Q-BALLS IN THE REGULARIZED SIGNUM-GORDON MODEL

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The regularized signum-Gordon potential has a smooth minimum and is linear in the modulus of the field value for higher amplitudes. The Q-ball solutions in this model are investigated. Their existence for charges large enough is demonstrated. In three dimensions numerical solutions are presented and the absolute stability of large Q-balls is proved. It is also shown, that the solutions of the regularized model approach uniformly the solution of the unregularized signum-Gordon model. From the stability of Q-balls in the regularized model follows the stability of the solutions in the original theory.

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1. Introduction

In the seminal paper [1] Coleman addressed the following problem. Consider a field theory with a symmetry in the internal field space. Then, due to the Noether theorem there is a charge Q in the system, a quantity constant in time. A legitimate problem is then what solution minimizes the energy E for a given Q. Coleman managed to answer this question for a class of "acceptable" field potentials and gave a recipe how to find the relevant solutions. He dubbed them Q-balls. The time dependence of Q-balls is confined to the space of the field symmetry, so that the energy and charge density do not evolve in time (actually up to Lorentz boosts). The space distribution of the field is given by a spherically symmetric, positive and monotone (as a function of the radial coordinate) decreasing function. These solutions are of physical importance and much attention has been paid to them, see [4]. The Q-ball Ansatz may be also useful in models spoiling the prerequisites given by Coleman (see *e.g.* [2,8]). Then, the status of the

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solutions is not clear; they may be just unstable configurations or absolutely stable solutions. In this paper we show, that for the scalar complex field with the "unacceptable" potential $V(\Phi) \sim \sqrt{|\Phi|^2 + \varepsilon^2} - \varepsilon$ such solutions may fall into the second category. In what follows we refer to this potential as regularized signum-Gordon one. This name traces back to original motivation.

Recently considerable work has been done in exploring the signum-Gordon model ([2,3]), where the field potential $V(\Phi) \sim |\Phi|$. One of the intriguing characteristics of such field theory is the absence of the linear regime ("infinite" mass). It results in the compactness of solutions, the Q-balls described in [2] are paradigmatic. In this reference it is also pointed out, that the signum-Gordon model may be regarded as a limiting case for the regularized one when $\varepsilon \to 0$. The considerations presented below support the suggestion: the Q-balls in regularized theory tend uniformly with ε to these ones known from the "sharp" potential. Consequently, the global quantities: charge and energy approach the relation found in Ref. [2].

The paper is organized as follows. The next section is devoted to Q-balls in the regularized signum-Gordon model. Numerical results in the case of the three spatial dimensions are reviewed in Sec. 3. In Sec. 4 we examine briefly the equation motivated by the Q-ball Ansatz in the signum-Gordon model in any number of space dimensions. To this end, we recall and supplement the results presented in Ref. [2]. In Sec. 5 the limit $\varepsilon \to 0$ is taken for solutions of the regularized model. Finally, in Sec. 6 we adapt the Coleman's proof of Q-ball stability for the regularized model (also in three dimensions). An immediate consequence of the stability of the Q-balls in the regularized model is the stability of the solutions in the original model. In the last section we summarize the results and discuss some open problems.

2. The regularized model

The Lagrangian defining the theory of interest has the form

$$L = \partial_\mu \Phi \partial^\mu \bar{\Phi} - \lambda \sqrt{\varepsilon^2 + \Phi\bar{\Phi}} + \lambda\varepsilon \,, \tag{1}$$

where Φ and $\bar{\Phi}$ denote the scalar field and its complex conjugation, ε and λ are positive real numbers. The Lagrangian respects Lorentz symmetry. The space dimension n does not need to be specified now. The field Φ, the space-time coordinates x^μ and the constants λ and ε are considered dimensionless. A global change of the field phase does not affect the Lagrangian giving rise to the Noether charge

$$Q = \frac{1}{2i} \int d^n x \left[\partial_t \Phi \bar{\Phi} - \Phi \partial_t \bar{\Phi}\right] . \tag{2}$$

This makes the Coleman's question about field configuration minimizing the energy E for a given charge Q relevant. Although the theory (1) is not an "acceptable" one (a discussion of the acceptability is postponed to Sec. 6), we plug the Q-ball Ansatz into the field equations. The Q-ball Ansatz for the complex scalar field has the form

$$\Phi(t, \vec{x}) = F(r) \exp(i\omega t) \,, \tag{3}$$

where F is a real valued function of the radial coordinate r and $\omega > 0$. After rescaling of the radial variable $y = \omega r$ and the profile function $f_\delta(y) = (2\omega^2/\lambda)F$ the following equation is obtained

$$f_\delta'' + \frac{n-1}{y} f_\delta' + f_\delta = \frac{f_\delta}{\sqrt{\delta^2 + f_\delta^2}} \,. \tag{4}$$

The $'$ stands for derivative with respect to y and $\delta = 2\omega^2\varepsilon/\lambda$. The physically meaningful solution obeys the conditions: $f_\delta'(0) = 0$ and $f_\delta(\infty) = 0$. Such a solution of the above equation is denoted as $\hat{f}_\delta$.

Now, we argue that equation (4) has at least one meaningful solution. The above equation may be interpreted in terms of classical mechanics — it corresponds to a point particle moving in a potential $f_\delta^2/2 - \sqrt{f_\delta^2 + \delta^2}$ and subject to the time dependent friction — y is regarded here as time. The potential changes qualitatively for $\delta = 1$. If $\delta > 1$, it has one global minimum and only the trivial solution ($f_\delta = 0$) satisfies equation (4) and boundary conditions. For $0 < \delta < 1$ the potential has a local maximum at $f = 0$ and two symmetric global minima for $f = \pm\sqrt{1 - \delta^2}$. In this range of the parameter δ equation (4) admits nontrivial solutions. The heuristic reasoning for their existence uses the continuity argument as follows: it is possible to find such $f_\delta(0)$ that the particle cannot pass the local maximum and oscillates forever around one of the minima. However, for another $f_\delta(0)$, larger then the previous one, the particle may cross the local maximum and dip on its other side. In between the two families of solutions the sought after solution is expected. The fact of the existence of the two families is demonstrated in Sec. 5.2. The exact solutions of equation (4) are not known, so we have to resort to the numerics. The results are presented in Sec. 3.

The mechanical analogy is useful to demonstrate that the one-dimensional model has the relevant solutions. In this case there is no friction. Hence, the equation gains an integral of motion corresponding to the mechanical energy

$$E_{\mathrm{mech}} = \tfrac{1}{2} {f_\delta'}^2 + \tfrac{1}{2} f_\delta^2 + \delta - \sqrt{\delta^2 + f_\delta^2} \,.$$

The Q-ball solution emerges for $E_{\mathrm{mech}} = 0$, the inverse function has then the form

$$y(f_\delta) = \int\limits_{2\sqrt{1-\delta}}^{f_\delta} \frac{dx}{\sqrt{2\sqrt{x^2+\delta^2} - x^2 - 2\delta}}\,.$$

A detailed analysis of the above formula reveals the relation between the Q-balls in the regularized and the "sharp" model. The same may be achieved with methods presented in Sec. 5 (some obvious modifications are in order then). In the sequel we will no more return to the one-dimensional case.

Except for $n = 2$, we can give a more reliable argument for existence of the required solution. The argument follows from a theorem proved in Ref. [5]. That theorem states that the equation

$$\Delta\psi = \frac{dU(\psi)}{d\psi}\,, \tag{5}$$

has at least one spherically symmetric positive, monotone and vanishing in infinity solution. What is more, the integrability of the terms $U(\psi)$ and $(\nabla\psi)^2$ is also granted. In the above equation ψ denotes a real valued function and Δ stands for a Laplacian in $n > 2$ dimensions. It holds, if U satisfies four conditions:

(1) U is continuously differentiable for all ψ;

(2) $U(0) = U'(0) = 0$;

(3) U is somewhere negative;

(4) There exist positive numbers a, b, α and β such that $\alpha < \beta < 2n/(n-2)$ and
$$U \geq a|\psi|^\alpha - b|\psi|^\beta\,.$$

Let us consider
$$U(\psi) = \left(\sqrt{\psi^2+\delta^2} - \delta\right) - \tfrac{1}{2}\,\psi^2\,.$$

This U satisfies the above requirements (*e.g.* for $\delta < 0.95$ one can take $\alpha = 2$, $\beta = 3$, $a = (1-\delta)/2$, $b = 4(1-\sqrt{\delta})\delta^{-1}$). Thus, the theorem applies. It is clear, that the solution described in the theorem corresponds to the Q-ball solution of the Eq. (4). Alas, in two spatial dimensions we have nothing but the heuristic argument.

3. Numerical results

The numerical analysis of the regularized signum-Gordon model is done for $n = 3$ space dimensions. Some profile functions f_δ for various δ are depicted in Fig. 1. The relevant solution of the original signum-Gordon model is also plotted in this figure. Such a presentation supports the supposition that the solution of the "sharp" potential is a limiting case for the solutions of the regularized problem. The relations between the charge, energy and

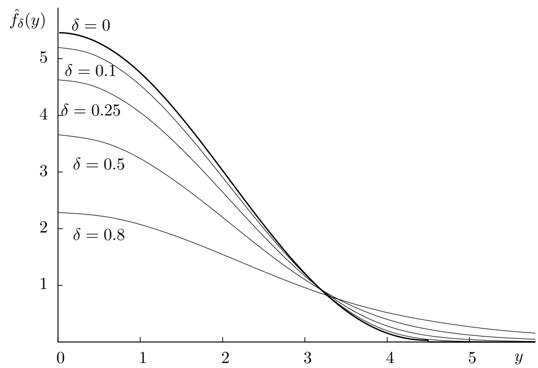

Fig. 1. Profile function for various δ values. The solution of signum Gordon model is marked as $\delta = 0$.

the parameter δ are most interesting from the physical viewpoint. Plugging the Ansatz (3) into the definition of the charge (2) we obtain

$$Q = \frac{\pi}{\lambda}\left(\frac{2\varepsilon}{\delta}\right)^3 \int \hat{f}_\delta^2 r^2 \; dr = \frac{(2\varepsilon)^3}{\lambda}\bar{Q}(\delta)\,. \tag{6}$$

The energy is given by the formula

$$E = \frac{\pi}{\sqrt{\lambda}}\left(\frac{2\varepsilon}{\delta}\right)^{5/2} \int dr \; r^2 \left[(\hat{f}_\delta{}')^2 + \hat{f}_\delta^2 + 2\left(\sqrt{\hat{f}_\delta^2 + \delta^2} - \delta\right)\right] = \frac{(2\varepsilon)^{5/2}}{\sqrt{\lambda}}\bar{E}(\delta)\,. \tag{7}$$

$\bar{Q}$ and $\bar{E}$ are functions of the parameter δ only. The relation $\bar{E}(\bar{Q})$ is shown in Fig. 2.

Quite a general feature in theories with Q-balls is the existence of a solution with minimal possible charge and energy values. This is not the case of the "sharp" signum-Gordon model, where

$$E = \left(\frac{5\pi}{6}\right)^{1/6} \frac{12\sqrt{y_0}\;\lambda^{1/3}}{5}\, Q^{5/6} \tag{8}$$

for any charge $Q > 0$. In the above formula $y_0 \approx 4.49$ is a numerical constant, see [2]. The $E(Q)$ relation in the regularized model inherits both from the

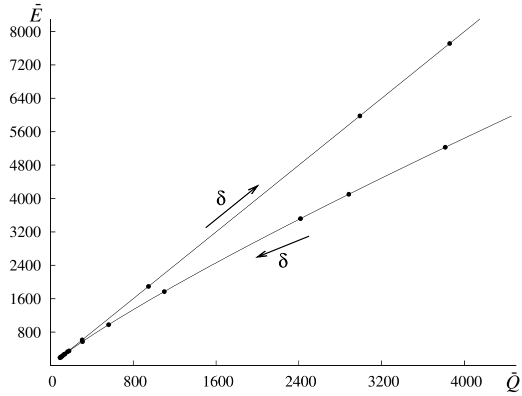

Fig. 2. Relation $\bar{E}(\bar{Q})$ for the regularized signum-Gordon model. For a given charge there exist two different Q-balls with different energies resulting in a cusp on the graph. The arrows show how δ changes along the curve.

ordinary models and the signum-Gordon one. As in most models, there is a Q-ball with the smallest possible charge and energy (the corresponding solution is found for $\delta = 0.96$, see Table I). Two branches of the relation $E(Q)$ originate from the point corresponding to this solution, see Fig. 2. The branch corresponding to larger δ's has larger energy values. Physically more favorable are solutions with smaller δ — they may be absolutely stable. This lower branch of solutions reproduces the power dependence $E(Q)$ known from the signum-Gordon model, see Fig. 3. The energy and charge are smooth functions of $0 < \delta < 1$.

TABLE I

Data of some exemplary solutions in the regularized model.

δ	$f_\delta(0)$	$\bar{Q}$	$\bar{E}$
0.9999	0.0613	947.16	1894.41
0.999	0.1928	306.32	612.95
0.99	0.5844	117.44	235.75
0.97	0.9629	93.62	188.49
0.96	1.0948	92.62	186.53
0.95	1.2094	93.59	188.43
0.8	2.2862	179.77	348.21
0.5	3.6589	1100.09	1770.26
0.25	4.6276	11668.8	13653.2
0.1	5.1988	213385	160261

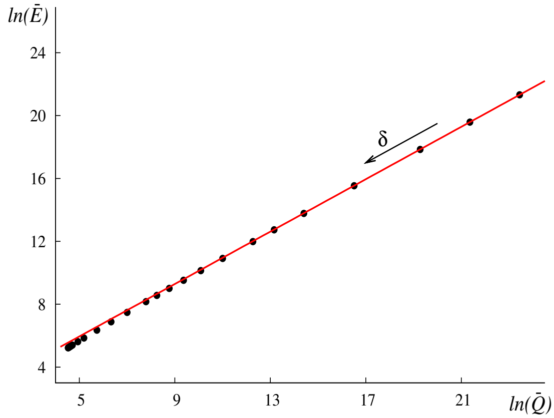

Fig. 3. The points come from the numerics of regularized model. The solid line illustrates the relation (8). The agreement is excellent.

The corner stone for the Q-ball theory is their absolute stability. In Sec. 6 it is shown, that the criterion ensuring this reads $E < Q\sqrt{2\lambda/\varepsilon}$, where the proportionality coefficient between E and Q is the mass parameter of the theory. In the case of our model the inequality may be written in a dimensionless form

$$2 > \frac{\bar{E}}{\bar{Q}}\,. \tag{9}$$

The numerics indicates, that the inequality is violated if $\delta > 0.91$. It means, that almost all solutions from the lower branch are absolutely stable. The solutions lying on the upper branch cannot be absolutely stable, however they seem to be very close to the relation $\bar{E} = \sqrt{2\lambda/\varepsilon}\bar{Q}$. There are two more types of stability of Q-balls (see [6]): linear (classical) stability and stability against fission. The classical stability is granted, if

$$\frac{\omega}{Q}\frac{dQ}{d\omega} \leq 0\,,$$

where ω denotes the same quantity as in (3). In the case of our model, where $\omega \sim \sqrt{\delta}$, the solutions from the lower branch of the relation $E(Q)$ satisfy the condition. It turns out, rather unexpectedly, that the condition for the stability against fission coincides with this for linear stability. Thus, the solutions from the lower branch are physically relevant, although not all of them are absolutely stable. Some data useful for numerical analysis are given in Table I.

4. The signum-Gordon model

The defining feature of the signum-Gordon model is the field potential term in the Lagrangian given by $\lambda|\Phi|$. We plug the Q-ball Ansatz (3) into the equation of motion, rescale the radial coordinate $y = \omega r$ and introduce $f(y) = (2\omega^2/\lambda)F$ as previously (f without δ in subscript relates to the signum-Gordon model). This leads to equation [2]

$$f'' + \frac{n-1}{y} f' + f = \mathrm{sign}(f) \,. \tag{10}$$

Due to the symmetry $f \to -f$ we can consider only the solutions with $f(0) > 0$. Then, the above equation is a linear equation with the source term equal to unity. The homogeneous part is solved with the substitution $f(y) = y^{-\alpha} R(y)$, where $\alpha = (n-2)/2$. In this way the equation transforms into the Bessel's differential equation of the order of α. Two linearly independent solutions of (10) u_1 and u_2 may be written in terms of the Bessel functions of the first $J_\alpha(y)$ and the second $Y_\alpha(y)$ kind [7]:

$$u_1(y) = y^{-\alpha} J_\alpha(y) \,, \qquad u_2(y) = y^{-\alpha} Y_\alpha(y) \,. \tag{11}$$

For small values of y the first solution behaves like $u_1 \approx a - by^2$ with $a, b > 0$. On a larger scale it is oscillating function with decreasing amplitude. u_2 behaves in the vicinity of the origin like $y^{-2\alpha}$. Thus, the solution of equation (10) obeying the conditions $f(0) > 0$ and $f'(0) = 0$ has the following form:

$$f(y) = \frac{f(0) - 1}{u_1(0)} u_1(y) + 1 \,. \tag{12}$$

Strictly speaking, this function solves (10) as long as $f(y) > 0$. It has a simple structure: the function u_1 is rescaled and then shifted by the term $+1$. Thus, the positions of the extrema of f do not depend on the starting point $f(0)$. If $f(0) > 1$ (as is supposed in the sequel) the first extremum is minimum. We denote the argument of this minimum with y_0 and note that $u_1(y_0) \neq 0$. Let us also define

$$f_0 = 1 - \frac{u_1(0)}{u_1(y_0)} \,, \tag{13}$$

and point out that if $f(0) = f_0$, then $f(y_0) = 0$. If $f(0) < f_0$, f is valid solution for all arguments as it stays positive for all y. For $f(0) > f_0$ the value of f at the first minimum is negative. Thus, for some $y < y_0$ the function f changes its sign and ceases to solve (10). Following the solution with $f(0) = f_0$ an ambiguity is encountered for $y = y_0$. The equation admits three ways of continuation for $y > y_0$: a valid solution may follow the r.h.s.

of (12) with plus or minus sign or may be set to zero. It is our choice motivated by the field theoretical context to stick to the last option. The resulting function composed of two pieces corresponds to the Q-ball profile function and is further denoted as $\hat{f}$. We will show that the regularization of the potential supports the choice.

5. The limit $\delta \to 0$

The numerical results from Sec. 3 suggest, that the Q-balls in the regularized model approach the solution of the signum-Gordon model. Now, we can give some analytical arguments for this. The spatial dimension $n > 1$ does not need to be specified in what follows. First, we will show that for profile functions in the regularized model $\hat{f}_\delta(0) \to f_0$ when δ tends to zero. Then, we will find an upper bound for the modulus of the difference between the solution of the model with $\delta > 0$ and the one with $\delta = 0$. As the bound tends to zero, the solutions of the regularized models approach uniformly the solution characterized in Sec. 4.

5.1. General setting

Now we are in a position to explore the solutions of equation (4) for small values of the parameter δ. To deal with the limit we set the notation and give some general estimates used later.

Let us consider the difference between solutions of (4) and (10)

$$\eta(y) = f(y) - f_\delta(y)\,. \tag{14}$$

The pertinent solutions obey the same initial conditions $f_\delta(0) = f(0)$ and $f'_\delta(0) = f'(0) = 0$. Let us make it clear, that at this stage we investigate solutions of the equations that can spoil the condition $f(\infty) = f_\delta(\infty) = 0$. As long as $\mathrm{sign}(f(y)) = +1$ the differential equation holds

$$\eta'' + \frac{n-1}{y}\eta' + \eta = \frac{\delta^2}{\left(\sqrt{\delta^2 + f_\delta^2} + f_\delta\right)\sqrt{\delta^2 + f_\delta^2}}\,. \tag{15}$$

It is supplemented with the initial conditions $\eta(0) = 0$ and $\eta'(0) = 0$. For notational convenience let us call the r.h.s. in the above equation $\varphi(f_\delta(y))$ or shorter $\varphi(y)$. φ seen as a function of positive f_δ is a positive and monotone decreasing function. Some algebra makes evident that $\varphi(f_\delta) < \delta^{2/3}/g$ (with g being a positive constant) as long as

$$f_\delta > \delta^{2/3}\sqrt{\frac{g}{2}}\frac{1 - \frac{\delta^{2/3}}{g}}{\sqrt{1 - \frac{\delta^{2/3}}{2g}}} = \delta^{2/3}\sqrt{\frac{g}{2}} + o(\delta)\,. \tag{16}$$

Equation (15) is an inhomogeneous linear equation. The homogeneous part is the same as in (10). Then, using u_1 and u_2 the solution may be written in the form

$$\eta(y) = \int_0^y G(y,s)s^{n-1}\varphi(s)ds\,, \tag{17}$$

where $G(y,s)$ is a Green function (see [7]) and does not depend on the parameter δ. It has the form

$$G(y,s) = \frac{u_1(s)u_2(y) - u_2(y)u_1(s)}{y^{n-1}\left(u_2'(y)u_1(y) - u_1'(y)u_2(y)\right)}\,.$$

A priori any combination of the functions u_1 and u_2 could be added to the solution (17), but the boundary conditions exclude such terms. The integral does not give rise to any ambiguity or difficulty for $y \to 0$. The above form of η gives the following bound

$$|\eta(y)| \leq \max_{s\in(0,y)}\{\varphi(s)\}\int_0^y \left|G(y,s)s^{n-1}\right| ds\,. \tag{18}$$

Hence, for all $y < y_0 + 1$ the inequality holds

$$|\eta(y)| \leq \max_{s\in(0,y)}\{\varphi(s)\}\int_0^{y_0+1} \left|G(y,s)s^{n-1}\right| ds = g_1 \max_{s\in(0,y)}\{\varphi(s)\}\,, \tag{19}$$

where the last equality defines g_1. In order to get another helpful observation it is convenient to rewrite equation (15) in the form

$$y^{n-1}\eta'(y) = \int_0^y s^{n-1}\left(\varphi(s) - \eta(s)\right) ds\,.$$

Together with (19) this gives the bound on the derivative $\eta'(y)$ for all $y < y_0 + 1$:

$$|\eta'(y)| \leq y^{1-n}\int_0^y s^{n-1}\left(|\varphi(s)| + |\eta(s)|\right) ds \leq g_2 \max_{s\in(0,y)}\{\varphi(s)\}\,, \tag{20}$$

where $g_2 = (y_0 + 1)(1 + g_1)/n$.

5.2. The limit $f_\delta(0)$ with $\delta \to 0$

Now, we demonstrate that a solution with $f_\delta(0)$ far from f_0 cannot be the Q-ball profile function — it either has a minimum or changes its sign. First, we deal with the solutions $f_\delta(0) = f(0) = f_0 - \xi$, $\xi > 0$. It is shown in Sec. 4, that then equation (15) and the solution (17) hold for any argument y. Assume $\eta(y)$ does not tend to zero for $0 < y < y_0 + 1$ when δ gets smaller and smaller. The inequality (19) makes clear that it may be true only if $\varphi(f_\delta) \geq \delta^{2/3}/g_1$ on a finite segment. Equivalently, it means that $f_\delta < \delta^{2/3}\sqrt{g_1/2}$ on this segment as follows from (16). For continuity reason f_δ has to reach this value for the first time at a point y_s. At this point the relation holds

$$|f(y_s)| \leq |f_\delta(y_s)| + |\eta(y_s)| \leq \delta^{2/3}\sqrt{\frac{g_1}{2}} + \delta^{2/3}\,. \tag{21}$$

This inequality may be combined with another one: $f(y_0) \leq f(y_s)$, what restricts the initial conditions allowing the reasoning

$$0 \leq \xi \leq \left|\frac{u_1(0)}{u_1(y_0)}\right| \delta^{2/3}\left(\sqrt{\frac{g_1}{2}} + 1\right)\,. \tag{22}$$

If this inequality is spoiled, $|\eta|$ is smaller then $\delta^{2/3}/g_1$ on the whole segment. From this we can infer the existence of a minimum of f_δ; it suffices that the function f takes a value bigger then $f(y_0) + \delta^{2/3}$ twice in the segment (eventually one can consider a larger segment instead of the arbitrarily taken $y_0 + 1$). Thus, for δ small enough, the solution interpreted as Q-ball cannot start with $f_\delta(0) < f_0 - \delta^{2/3}(\sqrt{g_1/2} + 1)$.

Let us analyze the case $f_\delta(0) = f(0) = f_0 + \xi$. It is argued in the Sec. 4, that there exists a point $y_1 < y_0$ for which $f(y_1) = 0$ and $f'(y_1) < 0$. Consider another point for which $f_\delta(y_z) = \delta^{2/3}g_1^{1/3}$. It means, that $|\eta(y_z)| < \delta^{2/3}g_1^{1/3}/2$. Such choice of the function value ensures that $0 < f(y_z)$. To see, that the function f_δ reaches the requested value assume the contrary: f_δ does not. As it is a continuous function, it is always bigger then this (*i.e.* $\delta^{2/3}g_1^{1/3}$) value. Then, η is small enough to ensure that $f \neq 0$ for any value $0 < y < y_0$, what is false. Now, we can show that f_δ changes its sign if δ is sufficiently small. To this end we make use of (20) to get

$$f'(y_z) - \frac{g_2 g_1^{-2/3}}{2}\,\delta^{2/3} < f'_\delta(y_z) < f'(y_z) + \frac{g_2 g_1^{-2/3}}{2}\,\delta^{2/3}\,. \tag{23}$$

We consider such values of δ, that $|f'(y_z)| > \delta^{2/3}g_2 g_1^{-2/3}/2$. Let us solve the following equation for y_2

$$f_\delta(y_2) = -\delta^{1/3}$$

using the Taylor expansion

$$f_\delta(y) = f_\delta(y_z) + f'_\delta(y_z)(y - y_z) + \ldots .$$

The solution reads

$$y_2 = y_z + \frac{\delta^{1/3} + \delta^{2/3} g_1^{1/3}}{|f'_\delta(y_z)|} .$$

Trading $f'_\delta(y_z)$ for $f'(y_z)$ in the above relation changes the result with a term of the order of δ^1, what is negligible. To ensure validity of the solution the reminder of the Taylor expansion R has to be shown irrelevant. It has the form

$$R = \frac{f''_\delta(s)}{2 f'^2_\delta(y_z)} \; (y_2 - y_z)^2 ,$$

where $s \in (y_z, y_2)$. Equation (4) does not touch the quantity

$$E_{\mathrm{mech}} = (n-1) \int\limits_{y_s}^{y} \frac{f'^2_\delta(r)}{r} \; dr + \frac{1}{2}\left(f'^2_\delta(y) + f^2_\delta(y) \right) - \sqrt{f^2_\delta(y) + \delta^2} ,$$

which is interpreted as the mechanical energy (see Sec. 2) at "time" y plus the energy lost on the "time" interval $[y_s, y]$. From this we can get a bound on f'_δ in terms f_δ for all $y > y_s$. Plugging this into equation (4) a bound for f''_δ is found. Hence, if δ is small enough, the solution starting with $f_\delta(0) > f_0$ cannot correspond to a Q-ball profile function as it changes its sign.

A crude estimation of ξ allowing the above reasoning gives $\xi \sim \delta^{4/3}$. This is obtained by finding y_z by Taylor expansion of f around y_0 and by checking the condition $|f'(y_z)| > \delta^{2/3} g_2 g_1^{-2/3}/2$.

The succinct conclusion of this section is

$$\lim_{\delta \to 0} \hat{f}_\delta(0) = f_0 . \tag{24}$$

5.3. The limit $\hat{f}_\delta$ with $\delta \to 0$

To investigate the difference between the Q-ball solutions in the regularized signum-Gordon model and the original one it is convenient to use the method from the previous section. First, we denote

$$\hat{\eta}(y) = \hat{f}(y) - \hat{f}_\delta(y) \tag{25}$$

and for further convenience

$$r(\delta) = g_1 \delta^{2/3} + \left| f_0 - \hat{f}_\delta(0) \right| .$$

For $y < y_0$ the equation for $\hat{\eta}$ has the same form as (15). As $\hat{\eta}(0) \neq 0$ the solution of this equation differs slightly from (17), it has the following form

$$\hat{\eta}(y) = \int_0^y G(y,s)s^{n-1}\varphi\left(\hat{f}_\delta(s)\right) ds + \frac{f_0 - \hat{f}_\delta(0)}{u_1(0)}\, u_1(y)\,. \tag{26}$$

Consider a point y_3 such that $\hat{f}_\delta = \delta^{2/3}\sqrt{2}$. Assume, that $y_3 \leq y_0$. The term coming from the integration in (26) is not larger then $g_1\delta^{2/3}$, see (16), (18) and (19). As the amplitude of u_1 decreases, the second term in the solution is bounded by $|f_0 - \hat{f}_\delta(0)|$. Thus

$$\hat{f}(y_3) \leq \frac{\delta^{2/3}}{\sqrt{2}} + r(\delta)\,.$$

The two functions are positive, so the difference between them for any y is equal to, or lesser then $\max\{\hat{f}(y), \hat{f}_\delta(y)\}$. They are also decreasing, hence such bound for $|\hat{\eta}|$ is valid for all arguments larger then the one used in estimation. In that way we obtain the relation

$$|\hat{\eta}(y)| < \frac{\delta^{2/3}}{\sqrt{2}} + r(\delta)\,, \tag{27}$$

valid for all $y > 0$. If $y_3 > y_0$, the above estimate remains valid. To see this, note that the previous bound for $\hat{\eta}$ holds for all $y < y_0$ as $\hat{f}_\delta(y_0) > \delta/\sqrt{2}$. For $y > y_0$ we have the identity $\hat{f}_\delta(y) = \hat{\eta}(y)$, hence $\hat{\eta}$ decreases. This completes the proof of one of the main results of this paper: the Q-ball solutions in the regularized signum-Gordon model approach the solution of the "sharp" model uniformly. In consequence, the charge and energy computed in regularized model tend to the value known from the original one as $\delta \to 0$.

5.4. The limit $\delta \to 0$ for energy and charge

The numerical results from Sec. 3 point to the agreement between the relation $E(Q)$ in both models of interest. Now we can show, that this is not an accidental coincidence. The fact, that the integration $\int d^n x\ f_\delta^2$ approaches the value known from the signum-Gordon model follows immediately from the uniform convergence of the functions $\hat{f}_\delta$. It is natural to write the result of the integration in the form $q_0 + q(\delta)$, where the first term is the limiting value, the second reports on the δ-dependent corrections. Plugging this into the original formula for charge (2) and trading δ for the original parameters of the model we get the formula

$$Q = \frac{\pi\lambda^2}{\omega^{n+3}}\left(q_0 + q\left(\frac{2\varepsilon\omega^2}{\lambda}\right)\right)\,. \tag{28}$$

In the leading order it is the same formula as in the "sharp" model, the effect of the regularization is negligible both for large charges (small ω) and tiny regularization parameter ε.

The same result is true for energy. However, to see this, more work is needed. In terms of $\hat{f}_\delta$ the energy functional has the following form

$$E = \frac{\pi\lambda^2}{\omega^{n+2}} \int dr\ r^{n-1} \left[(\hat{f}_\delta{}')^2 + \hat{f}_\delta^2 + 2\left(\sqrt{\hat{f}_\delta^2 + \delta^2} - \delta \right) \right] . \tag{29}$$

First, the derivative $\hat{f}_\delta'$ approaches $\hat{f}'$; their difference $\hat{\eta}'$ may be bounded analogously to $\hat{\eta}$. Next, we deal with the potential energy of the field, as the dependence on δ is explicit there. We separate the δ independent part in this integral

$$\int dy\, y^{n-1} \left(\sqrt{\hat{f}_\delta^2 + \delta^2} - \delta \right) = \int dy\, y^{n-1} \hat{f}_\delta - 2\delta \int dy\, y^{n-1} \frac{\hat{f}_\delta}{\sqrt{\hat{f}_\delta^2 + \delta^2} + \delta + \hat{f}_\delta}$$

and show that the term depending explicitly on δ tends to zero. In the above formula the positivity of $\hat{f}_\delta$ is taken into account. Let us split the area of integration into two parts. First, we treat the integration in a compact volume

$$2\delta \int\limits_0^{y_0} dy\, y^{n-1} \frac{\hat{f}_\delta}{\sqrt{\hat{f}_\delta^2 + \delta^2} + \delta + \hat{f}_\delta} < 2\delta \int\limits_0^{y_0} dy\, y^{n-1} \frac{1}{2} \to 0\,.$$

The integration in the remaining volume is also negligible

$$0 \le 2\delta \int\limits_{y_0} dy\, y^{n-1} \frac{\hat{f}_\delta}{\sqrt{\hat{f}_\delta^2 + \delta^2} + \delta + \hat{f}_\delta} < 2\delta \int\limits_{y_0} dy\, y^{n-1} \frac{\hat{f}_\delta}{2\delta} \to \int\limits_{y_0} dy\, y^{n-1} \hat{f} = 0\,,$$

as expected.

Denoting the results of integrations in the energy definition with $\kappa_0 + \kappa(\delta)$ (analogously to the results of integration in charge definition) we obtain

$$E = \frac{\pi\lambda^2}{\omega^{n+2}} \left(\kappa_0 + \kappa\left(\frac{2\omega^2\varepsilon}{\lambda} \right) \right) . \tag{30}$$

Again, the formula in the leading order is the same as in the model without regularization. This explains the agreement seen in Fig. 3 — the dependence on regularization parameter ε practically factors out in the relation $E(Q)$.

6. The absolute stability of Q-balls

As already mentioned, the potential in (1) does not fall into a class of "acceptable" ones. For the class Coleman showed in [1] that the Q-ball solutions are absolutely stable, *i.e.* for a given charge value no configuration can have a lesser energy. The status of Q-balls in the regularized signum-Gordon model is at the moment unclear. In this section we are about to adapt the Coleman's proof to the theory set by (1). To this end we follow closely his arguments. As originally, our proof is done in three space dimensions.

To begin with, we define the Q-ball initial data. A set of initial data is said to be of this type if the spatial distribution of the field is given with a real, positive, spherically symmetric and monotone decreasing to zero function F. The condition for time derivative is $\partial_t \Phi(t=0,r) = i\omega F(r)$, and ω is a positive constant. The first step in the proof is very general and we just straightforwardly quote it. It states, that for any set of initial values there exists a set of Q-ball type having the same charge Q and equal or lesser value of energy E. As a result, we are allowed to constrain the investigation to the energy functional written in the form

$$E_Q = \int d^3x \left[(\nabla F)^2 + U(F)\right] + \frac{Q^2}{I}\,, \tag{31}$$

where F is a function giving the spatial distribution of the initial data of Q-ball type and $I[F] = \int d^3x F^2$,

$$U(F) = \lambda\left(\sqrt{F^2+\varepsilon^2} - \varepsilon\right)\,.$$

In this form the energy is a function of F and Q is a parameter, $\omega = Q/I$. The aim of the proof is to show, that the minimum of the functional may be reached. Before we proceed, let us discuss the definition of an "acceptable" potential. A field potential U is "acceptable", if

(1) $U(0) = 0$ and U is positive everywhere else. U is twice continuously differentiable, $U'(0) = 0$ and $U''(0) = \mu^2$.

(2) The minimum of U/F^2 is attained for some $F_0 \neq 0$.

(3) There exist three positive numbers a, b and $c > 2$, such that

$$\tfrac{1}{2}\,\mu^2 F^2 - U(F) \leq \min(a, b|F|^c)\,. \tag{32}$$

The signum-Gordon model spoils all these three conditions. Its regularized version fails to satisfy the second and the third point. As for the second condition, one can say, that the minimum in both models is attained for $\Phi = \infty$. Instead of this requirement it suffices, that for some Q there exists a function F, for which $\sqrt{2}\mu Q > E$ (for our convention in Lagrangian $\sqrt{2}$ appears occasionally). In the regularized model the Q-ball solutions meet this criterion for charges large enough. It follows from the relation $E(Q)$ in the model with "sharp" potential [2]

$$E \sim Q^{\frac{n+2}{n+3}}\,.$$

The meaning of this relation in the regularized potentials is explained in Sec. 5.4. The third condition for acceptability of the potential is a technical one, useful for some estimates. Happily, we are able to bypass the requirement without any harm to the proof.

Let us define:

$$K[F] = \int d^3x \ \left(\nabla F\right)^2 \,,$$

$$V[F] = \int d^3x \ U(F) = \lambda \int d^3x \left(\sqrt{F^2+\varepsilon^2} - \varepsilon\right)\,,$$

and

$$W[F] = U[F] - \tfrac{1}{2}\,\mu^2 I[F]\,,$$

with $\mu^2 = \lambda/\varepsilon$. This quantity satisfies a nice identity

$$W[F] = -\frac{\mu^2}{2\lambda^2}\int d^3x \ U^2(F)\,. \tag{33}$$

Hence, $W[F]$ is negative for any F. Two decompositions of energy are useful. The first of them is given by (31), the second one is

$$E_Q = K + \frac{\mu^2}{2} I + W + \frac{Q^2}{I}\,. \tag{34}$$

Now we discuss a meson argument: any spatial distribution of the field vanishing in the infinity may be modified by adding a function h of compact support

$$F(r) \to F(r) + L^{-3/2} h\left(\frac{r-d}{L}\right)\,. \tag{35}$$

If L and d are taken large enough the integrals $K[F]$ and $W[F]$ stay unaffected and simultaneously $I[F]$ increases by a constant amount. It may be surprising, that this is true for the regularized potential, no matter how small δ is. This argument however does not work in the case of the "sharp" potential. Equation (34) may be written in the form

$$E_Q - K - W = \frac{\mu^2}{2} I + \frac{Q^2}{I}\,. \tag{36}$$

The r.h.s. has a minimal value $\sqrt{2}\mu Q$ for $I = \sqrt{2}Q/\mu$. Hence, we can arrange to add ΔQ to the charge value and $\sqrt{2}\mu\Delta Q$ to the energy. Consequently if $\inf E_Q < \sqrt{2}\mu Q$, there must exist a minimal charge $Q_{\min} \geq 0$ for which this inequality holds.

Consider a sequence of functions $\{F_i\}_{i=1}^{\infty}$ for which $\lim E_Q[F_i] = \inf E$. The existence of such a sequence is guaranteed by the definition of infimum. We can choose F_i to be positive, spherically symmetric and monotone decreasing to zero functions. K is a positive quantity bounded above by the energy. Thus there is a subsequence such that K has a limit. The same reasoning holds for V. If E, K and V converge, so does also I and W. The limiting values are denoted with tildes, *e.g.*

$$\tilde{K} = \lim_{i\to\infty} K[F_i]\,.$$

What is more, we can choose such subsequence, that is bounded uniformly in i for all the quantities: E, K, V and W. We assume this to be done. We will need the inequality

$$\tilde{I} > \frac{\sqrt{2}Q}{\mu}\,. \tag{37}$$

Assume, that $\tilde{I} < \sqrt{2}Q/\mu$. Then adding mesons at infinity to the sequence, so that $I = \sqrt{2}Q/\mu$ for almost all F_i, results in a sequence converging to energy lower then $\tilde{E} = \inf E_Q$, absurdity. If we assume $\tilde{I} = \sqrt{2}Q/\mu$, then $\tilde{W} < -\tilde{K}$ (as $\tilde{E} < \sqrt{2}\mu Q$). The scaling transformation

$$F_i(y) \to F_i\left(y(1+\alpha)\right)\,,$$

with small α parameter. Then the energy transforms

$$\tilde{E} \to \tilde{E} - \alpha\tilde{K} - 3\alpha\tilde{W} + \dots\,,$$

where the omitted terms are of the order of α^2, $\tilde{I}$ is in its stationary point and does not contribute in the first order. Taking α small and negative we could in this way again lower the energy below its infimum. It is convenient to introduce functions $f_i(r) = rF_i(r)$, where r is the radial coordinate. With no additional prerequisites we are able to show, that this functions form a uniformly bounded sequence of equicontinious functions. To see this we note, that

$$K[F_i] = 4\pi \int dr\ \left(\frac{df_i}{dr}\right)^2$$

and

$$I[F_i] = 4\pi \int dr\, f_i^2\,.$$

By Schwarz inequality we obtain

$$f_i^2(r) = -\frac{1}{2}\int\limits_r^\infty dr\, f_i\frac{df_i}{dr} \le \frac{1}{8\pi}\sqrt{I[F_i]K[I_i]} \tag{38}$$

and

$$|f_i(r_1) - f_i(r_2)| = \left|\int\limits_{r_1}^{r_2} dr\frac{df_i}{dr}\right| \le \sqrt{\frac{[K[F_i]|r_1 - r_2\,|}{8\pi}}\,. \tag{39}$$

This inequalities legitimate the above statement. Hence, by Ascoli's theorem there exists a subsequence of $\{f_i\}$ which is pointwise convergent everywhere and uniformly convergent on any finite interval. This implies the same for $\{F_i\}$, except for $r = 0$. The limit of the convergent subsequence is denoted with $\tilde{F}$. The task is now to show, that $E_Q[\tilde{F}] = \tilde{E}$.

K defines a Hilbert — space norm under which the Fs are bounded family of vectors. Such a bounded family has always a weakly converging subsequence. The norm of the weak limit is always less then or equal to the limit of norms. Thus,

$$K[\tilde{F}] \le \tilde{K}\,. \tag{40}$$

Analogously,

$$I[\tilde{F}] \le \tilde{I}\,. \tag{41}$$

As for W we take two positive numbers $0 < r_\_ < r_+$ and keeping in mind the relation (33) and (38) we note, that

$$\frac{2\pi\mu^2}{\lambda^2}\int\limits_0^{r_\_} dr\; r^2U^2(F_i) \le \frac{2\pi\mu^2}{\lambda^2}\int\limits_0^{r_\_} dr f_i^2(r) \le \frac{\mu^2}{4\lambda^2}\sqrt{K[F_i]I[F_i]}r_\_$$

and

$$\frac{2\pi\mu^2}{\lambda^2}\int\limits_{r_+}^\infty dr\; r^2U^2(F_i) \le \frac{2\pi\mu^2}{\lambda^2}\int\limits_{r_+}^\infty dr\; r^2F_iU(F_i) \le \frac{\mu^2\sup f_i}{2\lambda^2 r_+}V[F_i]\,.$$

Thus, taking $r_\_$ and r_+ appropriately we can make the above integrals as small as we want. As F_i converges uniformly to $\tilde{F}$ in this interval, we get

$$\lim_{i\to\infty} W[F_i] = W[\tilde{F}]\,.$$

Finally, we show $\tilde{I} = I[\tilde{F}]$. Assume, that $I[\tilde{F}] < \tilde{I}$. Then, by adding meson at infinity we can construct a new function F' such that $W[F'] = W[\tilde{F}]$,

$K[F'] = K[\tilde{F}]$ and $I[F']$ anywhere in between $I[\tilde{F}]$ and $\tilde{I}$. Using (37) we can take, that

$$\tilde{I} > I[F'] > \frac{\sqrt{2}Q}{\mu}\,.$$

This implies, that

$$\frac{Q^2}{I[F']} + \frac{\mu^2}{2} I[F'] < \frac{Q^2}{\tilde{I}} + \frac{\mu^2}{2}\tilde{I}\,.$$

Together with equation (36) it results in a contradiction: $E_Q[F'] < \tilde{E}$. Thus, $\tilde{I} = I[\tilde{F}]$. By (34) and (40) and taking into account the last result we obtain $E_Q[\tilde{F}] \leq \tilde{E}$. Since $E_Q[\tilde{F}] < \tilde{E}$ is impossible, we conclude, that $E[\tilde{F}] = \tilde{E}$. Having granted the existence of the minimum of the functional, we are legitimate to claim that it corresponds to the solution of the equation

$$\frac{\delta E_Q[F]}{\delta F} = 0\,,$$

i.e. equation (4).

The absolute stability of the Q-balls in the regularized models suggests the stability of the Q-balls in the signum-Gordon model. It follows from a simple argument. Consider a set of initial data given by a function F to the charge value Q and the energy E' lower then the energy E of the corresponding Q-ball in the signum-Gordon model. The energy E obeys the relation (8). We can plug F into the energy functional of an regularized model (31) with a parameter ε and the charge Q. As $|F| \geq \sqrt{F^2+\varepsilon^2} - \varepsilon$, the energy functional with any regularization yields then a smaller value then in the case of the "sharp" potential. For ε small enough the energy of the corresponding Q-ball may be as close to E as needed, see Sec. 5.4. Hence, for ε tiny enough the function F results in the energy value smaller then that of the related Q-ball, what has been already proven impossible.

7. Conclusions

We have shown, that the Q-balls are physically relevant solutions of the regularized signum-Gordon model in three spatial dimensions. They are absolutely stable for large values of charge. What is more, we have demonstrated that Q-balls in the regularized signum-Gordon model approach the solution known from the "sharp" model. It holds both for profile functions and their global characteristics and is well illustrated by the numerical solutions. For the first time the parabolic approach to the vacuum known in the signum-Gordon model emerged in the limiting procedure. The stability of the solutions in the regularized model guarantees the stability of the Q-balls in the original model.

We have shown, that the regularization does not change some characteristics of the model drastically. We hope that parallel exploration of both models will shed light on some tough issues, let us mention only the propagation of a perturbation in the model with the "sharp" potential.

Finally, let us point to a very intriguing direction of investigation, *i.e.* quantization of the models. The question about the role played by the quantum counterparts of Q-balls is both intriguing and hard.

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