled junctions, and discuss the sustainability of the synchronous state and its effect on the directed transport using the coupled rotators model. Taking into account the symmetry properties of the system and parameters like the incommensurate frequencies of the force as well as the junctions' damping factors, the dynamical behaviour is accordingly investigated.

2. The chaotic ratchet coupled junctions model

The phase θ of an underdamped Josephson junction can be described by the following nonlinear differential equation (see for more details Ref. [14]):

$$\omega_p^{-2}\frac{d^2\theta}{dt^2} + \omega_c^{-1}\frac{d\theta}{dt} + \frac{dU(\theta)}{d\theta} = F(t)\,, \tag{1}$$

where ω_p, ω_c are the plasma and characteristic frequency respectively, and linked to the junction parameters, such as the critical current, shunt resistance and capacitance. The third term on the left in Eq. (1) represents the derivative of the potential $U(\theta)$ ascribed to the Josephson tunnelling fluxons, which is considered asymmetric of the ratchet type. $F(t)$ is the external time-dependent excitation that is normally taken in the form of current. After having performed a proper normalisation [14], Eq. (1) could be assimilated, in the absence of any noise components, to the inertial equation of a particle moving in one dimension under the influence of a quasiperiodic external driving in a ratchet potential, such that

$$\ddot{x} + \alpha\dot{x} + \frac{dU(x)}{dx} = a_1\cos(\omega_1 t) + a_2\cos(\omega_2 t)\,, \tag{2}$$

where α is the damping parameter, a_1, a_2 are the amplitudes of the quasiperiodic excitation,ω_1, ω_2 are its frequencies, normalised by a small characteristic frequency ω_0 of the system. The ratchet potential (see Fig. 1) is given by [18]

$$U(x) = C - \frac{1}{4\pi^2\delta}\left[\sin(2\pi\chi) + \frac{1}{4}\sin(4\pi\chi)\right]\,,$$

where $\chi = x - x_0$ is a quantity that locates the minimum of the potential at the origin due to x_0 shift; C, δ are constants set to 0.0173 and 1.6, respectively. Since Eq. (2) is inhomogeneously differential, depending explicitly on time and also nonlinear, periodic and chaotic dynamics occurring in three-dimensional phase space are anticipated. Moreover, since we consider a bi-frequency quasiperiodic force, the incommensurability of frequencies is potentially influential on the system dynamical evolution. Since we can set $\omega_1/\omega_2 = n$, for which n can be rational or irrational depending on $\omega_{1,2}$ values, we can reformulate Eq. (2) as a three-dimensional system in (x, y, z), so that it yields

$$\begin{aligned}
\dot{x} &= y\,,\\
\dot{y} &= a_1\cos nz + a_2\cos z - \alpha y - \frac{dU(x)}{dx}\,,\\
\dot{z} &= \omega_2\,.
\end{aligned} \tag{3}$$

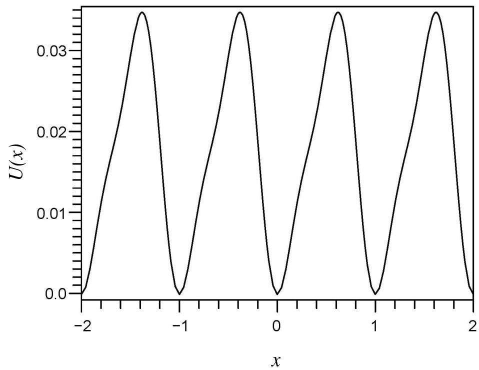

Fig. 1. The dimensionless ratchet potential for $C = 0.0173$, $\delta = 1.6$.

Eq. (3) is autonomous but not stiff, and can be solved using numerical methods, such as the Livermore Stiff Ordinary Differential Equations (LSODE) algorithm [19], which includes an improved dynamic step size. This numerical integrator provides rapid and higher accuracy of solutions for stiff and non-stiff equations via invoking the option "stiff = false or true".

On the other hand, the two coupled rotators model [18], allows writing the following differential equations

$$\ddot{\theta}_1 + \beta_1\dot{\theta}_1 + f_1(\theta_1) = F_1(t) + \varepsilon\left(\dot{\theta}_2 - \dot{\theta}_1\right), \tag{4}$$

$$\ddot{\theta}_2 + \beta_2\dot{\theta}_2 + f_2(\theta_2) = F_2(t) + \varepsilon\left(\dot{\theta}_1 - \dot{\theta}_2\right). \tag{5}$$

Here $f_{1,2}$ are 2π-periodic functions of the system, $\theta_{1,2}$ are phases of the rotators, and ε represents the coupling parameter, which controls the strength of feedback into the system. The system is driven by the time-dependent forces $F_{1,2}(t)$, where $\beta_{1,2}$ are damping constants. The model applies to the resistively coupled Josephson junctions, which are acted upon by the external currents having DC and AC components [20]. Hence for the bi-directionally coupled chaotic ratchet junction (as $\omega_1/\omega_2 = n$), Eqs. (4) and (5) can be expressed in the form

$$\ddot{x}_1 + \alpha_1\dot{x}_1 + \frac{dU(x_1)}{dx_1} = a_1\cos(\omega_1 t) + a_2\cos(\omega_2 t) + \varepsilon\left(\dot{x}_2 - \dot{x}_1\right), \tag{6}$$

$$\ddot{x}_2 + \alpha_2\dot{x}_2 + \frac{dU(x_2)}{dx_2} = a_3\cos(\omega_3 t) + a_4\cos(\omega_4 t) + \varepsilon\left(\dot{x}_1 - \dot{x}_2\right). \tag{7}$$

In Eq. (7), a_3 and a_4 are the amplitudes of the time-dependent force driving the second junction, and ω_3, ω_4 are its frequencies. We also require $\omega_3/\omega_4 = m$, so that their ratio m can be rational or irrational, according to $\omega_{3,4}$ values. The coupled equations (6) and (7) are expressed similarly to (3) and solved using the LSODE method. The choice of a quasiperiodic excitation and incommensurability has been made with respect to its significant influence on dynamical systems. As known, quasiperiodicity (a possible route to chaos) is a behaviour caused by two or more simultaneous periodicities whose frequencies are out of phase (not commensurate) with one another. A trajectory of such system in this case does not repeat itself, and it has two different frequencies. The irrationality of the field winding number, *i.e.* the ratio of the frequencies, provides also stability to the orbit [24].

3. Phase synchronisation of the chaotic junctions

The coupled equations (6) and (7) describe the dynamical behaviour of the chaotic ratchet junctions. As stated earlier, the LSODE algorithm has been employed to obtain the dynamics with a computational time step that can be dynamically set via controlled minimum and maximum step size [19]. In our simulations the following parameter values of the coupled system have been chosen:

$$\begin{aligned} \alpha_1 &= \alpha_2 = 0.01\,, \qquad a_1 = a_2 = 0.08092\,, \qquad a_3 = a_4 = 0.08092\,, \\ n &= \omega_1/\omega_2 = \sqrt{\pi}\,, \quad m = \omega_3/\omega_4 = \sqrt{\pi}\,, \end{aligned}$$

whereas the coupling strength ε, has been varied throughout to achieve a synchronous state between the coupled junctions. As plainly obvious, we consider the ideal case for which the parameters of both junctions have been similarly set. Further, the frequencies of the external field are incommensurable having ratios equal to the square root of the transcendental and irrational number π. We hence examine phase synchronisation between both interacting ratchet junctions when they evolve from different initial conditions. The effect of synchronisation on directed transport in ratchet systems may have serious practical consequences. To explore this phenomenon, we start by presenting the dynamical phase of a single non-interacting Josephson junction for initial conditions $x_1(0) = 0, \dot{x}_1(0) = 0$ (Fig. 2). The trajectory, as can be seen is intermittent signalling the onset of chaos ($\alpha_1 = 0.01, a_1 = a_2 = 0.08092, n = \sqrt{\pi}$) as in Fig. 2(a), and the junction is very sensitive to initial conditions. The autocorrelation function in Fig. 2(b) which has been calculated from a time series data set of the phase $x_1(t)$ illustrates further this uncorrelated behaviour. Now for the coupled interacting system in asymmetric potential, the relevant trajectories are shown for three values of the coupling strength; $\varepsilon = 0, 0.32, 0.66$ in (a), (b), and (c) of Fig. 3, respectively. The initial conditions for junction 2 are, $x_2(0) = 0$, $\dot{x}_2(0) = 0.5$ and the other parameters are kept analogous to those of junction 1. The quantity that concerns us in achieving a synchronous

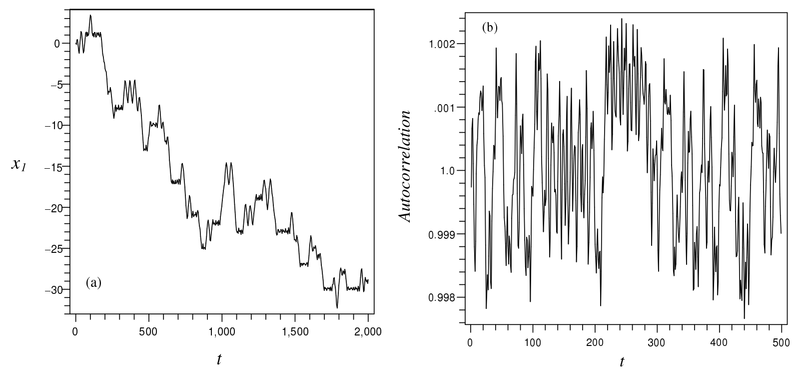

Fig. 2. (a) The intermittent trajectory of a single decoupled junction, $\alpha_1 = 0.01$, $a_1 = a_2 = 0.08092$, $n = \sqrt{\pi}$. (b) The autocorrelation function of the phase $x_1(t)$ of (a) showing the uncorrelated and chaotic behaviour.

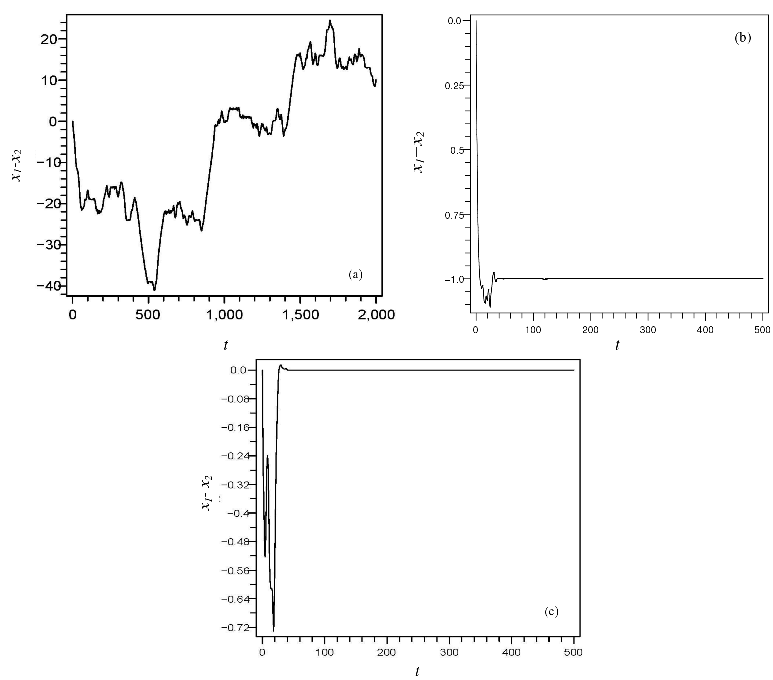

Fig. 3. (a) The phase difference dynamics of the decoupled ratchets ($\varepsilon = 0$), $\alpha_1 = \alpha_2 = 0.01$, $a_1 = a_2 = 0.08092$, $a_3 = a_4 = 0.08092$, $n = \omega_1/\omega_2 = \sqrt{\pi}$, $m = \omega_3/\omega_4 = \sqrt{\pi}$. (b) The phase difference dynamics for $\varepsilon = 0.32$, the phase is bounded and settled at a negative direction before the fulfilment of synchronisation. (c) Complete phase synchronisation at $\varepsilon = 0.66$ after nearly 40 s transient time, and for the same parameters as in (a) and (b).

state between the coupled ratchet junctions is $\Delta x = x_1 - x_2$ which represents the phase difference of both junctions. When the system is uncoupled for which $\varepsilon = 0$ as in Fig. 3(a), its trajectories show dynamically random and fluctuating behaviour, while the behaviour of both junctions in the phase space is highly uncorrelated as in Fig. 4(a). As the coupling is raised the

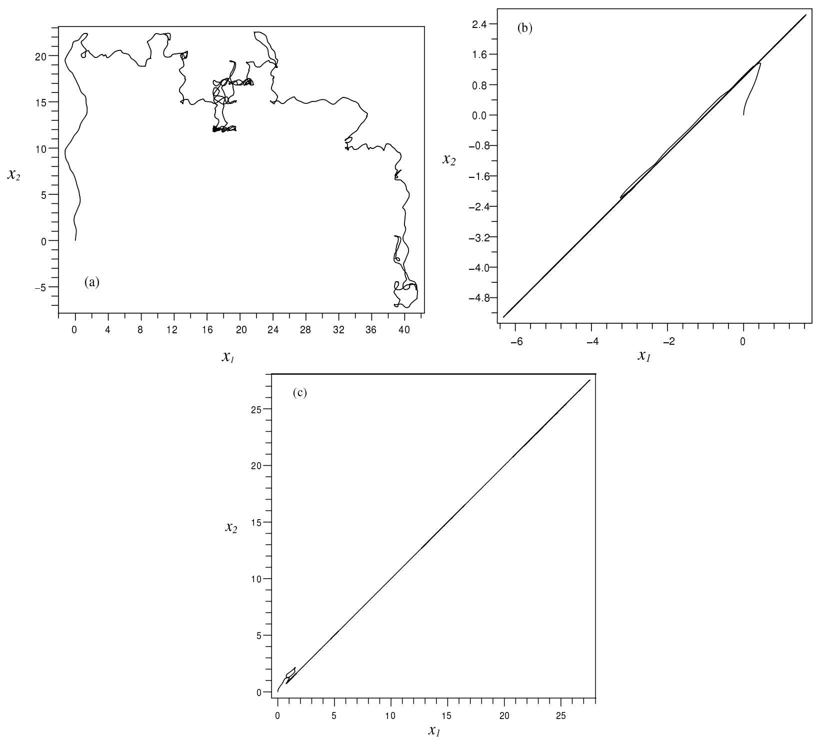

Fig. 4. Phase space plots for the coupled ratchet junctions: (a) For $\varepsilon = 0$, both phases x_1 and x_2 are divergent and non-diffused maintaining chaotically desynchronised state. System parameters same as in Figs. 2 and 3. (b) Phase space corresponding to Fig. 3(b), $\varepsilon = 0.32$. (c) Synchronous regime for $\varepsilon = 0.66$, where x_1 and x_2 are overly diffused.

system starts approaching synchronisation. This is remarkable in Figs. 3(b) and 3(c), after retrieving from a few transient times the quantity Δx progressively dwindles and attains a nil value as ε is increased between 0.32 and 0.66, for which phase–phase locking is retained. For quasiperiodically and irrationally driven excited ratchet junctions, simulations undertaken in this work reveal the persistence of a synchronised state between the two interacting systems for $\varepsilon > 0.66$. The coupled junctions rather break phase locking for $\varepsilon < 0.32$, and this phenomenon as we believe is due to the asymmetry associated with the ratchet potential and quasiperiodic field applied to the system, which is discussed later. The behaviour can be further illustrated when viewing the phase space plots in Fig. 4. For coupling parameter $\varepsilon = 0$, both phases x_1 and x_2 are divergent and non-diffused maintaining chaotically desynchronised state (Fig. 4(a)). Once ε increases, synchronisation occurs as revealed by phase space plots 4(b) and 4(c), where x_1 and x_2 tangibly diffuse and asymptotically approach the manifold $x_1 = x_2$. When the junctions are decoupled ($\varepsilon = 0$), it can be obviously noticed that the attractors are disparate, nonetheless strongly stochastic. The coupled system retains this behaviour as the coupling is increased until eventually and once phase–phase locking between both junctions is achieved, the attractors become identical ($\varepsilon = 0.66$) however diffused. The observed transition from non-synchronised to synchronous state via an interior crisis is not associated with identical attractors forming and disappearing. The attractors are initially non-identical when the junctions are decoupled, become identical and get more prominently diffused with increasing coupling.

4. Symmetry considerations and discussion

The general equation of motion of the uncoupled junction under a quasiperiodic field may be expressed as

$$\begin{aligned} \dot{x}_1 &= y_1 \,, \\ \dot{y}_1 &= -\alpha y_1 + f(x_1) + F(\theta_1, \theta_2) \,, \\ \dot{\theta}_1 &= \omega_1 \,, \\ \dot{\theta}_2 &= \omega_2 \,, \end{aligned} \tag{8}$$

where $f(x_1)$ is related to the asymmetric ratchet potential, and $F(\theta_1, \theta_2)$ is the quasiperiodic force expressed in terms of the phases θ_1 and θ_2. The argument in this section is extended to incorporate the symmetry considerations introduced in [21] to discuss the transport properties of Josephson junctions in symmetric periodic potentials. As $f(x_1)$ represents the potential of the tunnelling flux quanta in the junction 1, it is 2π-periodic and bounded *i.e.* $f(x_1 + 2\pi) = f(x_1)$. The symmetry characteristics of f contribute markedly to equation (8), for which one may examine the symmetry characteristics of the external perturbation $F(\theta_1, \theta_2)$ as well. The symmetry principles imply that if we carry out a reflection operation *i.e.*, substituting x_1 with $-x_1$ yields f symmetric if $f(x_1) = f(-x_1)$, and anti-symmetric otherwise. Since f is a ratchet, the reflection symmetry is thus lacking and it yields anti-symmetric properties when replaced by the ratchet potential (namely its derivatives) as shown by Eq. (8). It can be also demonstrated that $F(\theta_1, \theta_2)$ may have three symmetry modes [21] depending on the possible phase shifts of the phases θ_1 and θ_2. External fields like F in many cases might break the integrability of certain systems and give rise to deterministic chaos or unpredictability. This is another reason why it might be of significance studying such systems, particularly the ratchet ones. Thus expanding $F(\theta_1, \theta_2)$ into a Fourier series

$$F(\theta_1, \theta_2) = \sum_{j,k} F_{j,k} e^{i(j\theta_1 + k\theta_2)} \tag{9}$$

one may permit either phases or both to shift by π and easily check whether F remains invariant or not.

These inherent symmetries in the quasiperiodic field combined with $f(x_1) \neq f(-x_1)$ of a ratchet have been shown to entail important consequences on the transport properties. In earlier publications [14, 15] we investigated the effect of quasiperiodicity and asymmetry on the dynamical and associated transport characteristics of underdamped Josephson junctions. In our system, under consideration here, one has two coupled ratchet junctions studied in the weak field and low damping conditions, Fig. 3(a) and Fig. 4(a) demonstrated, as expected, the chaotic and strongly uncorrelated behaviour for $\varepsilon = 0$. Once the junctions are coupled, an extra term $\varepsilon(\dot{x}_2 - \dot{x}_1)$ (or $\varepsilon(\dot{x}_1 - \dot{x}_2)$) has to contribute to Eq. (8), for which synchronisation is initiated. The latter term (which is a velocity difference) related to the frequencies of the external field $\langle \dot{x}_1 \rangle = \frac{p_1}{q_1}\omega_1 + \frac{p_2}{q_2}\omega_2$ (or $\langle \dot{x}_2 \rangle = \frac{p_3}{q_3}\omega_3 + \frac{p_4}{q_4}\omega_4$) suppresses the massive phase fluctuations associated with broken symmetry contributions from the ratchet potential f and external field F, already mentioned, where $p_1, q_1, p_2, q_2, p_3, q_3, p_4, q_4$ are rotation numbers having irrational relationship. This allows for synchronisation (Fig. 3(b), (c) and Fig. 4(b), (c)) to be achieved, as phase–phase locking between both junctions is ensued. A transition also from a regime of which the phases rotate with different velocities to synchronised chaos, where the phase difference does not increase with time is occurring as the coupling goes higher. This can further be substantiated taking into account that stabilisation of ratchet dynamics due to applied weak perturbing field has been reported [22].

5. Conclusions

Synchronisation between two coupled ratchet Josephson junctions under the influence of a quasiperiodically varying field have been examined. The system has been modelled using the coupled rotators equations in the weak perturbation and low damping limit, for which irrational relation between the frequencies has been considered. For zero coupling, the dynamics were highly uncorrelated with initially non-identical and non-diffused attractors, becoming identical but diffused as the coupling ε reached 0.66. The latter transition from non-synchronised to synchronous chaotic state is not associated with attractors appearing and abolishing as our results show. The synchronisation state also permanently and markedly retains for higher than 0.66 coupling strength. The analysis of the symmetry properties related to the external excitation and ratchet potential of tunnelling flux quanta has further demonstrated that the symmetry of the system is broken. The corresponding enormous phase fluctuations prevailing for the case of decoupled junctions can be quenched as the coupling is gradually raised in the system, allowing phase–phase locking to take effect and synchronisation is thereby achieved. We put emphasis on exploring chaos synchronisation in soliton Josephson junctions, for which spatial contributions may be influential on the transport characteristics. Controlling chaos in such systems is of mounting interest especially for the implementation of directed transport in inertial soliton ratchets [23].

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