THE MINIMUM LENGTH PROBLEM OF LOOP QUANTUM COSMOLOGY

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(Received September 23, 2009)

The appearance of the Big Bounce (BB) in the evolution of the Universe is analyzed in the setting of loop quantum cosmology (LQC). Making use of an idea of a minimum length turns classical Big Bang into BB. We argue why the spectrum of the kinematical area operator of loop quantum gravity cannot be used for the determination of this length. We find that the fundamental length, at the present stage of development of LQC, is a free parameter of this model.

PACS numbers: 98.80.Qc, 04.60.Pp

1. Introduction

Observational cosmology strongly suggests that our Universe emerged from a state with extremely high energy densities of physical fields, called the initial Big Bang *singularity*. Most of all models of the Universe obtained within the general relativity (GR) also predict the initial singularity [1–4]. It is commonly believed that the singularity may be understood in a theory which unifies gravity and quantum physics. Recent analyses done within the loop quantum cosmology (LQC) concerning homogeneous isotropic universes of the Friedmann–Robertson–Walker (FRW) type, strongly suggest that the evolution of these universes does not suffer from the classical singularity: the *Big Bang* is replaced by *Big Bounce* (with finite energy density of matter) owing to strong quantum effects at the Planck scale [5–9].

The goal of this paper is the *revision* of the foundation of LQC concerning the *minimum length*, μ_o, which is responsible for the resolution of the cosmological singularity. We would like to attract an attention of the LQC community to the problem of the *determination* of μ_o. It has basic meaning since its numerical value specifies the energy scale of the Big Bounce transition. At the present stage of development of LQC the minimum length is a *free parameter*.

For simplicity of exposition we restrict ourselves to the quantization problem of the flat FRW model with massless scalar field. This model of the Universe unavoidably includes the initial cosmological singularity and has been intensively studied recently within LQC.

2. Hamiltonian

The gravitational part of the classical Hamiltonian, H_g, of GR is a linear combination of the first-class constraints, and reads [10–12]

$$H_g := \int_{\Sigma} d^3x \left(N^i C_i + N^a C_a + N C\right) , \tag{1}$$

where Σ is the space-like part of space-time $\mathbb{R} \times \Sigma$, (N^i, N^a, N) denote Lagrange multipliers, (C_i, C_a, C) are the Gauss, diffeomorphism and scalar constraint functions. In our notation $(a, b = 1, 2, 3)$ are spatial and $(i, j, k = 1, 2, 3)$ internal SU(2) indices. The constraint functions must satisfy a specific algebra. It is known that this algebra (for constraints smeared with test functions) is not a Lie algebra, but a Poisson algebra because it includes structure functions instead of structure constants (see, *e.g.* [10]).

In the case of flat FRW type universe with massless scalar field, and with fixed local gauge and diffeomorphism freedom, the classical Hamiltonian reduces to the scalar constraint and can be shown (see, *e.g.* [6]) to be

$$H_g = -\gamma^{-2} \int_{\mathcal{V}} d^3x \, e^{-1} \varepsilon_{ijk} E^{aj} E^{bk} F^i_{ab} \, , \tag{2}$$

where γ is the Barbero–Immirzi parameter, $\mathcal{V} \subset \Sigma$ is an elementary cell[1], $e := \sqrt{|\det E|}$, ε_{ijk} is the alternating tensor, E^a_i is a densitised vector field, and where F^i_{ab} is the curvature of an SU(2) connection A^i_a.

The resolution of the singularity, obtained within LQC, is based on rewriting the curvature F^k_{ab} in terms of holonomies around loops. The curvature F^k_{ab} can be determined [6] by making use of the formula

$$F^k_{ab} = -2 \lim_{\mathrm{Ar}\,\Box_{ij} \to 0} \mathrm{Tr} \left(\frac{h^{(\lambda)}_{\Box_{ij}} - 1}{\lambda^2 V_o^{2/3}} \right) \tau^k \, {}^o\omega^i_a \, {}^o\omega^j_a \, , \tag{3}$$

[1] In the case Σ is a non-compact manifold one introduces compact submanifold $\mathcal{V}$ to give precise mathematical meaning of the integrals.

where

$$h^{(\lambda)}_{\Box_{ij}} = h^{(\lambda)}_i h^{(\lambda)}_j (h^{(\lambda)}_i)^{-1} (h^{(\lambda)}_j)^{-1} \tag{4}$$

is the holonomy of the gravitational connection around the square loop $\Box_{ij}$ whose edges are parallel to the i- and j-directions and of coordinate length $\lambda V_o^{1/3}$ with respect to the flat fiducial metric ${}^o q_{ab} := \delta_{ij}\, {}^o\omega^i_a\, {}^o\omega^j_a$; fiducial triad ${}^o e^a_k$ and co-triad ${}^o\omega^k_a$ satisfy ${}^o\omega^i_a\, {}^o e^a_j = \delta^i_j$; spatial part of FRW metric is $q_{ab} = a^2(t)\, {}^o q_{ab}$; $\mathrm{Ar}\,\Box_{ij}$ denotes the area of the square; $V_o = \int_{\mathcal{V}} \sqrt{{}^o q} d^3x$ is the fiducial volume of $\mathcal{V}$; in what follows we set $V_o = 1$ as its value is not essential for our analysis.

The holonomy along straight edge of length λ in the k-direction (in the $j = 1/2$ representation of SU(2)) may be found [6] to be

$$h^{(\lambda)}_k(c) = \cos(\lambda c/2)\, \mathbb{I} + 2\, \sin(\lambda c/2)\, \tau_k\,, \tag{5}$$

where $\tau_k = -i\sigma_k/2$ (σ_k are the Pauli spin matrices). It is clear that matrix elements of (5) can be rewritten in terms of $\exp(i\lambda c/2)$ which we denote by $N_\lambda(c)$.

In what follows we apply the 'old' quantization scheme [6], despite the fact that the 'improved' scheme [7] is commonly used by LQC community. The reason is that mathematics underlying the old scheme has been presented *clearly* in a comprehensive paper [5]. However, our results concern both methods.

One can show [6] that H_g may be rewritten as

$$H_g = \lim_{\lambda \to 0}\; H^{(\lambda)}_g\,, \tag{6}$$

where

$$H^{(\lambda)}_g = -\frac{\mathrm{sgn}(p)}{2\pi G\gamma^3\lambda^3} \sum_{ijk} \varepsilon^{ijk} \mathrm{Tr}\left(h^{(\lambda)}_i h^{(\lambda)}_j \left(h^{(\lambda)}_i\right)^{-1} \left(h^{(\lambda)}_j\right)^{-1} h^{(\lambda)}_k \left\{ (h^{(\lambda)}_k)^{-1}, V \right\} \right), \tag{7}$$

and where $V = |p|^{3/2}$ is the volume of the elementary cell $\mathcal{V}$. The conjugate variables c and p satisfy $\{c, p\} = 8\pi G\gamma/3$. They determine *connections* A^k_a and density weighted *triads* E^a_k due to the relations $A^k_a = {}^o\omega^k_a\, c$ and $E^a_k = {}^o e^a_k \sqrt{q_o}\, p$. However, c and p are not elementary variables in (7). The elementary functions (variables) are chosen to be *holonomies* (described in terms of N_μ) and *fluxes* (proportional to p).

The classical total Hamiltonian for FRW universe with a massless scalar field, ϕ, reads

$$H = H_g + H_\phi = 0\,, \tag{8}$$

where H_g is defined by (6). The Hamiltonian of the scalar field is known to be: $H_\phi = p_\phi^2 |p|^{-3/2}$, where ϕ and p_ϕ are the elementary variables satisfying $\{\phi, p_\phi\} = 1$. The relation $H = 0$ defines the *physical* phase space of considered gravitational system with constraints.

3. Quantization

In the Dirac quantization [13,14] we find a kernel of the quantum operator $\hat{H}$ corresponding to H, *i.e.*

$$\hat{H}\,\Psi = 0\,, \tag{9}$$

(since the classical Hamiltonian is a constraint of the system), and try to define a scalar product on the space of solutions to (9). This gives a starting point for the determination of the physical Hilbert space $\mathcal{H}_{\rm phys}$.

3.1. Kinematics

The classical elementary functions satisfy the relation

$$\{p, N_\lambda\} = -i\frac{4\pi G\gamma}{3}\lambda N_\lambda\,, \tag{10}$$

where G is the Newton constant. Quantization of the algebra (10) is done by making use of the prescription

$$\{\cdot,\cdot\} \longrightarrow \frac{1}{i\hbar}\,[\cdot,\cdot]\,. \tag{11}$$

The basis of the representation space is chosen to be the set of eigenvectors of the momentum operator [5] and is defined by

$$\hat{p}\,|\mu\rangle = \frac{4\pi\gamma l_p^2}{3}\,\mu\,|\mu\rangle\,, \qquad \mu \in \mathbb{R}\,, \tag{12}$$

where $l_p^2 = G\hbar$. The operator corresponding to N_λ acts as follows

$$\hat{N}_\lambda\,|\mu\rangle = |\mu+\lambda\rangle\,. \tag{13}$$

The quantum algebra corresponding to (10) reads

$$\frac{1}{i\hbar}[\hat{p}, \hat{N}_\lambda]\,|\mu\rangle = -i\frac{4\pi G\gamma}{3}\,\lambda\,\hat{N}_\lambda\,|\mu\rangle\,. \tag{14}$$

The carrier space, $\mathcal{F}_g$, of the representation (14) is the space spanned by $\{|\mu\rangle,\ \mu \in \mathbb{R}\}$ with the scalar product defined as

$$\langle\mu|\mu'\rangle := \delta_{\mu,\mu'}\,, \tag{15}$$

where $\delta_{\mu,\mu'}$ denotes the Kronecker delta.

The completion of $\mathcal{F}_g$ in the norm induced by (15) defines the Hilbert space $\mathcal{H}^g_{\rm kin} = L^2(\mathbb{R}_{\rm Bohr}, d\mu_{\rm Bohr})$, where $\mathbb{R}_{\rm Bohr}$ is the Bohr compactification of the real line and $d\mu_{\rm Bohr}$ denotes the Haar measure on it [5]. $\mathcal{H}^g_{\rm kin}$ is the kinematical space of the gravitational degrees of freedom. The kinematical Hilbert space of the scalar field is $\mathcal{H}^\phi_{\rm kin} = L^2(\mathbb{R}, d\phi)$, and the operators corresponding to the elementary variables are

$$(\hat{\phi}\psi)(\phi) = \phi\psi(\phi)\,, \qquad \hat{p}_\phi\psi = -i\hbar\frac{d}{d\phi}\psi\,. \tag{16}$$

The kinematical Hilbert space of the gravitational field coupled to the scalar field is defined to be $\mathcal{H}_{\rm kin} = \mathcal{H}^g_{\rm kin} \otimes \mathcal{H}^\phi_{\rm kin}$.

3.2. Dynamics

The resolution of the singularity [5–9] is mainly due to the peculiar way of defining the quantum operator corresponding to H_g. Let us consider this issue in more details.

Using the prescription $\{\cdot,\cdot\} \to \frac{1}{i\hbar}[\cdot,\cdot]$ and specific factor ordering of operators, one obtains from (7) a quantum operator corresponding to $H_g^{(\lambda)}$ in the form [5]

$$\hat{H}_g^{(\lambda)} = \frac{i\,{\rm sgn}(p)}{2\pi l_p^2\gamma^3\lambda^3}\sum_{ijk}\varepsilon^{ijk}{\rm Tr}\left(\hat{h}_i^{(\lambda)}\hat{h}_j^{(\lambda)}\left(\hat{h}_i^{(\lambda)}\right)^{-1}\left(\hat{h}_j^{(\lambda)}\right)^{-1}\hat{h}_k^{(\lambda)}\left\{(\hat{h}_k^{(\lambda)})^{-1},\hat{V}\right\}\right). \tag{17}$$

One can show [5] that (17) can be rewritten as

$$\hat{H}_g^{(\lambda)}|\mu\rangle = \frac{3}{8\pi\gamma^3\lambda^3 l_p^2}\Big(V_{\mu+\lambda} - V_{\mu-\lambda}\Big)(|\mu+4\lambda\rangle - 2|\mu\rangle + |\mu-4\lambda\rangle)\,, \tag{18}$$

where $|\mu\rangle$ is an eigenstate of $\hat{p}$ defined by (12), and where V_μ is an eigenvalue of the volume operator corresponding to $V = |p|^{3/2}$ which reads

$$\hat{V}|\mu\rangle = \Big(\frac{4\pi\gamma|\mu|}{3}\Big)^{3/2} l_p^3\,|\mu\rangle =: V_\mu\,|\mu\rangle\,. \tag{19}$$

The quantum operator corresponding to H_g is defined to be [5,6]

$$\hat{H}_g := \hat{H}_g^{(\lambda)}\,|_{\lambda=\mu_o}\,, \qquad \text{where} \qquad 0 < \mu_o \in \mathbb{R}\,. \tag{20}$$

Comparing (20) with (6), and taking into account (3) we can see that the area of the square $\Box_{ij}$ is not shrunk to *zero*, as required in the definition of the classical curvature (3), but determined at the *finite* value of the area.

The mathematical justification proposed in [5,6] for such regularization is that one cannot define the *local* operator corresponding to the curvature F^k_{ab} because the 1-parameter group $\hat{N}_\lambda$ is not weakly continuous at $\lambda = 0$ in $\mathcal{F}_g$ (dense subspace of $\mathcal{H}^g_{\rm kin}$). Thus, the limit $\lambda \to 0$ of $\hat{H}^{(\lambda)}_g$ does not exist. To determine μ_o one proposes in [5–7] the procedure which is equivalent to the following: We find that the area of the face of the cell $\mathcal{V}$ orthogonal to specific direction is $\mathrm{Ar} = |p|$. Thus the eigenvalue problem for the corresponding *kinematical* operator of an area $\widehat{\mathrm{Ar}} := |\hat{p}|$, due to (12), reads

$$\widehat{\mathrm{Ar}}\,|\mu\rangle = \frac{4\pi\gamma l_p^2}{3}\,|\mu|\,|\mu\rangle =: \mathrm{ar}(\mu)\,|\mu\rangle\,, \qquad \mu \in \mathbb{R}\,, \tag{21}$$

where $\mathrm{ar}(\mu)$ denotes the eigenvalue of $\widehat{\mathrm{Ar}}$ corresponding to the eigenstate $|\mu\rangle$. On the other hand, it is known that in LQG the *kinematical* area operator has *discrete* eigenvalues [15, 16] and the smallest nonzero one, called an area gap Δ, is given by $\Delta = 2\sqrt{3}\,\pi\gamma l_p^2$. To identify μ_o one postulates in [6] that μ_o is such that $ar(\mu_o) = \Delta$, which leads to $\mu_o = 3\sqrt{3}/2$. It is argued [5–8] that one cannot squeeze a surface to the zero value due to the existence in the Universe of the *minimum* quantum of area. This completes the justification for the choice of the expression defining the quantum Hamiltonian (20) offered by LQC.

It is interesting to notice that for the model considered here (defined on one-dimensional constant lattice) the existence of the minimum area leads to the reduction of the non-separable space $\mathcal{F}_g$ to its *separable* subspace. It is so because due to (13) we have

$$\hat{N}_{\mu_o}\,|\mu\rangle = |\mu + \mu_o\rangle\,, \tag{22}$$

which means that the action of this operator does not lead outside of the space spanned by $\{|\mu + k\,\mu_o\rangle,\ k \in \mathbb{Z}\}$, where $\mu \in \mathbb{R}$ is fixed.

Finally, one can show (see, *e.g.* [5, 6]) that the equation for quantum dynamics, corresponding to (9), reads

$$\begin{aligned} B(\mu)\partial^2_\phi\psi(\mu,\phi) \;&-\; C^+(\mu)\psi(\mu+4\mu_o,\phi) - C^-(\mu)\psi(\mu-4\mu_o,\phi) \\ &-\; C^0(\mu)\psi(\mu,\phi) = 0\,, \end{aligned} \tag{23}$$

where

$$B(\mu) := \left(\frac{2}{3\mu_o}\right)^6 \left[|\mu+\mu_o|^{3/4} - |\mu-\mu_o|^{3/4}\right]^6, \; C^0(\mu) := -C^+(\mu) - C^-(\mu)\,, \tag{24}$$

$$C^+(\mu) := \frac{\pi G}{9|\mu_o|^3} \left|\, |\mu+3\mu_o|^{3/2} - |\mu+\mu_o|^{3/2}\right|, \; C^-(\mu) := C^+(\mu-4\mu_o)\,. \tag{25}$$

Equation (23) has been derived formally by making use of states which belong to $\mathcal{F} := \mathcal{F}_g \otimes \mathcal{F}_\phi$, where $\mathcal{F}_g$ and $\mathcal{F}_\phi$ are dense subspaces of the kinematical Hilbert spaces $\mathcal{H}^g_{\text{kin}}$ and $\mathcal{H}^\phi_{\text{kin}}$, respectively. The space $\mathcal{F}$ provides an arena for the derivation of quantum dynamics. However, the *physical* states are expected to be in $\mathcal{F}^\star$, the algebraic dual of $\mathcal{F}$ (see, *e.g.* [5, 6] and references therein). It is known that $\mathcal{F} \subset \mathcal{H}_{\text{kin}} \subset \mathcal{F}^\star$. Physical states are expected to have the form $\langle \Psi| := \sum_\mu \psi(\mu,\phi)\langle\mu|$, where $\langle\mu|$ is the eigenbras of $\hat{p}$. One may give the structure of the Hilbert space to some subspace of $\mathcal{F}^\star$ (constructed from solutions to (23)) by making use of the group averaging method [17, 18] and obtain this way the *physical* Hilbert space $\mathcal{H}_{\text{phys}}$.

The singularity resolution refers, first of all, to the behavior of the expectation value of the matter density operator. Numerical calculations have shown [7] that the mean value of this operator is bounded from above on the states (vectors of the physical Hilbert space) which are semi-classical asymptotically. It is suggested in [8] that the bounce may occur for the states which are more general than semi-classical at late times, which demonstrates robustness of LQC results. Quantum evolution, described by (23), is deterministic across the bounce region. The Universe undergoes a bounce during the evolution from pre-Big Bang epoch to post-Big Bang epoch. These are main highlights of LQC (see, *e.g.* [19] for a complete list).

The argument ϕ in $\psi(\mu,\phi)$ is interpreted as an evolution parameter, μ is regarded as the physical degree of freedom. Let us examine the role of the parameter μ_o in (23). First of all, its presence causes that (23) is a difference-differential equation so its solution should be examined on a lattice. It is clear that some special role must be played by $\mu_o = 0$ as the coefficient functions of the equation, defined by (24) and (25), are singular there. One can verify [6] that as $\mu_o \to 0$ the equation (23) turns into the Wheeler–DeWitt equation

$$B(\mu)\frac{\partial^2}{\partial\phi^2}\psi(\mu,\phi) - \frac{16\pi G}{3}\frac{\partial}{\partial\mu}\sqrt{\mu}\frac{\partial}{\partial\mu}\psi(\mu,\phi) = 0\,, \text{ with } B(\mu) := \left|\frac{4\pi\gamma G\hbar}{3}\mu\right|^{-3/2}. \tag{26}$$

Equation (23) is not specially sensitive to any other value of μ_o. Thus, the determination of the numerical value of this parameter by making use of the mathematical structure of (23) seems to be impossible.

4. Minimum length problem

The singularity resolution offered by LQC, in the context of flat FRW universe, is a striking result. Let us look at the key ingredients of the construction of LQC which are responsible for this long awaited result:

Discussing the mathematical structure of the constraint equation we have found that μ_o must be a non-zero if we wish to deal with the regular (23) instead of the singular (26). However, the numerical value of μ_o cannot be determined from equation (23). It plays the role of a *free* parameter if it is not specified.

The parameter μ_o enters the formalism due to the representation of the curvature of the connection F^k_{ab} via the holonomy around a *loop* (3). The smaller the loop the better approximation we have. The size of the loop, μ_o, determines the quantum operator corresponding to the modified gravitational part of the Hamiltonian (20). One may determine μ_o by making use of an *area* of the loop (used in fact as a technical tool). Thus, the *spectrum* of the quantum operator corresponding to an area operator, $\widehat{\mathrm{Ar}}$, seems to be a suitable source of information on the possible values of μ_o. Section 3 shows explicitly that the construction of the quantum level is heavily based on the *kinematical* ingredients of the formalism. Thus, it is natural to explore the kinematical $\widehat{\mathrm{Ar}}$ of LQC. However, its spectrum (21) is *continuous* so it is useless for the determination of μ_o. On the other hand, the spectrum of kinematical $\widehat{\mathrm{Ar}}$ of LQG is *discrete* [15, 16]. Thus, it was tempting to use such a spectrum to fix μ_o postulating that the minimum *quantum* of area defines the minimum area of the loop defining (20). This way μ_o has been fixed.

The physical justification, however, for such procedure is doubtful because LQC *is not* the cosmological sector of LQG. The relationship between LQG and LQC, at the formalisms level, has been examined recently [20]: LQC is a quantization method *inspired* by LQG (a field theory with infinitely many degrees of freedom) used to the quantization of the simplest models of the Universe (with finitely many degrees of freedom) with high symmetries.

The inspiration consists mainly in applying the two ingredients of LQG: *(i)* modification of F^k_{ab} by loop geometry, and *(ii)* making use of the holonomy-flux algebra. In other words, LQC has not been *derived* from LQG. The construction of LQC has been carried out by *mimicry* of the construction of LQG, but nothing more. LQG and LQC are two *different* quantum models of two *different* systems. Therefore, Eq. (20) includes an insertion by *hand* of specific properties of the spectrum of $\widehat{\mathrm{Ar}}$ from LQG into LQC [23]. After all, the area gap of the spectrum of $\widehat{\mathrm{Ar}}$ of LQG is not a fundamental constant (like the speed of light, Planck's constant, Newton's constant) so its use in the context of LQC has poor physical justification.

The singularity problems should be analyzed in terms of the Dirac observables and physical states [20]. In our recent papers we solve the constraints already at the classical level, make the identification of the Dirac observables and find the physical phase space before the quantization process. Our non-standard LQC is complementary to the Dirac quantization method which underlies standard LQC. We have found that the energy density operator has a *continuous* bounded spectrum [21]. The *volume* operator has a *discrete* spectrum bounded from below [22]. A *quantum* of the volume is parameterized by the minimum length.

5. Conclusions

It is claimed (see, *e.g.* [6–8]) that the introduction of the quantum of area at the kinematical level of LQC has sound theoretical justification. We believe we have shown that it is an *ad hoc* assumption without physical justification (see [23] for another criticism of this assumption). Thus, the energy scale characteristic to the Big Bounce is unknown. Claiming that the Planck scale appears naturally in LQC is still illusive, in spite of the enthusiasm invoked by the LQC results.

An identification of the *energy scale* specific to the Big Bounce transition is a fundamental problem since it is supposed to be the energy scale for the unification of gravity with quantum physics.

The LQC calculations, done for flat FRW model with massless scalar field, have shown that making an assumption on the existence of a minimum fundamental length in quantum geometry one can impose quantum rules onto the expression for the classical constraint (Hamiltonian) in such a way that some solutions to the equation describing the evolution of the Universe lead to finite expectation value for the matter density at any value of the evolution parameter. It is an interesting result which demonstrates the powerfulness of LQC. However, further investigations are needed for finding solution to the *minimum length* problem. We suggest that the solution may come from *observational* cosmology. For instance, an identification of the microscale specific to a *foamy* structure of space would be helpful.

We are grateful to Tomasz Pawłowski and Łukasz Szulc for helpful discussions.

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