

ON THE WEYL GRAVITATIONAL CONJECTURE AND MASSIVE SPINOR THEORY

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(Received January 25, 2010)

Corrected October 1, 2014
according to Erratum *ibid.* **45**, 2021 (2014)

In the general theory of relativity, matter energy can be expressed geometrically, and Weyl put forward the hypothesis that a curved-space formulation of the Dirac equation $[i\gamma^k(\partial_k - ieA_k) - m]\psi = 0$ for a relativistic spin-1/2 field ψ would lead to a possible reinterpretation of the mass parameter m in purely geometrical terms. Here, we show how this idea can be realized in the general-relativistically covariant Dirac equation $[i\gamma^k(\partial_k - \Gamma_k - ieA_k) - m]\psi = 0$ due to Fock, where Γ_i is the spinorial connection, after rewriting m^2 as $m^2(\gamma^0)^2$ in comoving coordinates. Thus, m is replaced by the matrix mass $m\gamma^0$, which can then be set equal to $i\gamma^k\Gamma_k$ in an anti-de Sitter space-time background that can be attributed to fermionic zero-point vacuum fluctuations. This result is analogous to the reidentification of the gauge field eA_i with $\text{Im}\Gamma_i$ in a Majorana representation of the γ_i .

DOI:10.5506/APhysPolB.41.779

PACS numbers: 03.65.Sq, 04.40.-b, 11.10.Lm, 11.30.Rd

1. Introduction

This paper is concerned with the conjecture, which we shall shortly elucidate, due to Weyl[1], in the course of his attempt to relate gravitation and electromagnetism, where the idea of gauge invariance was first put forward. This symmetry expresses constancy under simultaneous transformation of the wave function ψ , obeying the Dirac equation for a spinor of rest mass m and charge e in Minkowski space,

$$\left[i\gamma^k (\partial_k - ieA_k) - m \right] \psi = 0, \quad (1)$$

and the electromagnetic four-vector potential A_i contained therein, according to (see [1] p. 331)

$$\psi \rightarrow \psi' = e^{i\theta} \psi, \quad A_i \rightarrow A'_i = A_i + \frac{1}{e} \frac{\partial \theta}{\partial x^i}, \quad (2)$$

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where $\theta(x^i)$ is an arbitrary real function of the space-time coordinates x^i . The Dirac gamma matrices satisfy the Clifford algebra

$$\{\gamma_i, \gamma_j\} = 2g_{ij}, \quad (3)$$

where g_{ij} is the four-metric. Paper [1] emphasized the fact that the exponent in the factor multiplying ψ in Eqs. (2) is purely imaginary (see [1] p. 331), the invariance under (2) acting as an adjunct to the covariance under general coordinate transformations embodied in Einstein's theory of gravitation — and also emphasized the general relativistic nature of the transformations (2), resulting from the arbitrariness in the x^i -dependence of θ (see [1] p. 331).

The starting point of the investigation [1] (see [1] p. 330) was the Dirac[2] theory (1) for the relativistic electron, in which ψ is represented as a four-component spinor that results, as is known, in a doubling of the number of energy levels (see [1] p. 331). Weyl therefore argued that, without giving up relativistic invariance, one should return to the Pauli[3] two-component theory, which, however, holds true only for massless spinors $\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ — the two two-component spinors ψ_1 and ψ_2 become linked together if the spinor is massive.

This follows essentially from the presence of off-diagonal elements in the γ_i , which occur in the kinetic term $i\gamma^k \partial_k \psi$, but not in the mass term $m\psi$. In the Weyl[1] (chiral) representation, we have

$$\gamma_0 = \begin{pmatrix} 0 & -\sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \quad \gamma_\alpha = \begin{pmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{pmatrix}, \quad (4)$$

where the generalized 2×2 Pauli matrices σ_i are defined by

$$\sigma_0 = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5)$$

As a result, Eq. (1) reduces to the two two-component equations

$$i[(\partial_0 - ieA_0) - \sigma^\alpha (\partial_\alpha - ieA_\alpha)] \psi_2 + m\psi_1 = 0 \quad (6)$$

and

$$i[(\partial_0 - ieA_0) + \sigma^\alpha (\partial_\alpha - ieA_\alpha)] \psi_1 + m\psi_2 = 0. \quad (7)$$

Thus, the spinors ψ_1 and ψ_2 only decouple if $m = 0$, in which case we also set $e = 0$ (although this does not change the argument), since no charged massless fermion is known. Equations (6) and (7) then reduce further to

$$i(\partial_0 - \sigma^\alpha \partial_\alpha) \psi_2 = 0 \quad (8)$$

and

$$i(\partial_0 + \sigma^\alpha \partial_\alpha) \psi_1 = 0, \quad (9)$$

respectively, which differ by a sign in the second term, thereby breaking the symmetry between ψ_1 and ψ_2 .

In physical terms, this means, as remarked in paper [1] (see [1] p. 332), that one has to give up the left-right chiral symmetry, since $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the representation (4), the left- and right-handed components under projection being precisely

$$\psi_L = \frac{1}{2}(1 + \gamma_5)\psi = \psi_1, \quad \psi_R = \frac{1}{2}(1 - \gamma_5)\psi = \psi_2. \quad (10)$$

Eq. (9) therefore describes a massless, left-handed neutrino, for example.

This result led Weyl to express the hope, since mass is ultimately of gravitational origin, that it would be possible, in a more complete, curved-space formulation of the Dirac equation, to replace m by a purely geometrical term (see [1] p. 331). He also emphasized, however, that quantization of the gravitational field equations themselves was a prerequisite to a complete understanding — which is still a subject of research.

The general-relativistically covariant Dirac equation, on the other hand, was developed soon thereafter by Fock[4], taking the form

$$\left[i\gamma^k (\partial_k - \Gamma_k - ieA_k) - m \right] \psi = 0, \quad (11)$$

which contains in addition to Eq. (1) the spinorial connection

$$\Gamma_k = -\frac{1}{4} \omega_{abk} \bar{S}^{ab}. \quad (12)$$

The spin connection and spin operator occurring in the definition (12) are constructed from the tetrad components t_i^a and Christoffel symbols

$$\Gamma_{jk}^l = \frac{1}{2} g^{lm} (\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}) \quad (13)$$

as

$$\omega_{abk} = -\omega_{bak} = t_{ja} t_{b;k}^j \equiv t_{ja} \left(\partial_k t_b^j + \Gamma_{kl}^j t_b^l \right) \quad (14)$$

and

$$\bar{S}^{ab} = \frac{1}{2} [\bar{\gamma}^a, \bar{\gamma}^b], \quad (15)$$

respectively. The metric g_{ij} , now describing a curved space-time, is related at every point to the Minkowski tangent space η_{ab} by

$$g_{ij} = t_i^a t_j^b \eta_{ab}, \quad \eta_{ab} = t_a^i t_b^j g_{ij}, \quad (16)$$

the tetrads obeying the orthogonality conditions

$$t_i^a t_a^j = \delta_i^j, \quad t_i^a t_b^i = \delta_b^a. \quad (17)$$

The derivation of Eq. (11) is discussed in detail in Ref. [5].

2. The evaluation of the spinorial connection

From formulae (12)–(16), we see that the ω_{abk} , and hence the Γ_k , are in general complicated functions of the metric g_{ij} , which it is therefore natural to simplify by imposing some degree of symmetry on the space-time. If the particle under consideration is regarded as a stationary extended object, then the total mass-energy is defined classically as a volume integral of the $\binom{0}{0}$ component of the Ricci tensor R_{ij} , accordingly to the formula of Nordström[6] (see also Tolman[7]),

$$m = \frac{2}{\kappa^2} \int d^3x \sqrt{-g} R_0^0, \quad (18)$$

assuming the space-time to be asymptotically flat at infinity, where $\kappa^2 \equiv 8\pi G_N$ is the gravitational coupling, $G_N \equiv M_P^{-2}$ being the Newton constant and $M_P = 1.221 \times 10^{19}$ GeV the Planck mass, and $g = \det g_{ij}$.

Application of this result to the Dirac theory would therefore result in the replacement of the differential equation (11) by an integro-differential equation in space-time on the atomic scale. In the absence of a precise understanding of how to formulate this problem, let us instead see what can be learnt from Eq. (11) as it stands. Evidently, this corresponds to a semi-classical approximation in which the wave function ψ of the electron is described quantum mechanically, while the spinorial connection Γ_k is defined from a classical, curved background space-time. In fact this should be a good approximation, since we are trying to investigate the structure of space-time within the Compton wavelength of the electron, $\lambda_C = \hbar/mc \approx 4 \times 10^{-11}$ cm, larger by a factor $\sim 2 \times 10^{22}$ than the Planck scale $l_P \equiv G_N^{1/2} \approx 2 \times 10^{-33}$ cm below which quantum gravitational effects can no longer be ignored.

We assume the electron to occupy a spherically symmetric region of radius $r_0 \sim \lambda_C$, far from which, at distances $r \gg r_0$, it can be regarded as a point mass and the space-time considered essentially flat. Within the electron, however, at radius $r \lesssim r_0$, space-time must become appreciably curved, raising the question whether it is possible to construct a contribution $i\gamma^k \Gamma_k$ to Eq. (11) in simulation of a real, constant scalar source-mass m .

Here, it is instructive[4, 5] that the term involving the gauge vector potential eA_k in Eqs. (1) and (11) can be completely absorbed into the definition of Γ_k via the transformation

$$\Gamma_k \rightarrow \Gamma'_k = \Gamma_k + ieA_k \mathbf{1}. \quad (19)$$

The imaginary gauge term ieA_k can be identified exclusively with the imaginary part $\text{Im}(\Gamma'_k)$ if we choose a Majorana representation for the gamma matrices in which all the γ_i are imaginary (see Ref. [8], for example). In this

case $e\gamma^k A_k$ is purely imaginary, while m , ω_{abk} and \bar{s}^{ab} are purely real, as a result of which Γ_k is real when $A_k = 0$, leading to the conjecture that m can be related to $\text{Re}(\Gamma'_k)$.

Since m is a constant, the most natural simplifying assumption for the interior of the particle is the maximally symmetric space-time generated by an effective cosmological constant Λ , so that the Riemann–Christoffel and Ricci tensors are

$$R_{ijkl} = \frac{1}{3}\Lambda(g_{il}g_{jk} - g_{ik}g_{jl}), \quad R_{ij} = -\Lambda g_{ij}. \quad (20)$$

These spaces constitute a subset of the Friedmann space-times

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2, \quad (21)$$

where t is comoving time and $a(t) \equiv a_0 e^{\alpha(t)}$ is the radius function of the three-space $d\mathbf{x}^2$, assumed to be flat. In this case the Γ_k are known[9, 10] and can be written in terms of the matrix-valued mass $\mathcal{M}(t)$, defined by

$$\mathcal{M}(t) \equiv m(t)\gamma^0 = i\gamma^k \Gamma_k, \quad (22)$$

where the scalar mass function, denoting $\bullet \equiv d/dt$, is

$$m(t) = \frac{3}{2} i \dot{\alpha}(t). \quad (23)$$

For the space-time (20), (21), we have

$$\alpha = \sqrt{\frac{\Lambda}{3}}t, \quad \dot{\alpha} = \sqrt{\frac{\Lambda}{3}}, \quad (24)$$

from which it follows that $\dot{\alpha}$ is real and constant in de Sitter space. Eq. (23) shows, however, that m is then imaginary and to generate a real m we have instead to choose anti-de Sitter space, where $\Lambda < 0$ and $\dot{\alpha}$ is imaginary. In the coordinate system (21), this means Wick rotation of the time,

$$t \rightarrow \tilde{t} = \pm it. \quad (25)$$

(It is alternatively possible to express the anti-de Sitter metric in the static, Lorentzian coordinate system (t', r, θ, ϕ) , when the line element reads

$$ds^2 = \left(1 - \frac{\Lambda r^2}{3}\right) dt'^2 - \left(1 - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (26)$$

which is sometimes more convenient.)

3. The definition of mass

The question now is whether the matrix mass \mathcal{M} defined in Eq. (22) can be converted into a scalar. Since \mathcal{M} plays the rôle of an operator acting on the wave function ψ in Eq. (11), this is equivalent to seeking a solution of the eigenvalue equation

$$\gamma^0 \psi = \lambda_0 \psi, \quad (27)$$

where λ_0 is non-trivial. The problematic nature of Eq. (27) emerges if we focus attention first on Weyl spinors, which are by definition only left- or right-handed. In this case ψ is an eigenfunction of one of the chirality operators (10), which we can write as

$$\frac{1}{2} (1 \pm \gamma_5) \psi_{\pm} = \psi_{\pm}. \quad (28)$$

The projection operators $\Pi_{\pm} \equiv \frac{1}{2}(1 \pm \gamma_5)$ do not commute with γ_0 , however, since γ_0 and γ_5 anti-commute, as a result of which we have the non-vanishing commutator

$$(\Pi_{\pm}, \gamma_0) = \pm \gamma_5 \gamma_0. \quad (29)$$

(Note that the same situation obtains in all even dimensionalities D , since the generalized $(D+1)$ -gamma symbols $\hat{\gamma}_{D+1} \equiv (-1)^{(D-2)/4} \hat{\gamma}_0 \hat{\gamma}_1 \dots \hat{\gamma}_{D-1}$ always anti-commute with $\hat{\gamma}_0$. In odd dimensionalities $\hat{\gamma}_{D+1}$ cannot be defined in this way, and Weyl spinors do not exist, because $D/2$ is non-integral.)

Consequently, a Weyl spinor cannot be an eigenfunction of γ^0 and Eq. (27) has no solution.

In flat space-time, $(\gamma^0)^2 = 1$, which would mean, multiplying Eq. (27) through by γ^0 , that $\lambda_0 = 1$ and hence $\gamma^0 = \mathbf{1}$. Then the anti-commutator $\{\gamma^0, \gamma^{\alpha}\}$ reduces to $2\gamma^{\alpha}$ rather than zero, and therefore Eq. (27) has no solution for any type of spinor.

This impasse naturally leads us to reappraise the exact notion of mass in the Dirac equation (1), which appears to be dependent upon representation in the spinor space, containing the γ_i *per se*. We should expect experimental results to be independent of representation, however, the most important examples being the phenomenon of the existence of positively charged electrons, that is positrons, and the explanation of the intrinsic spin of the electron.

Not surprisingly, the electronic spin cannot be extracted from Eq. (1) alone (although an anomalous contribution to the magnetic moment can be added via a Lagrangian

$$\delta L = -\frac{1}{2} i l_0 \sqrt{-g} \tilde{\psi} F_{ij} s^{ij} \psi, \quad (30)$$

where $\tilde{\psi} = \psi^+ \gamma^0$, $\psi^+ = \psi^T$, and l_0 is the anomaly parameter, introduced by Pauli[11]). Rather, Dirac[12] emphasized that it is necessary to square

the linear operator of Eq. (1), yielding a modified Klein–Gordon equation, which in flat space-time reads

$$\left[\square + m^2 - e^2 A_k A^k - ie \left(2A^k \partial_k + \frac{1}{2} F_{ij} s^{ij} \right) \right] \psi = 0, \quad (31)$$

assuming the Lorentz gauge

$$\partial_k A^k = 0. \quad (32)$$

Eq. (31) differs from Eq. (1) in two important respects: firstly, the field-free part of the operator, $\square + m^2$, no longer contains the γ_i explicitly, these having combined to produce the representation-independent metric; and secondly, Eq. (31) now contains the magnetic-moment term $F_{ij} s^{ij}$, which only appears as a result of the second operator $[i\gamma^k (\partial_k - ieA_k) - m]$ acting on the first. In writing Eqs. (30) and (31), we define the electromagnetic field tensor

$$F_{jk} = \partial_j A_k - \partial_k A_j \quad (33)$$

and the spin operator

$$s_{ij} = \frac{1}{2} (\gamma_i \gamma_j - \gamma_j \gamma_i). \quad (34)$$

As discussed in Ref. [12], Eq. (31) contains twice as many solutions as Eq. (1), due to its being invariant under reversal of the energy operator $p_0 \equiv -i\hbar\partial_0 \rightarrow -p_0$ (time-reversal invariance after quantization), implying the existence of both positive- and negative-energy solutions. According to the hole theory[13], the solutions of negative energy can be interpreted as particles with the opposite energy and charge, that is they have positive energy and positive charge.

Technically, these solutions can be obtained by taking the Hermitian conjugate of Eq. (1). The difficulty arising from the fact that γ_0 is Hermitian, while the γ_α are anti-Hermitian, can be resolved either by multiplying through by $\gamma_0 \equiv \beta$, which has the effect of converting the γ_α into the Hermitian matrices $\alpha_\alpha \equiv \beta\gamma_\alpha$, or by Euclideanizing the time coordinate, so that $\gamma_0 \rightarrow \tilde{\gamma}_0 = \pm i\gamma_0$, the metric becomes positive definite, and all the gamma matrices are anti-Hermitian. In the latter case, for example, we have $(i\gamma^k)^+ = i\gamma^k$, while $(\gamma^k A_k)^+ = -\gamma^k A_k$ (since the A_k are Hermitian), so that the Hermitian conjugate of Eq. (1) is

$$\left[i (\partial_k \psi^+) \gamma^k - m\psi^+ - e\psi^+ \gamma^k A_k \right] = 0, \quad (35)$$

which clearly describes a particle of the same energy but opposite charge.

Eq. (31) is also invariant under the transformation[14]

$$m \rightarrow -m, \quad (36)$$

raising the question why mass is always positive. In fact we can multiply m by any matrix square root of unity as well, resulting in the more general transformation, for example,

$$m \rightarrow \pm m \gamma^0, \quad (37)$$

since $(\pm \gamma^0)^2 = 1$, without changing any of the experimental predictions. Thus, Eq. (1) could be written equivalently as

$$\{i\gamma^0 [\partial_0 + i(m - eA_0)] + i\gamma^\alpha (\partial_\alpha - ieA_\alpha)\} \psi = 0. \quad (38)$$

The matrix-valued mass term $m\gamma^0$ occurring in Eq. (38) can now be completely simulated by the gravitational vacuum term $i\gamma^k \Gamma_k = \frac{3}{2} i \dot{\alpha} \gamma^0$ in an anti-de Sitter background space-time, thus realizing Weyl's conjecture, if we set

$$\Lambda = -\frac{4m^2}{3}. \quad (39)$$

4. Gravito-electromagnetism

It is interesting to examine Eq. (38) from the gravito-electromagnetic viewpoint. Suppose first, for simplicity, that $A_\alpha = 0$ and that $\psi = \psi(x^0)$, in which case Eq. (38) reduces to

$$\gamma^0 [i\partial_0 - (m - eA_0)] \psi = 0. \quad (40)$$

Now the line action for a test particle of rest mass m and charge e in interaction with external gravitational and electromagnetic fields g_{ij} and A_i , respectively, can be written as

$$S \equiv \int \mathcal{L} ds = - \int m ds + \int e A_i dx^i = \int \left(-m \frac{ds}{d\lambda} + e A_i \frac{dx^i}{d\lambda} \right) d\lambda, \quad (41)$$

where λ is an affine parameter. The path of the particle is given by $\delta S = 0$, that is

$$\delta \int \left[-m \left(g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right)^{1/2} + e A_i \frac{dx^i}{d\lambda} \right] d\lambda = 0, \quad (42)$$

which, since the integrand is a homogeneous function of first degree of the four variables $dx^i/d\lambda$, is equivalent to a variational problem with only three dependent variables x^α , and with global time x^0 as independent variable (see Courant and Hilbert[15]).

Therefore, the motion of the particle is determined by a Hamilton principle, and we may now write

$$\delta \int L(x^0, x^\alpha, dx^\alpha/dx^0) dx^0 = 0. \quad (43)$$

Further, expressing the line element in the form

$$ds^2 \equiv g_{ij} dx^i dx^j = \left(\sqrt{h} dx^0 - g_\alpha dx^\alpha \right)^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta = d\tau^2 - dl^2, \quad (44)$$

where, as shown by Landau and Lifschitz[16], the physical (chronometrically invariant[17]) three-space dl^2 is defined on the time slice $dx_0 = 0$ via the three-metric

$$\begin{aligned} \gamma_{\alpha\beta} &= -g_{\alpha\beta} + g_\alpha g_\beta, & \gamma^{\alpha\beta} &= -g^{\alpha\beta}, \\ g_\alpha &= -g_{0\alpha}/\sqrt{h}, & g^\alpha &= \sqrt{h} g^{0\alpha}, & h &= g_{00}, \end{aligned} \quad (45)$$

we can expand the Lagrangian L (see also Møller[18]) as

$$L = -m \frac{ds}{dx^0} + e A_i \frac{dx^i}{dx^0} = -m \left[\left(\sqrt{h} - \mathbf{g} \cdot \mathbf{w} \right)^2 - w^2 \right]^{1/2} + e (A_0 + A_\alpha w^\alpha). \quad (46)$$

Differentiation of L with respect to the velocities $w^\alpha \equiv dx^\alpha/dx^0$, keeping x^α fixed, thus yields

$$\frac{\partial L}{\partial w^\alpha} = m \Gamma(w) \left[w_\alpha + \left(\sqrt{h} - \mathbf{g} \cdot \mathbf{w} \right) g_\alpha \right] + e A_\alpha, \quad (47)$$

where the curved-space gamma factors are defined, setting $v = dl/d\tau$, as

$$\begin{aligned} \Gamma(w) &= \frac{dx^0}{ds} = \left[\left(\sqrt{h} - \mathbf{g} \cdot \mathbf{w} \right)^2 - w^2 \right]^{-1/2}, \\ \gamma(v) &= \frac{d\tau}{ds} = (1 - v^2)^{-1/2}. \end{aligned} \quad (48)$$

From expressions (46)–(48), we obtain the Hamiltonian

$$\begin{aligned} H &= w^\alpha \frac{\partial L}{\partial w^\alpha} - L \\ &= m \sqrt{h} \Gamma(w) \left(\sqrt{h} - \mathbf{g} \cdot \mathbf{w} \right) - e A_0 = m \sqrt{h} \gamma(v) - e A_0. \end{aligned} \quad (49)$$

In stationary background fields, for which $\partial_0 g_{ij} = \partial_0 A_i = 0$, L does not depend upon x^0 explicitly, and therefore the Hamiltonian (49) is a conserved quantity[16, 18]. (It is straightforward to show that the gravitational contribution to H is the zeroth component p_0 of the covariant mechanical four-momentum of the particle, defined by $p^i = m dx^i/ds$, and hence, from the geodesic equation, constant for a stationary gravitational field.)

The chronometrically invariant quantity

$$\mathcal{E} = m\gamma(v) \quad (50)$$

defines the locally measured (kinetic plus rest-mass) energy of the particle, while the gravito-electric and electric scalar potentials are identified as

$$\phi^{(g)} = \sqrt{h}, \quad \phi = -A_0, \quad (51)$$

so that the Hamiltonian can be written as

$$H = p_0 - eA_0 = \mathcal{E}\phi^{(g)} + e\phi. \quad (52)$$

Note that H is not chronometrically invariant, due to its h -dependence.

For a particle at rest in comoving coordinates, we have

$$v = 0, \quad \gamma(v) = 1, \quad h = 1, \quad (53)$$

whereupon Eq. (49) simplifies to

$$H = m - eA_0. \quad (54)$$

We now see that Eq. (40) can be written as

$$\gamma^0 (i\partial_0 - H) \psi = 0, \quad (55)$$

which is γ^0 times (the time-dependent Schrödinger equation). In this formulation, the Hamiltonian energy of the electron is thus incorporated into the spinor wave equation as a single entity. The two constituents m and $e\phi$ occur multiplied by the same matrix factor γ^0 , which seems reasonable on geometrical grounds, and suggests that in curved space-time one should use the full definition (49) for H in Eq. (55), rather than the flat-space limit (54).

Finally, restoring the vector potential A_α and the x^α -dependence of ψ , we can write the wave equation (38) as

$$[\gamma^0 (i\partial_0 - H) + \gamma^\alpha (i\partial_\alpha + eA_\alpha)] \psi = 0. \quad (56)$$

5. Vacuum fluctuations

In the previous sections, we have developed the idea that the mass parameter m in the Minkowski-space formulation of the Dirac equation for a spinor ψ can be simulated precisely by transforming to a curved, background anti-de Sitter space generated by a negative cosmological constant defined from m by Eq. (39). That is to say, a massless spinor in anti-de Sitter space-time behaves like a massive spinor in Minkowski space, in the

sense that the operator $i\gamma^0(\partial_0 + im)$ in Eq. (38) is replaced by the operator $i(\gamma^0\partial_0 - \gamma^k\Gamma_k)$ in Eq. (11) with $m = 0$. Further, by extension, the same reasoning applies in all dimensionalities $D \geq 2$ for which a non-vanishing spinorial connection exists.

The most natural way of explaining this result is to attribute the mass of the fermion to quantum-mechanical vacuum fluctuations. The general-relativistically invariant vacuum is necessarily a maximally symmetric space, as shown by Zel'dovich[19, 20] (only in this case is the energy-momentum tensor independent of the choice of unit time-like vector u^i). Zel'dovich[20] also found that zero-point fluctuations of the free fermionic vacuum give rise to a negative semi-definite energy-density, originating from the negative-energy solutions to the Dirac equation, and hence to a negative cosmological constant after imposing general covariance.

Indeed, Λ must be negative, producing an anti-de Sitter space, in order that the Nordström energy-density be positive,

$$\rho_N \equiv \frac{2R_0^0}{\kappa^2} \equiv T_0^0 - T_\alpha^\alpha \equiv \rho + 3p = -2\rho = -\frac{2\Lambda}{\kappa^2}, \quad (57)$$

where the energy density ρ and pressure p are given by

$$\rho = -p = \frac{\Lambda}{\kappa^2}, \quad (58)$$

for the gravitational mass measured at infinity will then be positive, according to Eq. (18). In other words, we have a completely self-consistent picture of the fermionic vacuum energy only if Λ is negative.

To understand this in more detail, consider a region of anti-de Sitter space that is spherically symmetric in both configuration and momentum space. The energy-density of the zero-point fluctuations is then defined by the integral[20]

$$\rho_v = -\frac{1}{2} 4 \frac{\hbar}{(2\pi)^3} \int_0^{k'} 4\pi k^2 (k^2 + m^2)^{1/2} dk = -\frac{\hbar}{\pi^2} \int_0^{k'} k^2 (k^2 + m^2)^{1/2} dk, \quad (59)$$

where $k \equiv 2\pi/\lambda$ is the three-momentum, λ being the wavelength, $k_0 \equiv -(k^2 + m^2)^{1/2}$ is the relativistic energy of the fermion, the coefficient $1/2$ is the standard prefactor, and the statistical weight 4 accounts for the two spins $\pm 1/2$ and helicities ± 1 . Strictly speaking, we have to carry out a regularization procedure and then let the upper limit $k' \rightarrow \infty$. To obtain a finite answer, however, it is obviously sufficient to impose an effective cut-off on the integral (59), which otherwise diverges.

Adopting a semi-phenomenological approach, we can now equate expressions (58) and (59), after substitution from Eq. (39), so that

$$\rho_v \equiv \frac{\Lambda}{\kappa^2} = -\frac{m^2 M_P^2}{6\pi}. \quad (60)$$

Since $m \ll M_P$ for all known particles, we shall find that $m \ll k'$, and can therefore approximate the vacuum integral (59) as

$$\rho_v \approx -\frac{\hbar}{\pi^2} \int_0^{k'} k^3 dk = -\frac{\hbar k'^4}{4\pi^2}. \quad (61)$$

Equating expressions (60) and (61), we obtain the solution

$$k' \approx \sqrt{\sqrt{\frac{2\pi}{3}} m M_P}, \quad (62)$$

which, applied to the electron mass $m_e = 0.511$ MeV, for example, yields

$$k'_e \approx 9.50 \times 10^7 \text{ GeV}. \quad (63)$$

Finally, we compare k' with the radius r_0 in configuration space, defined now using the static coordinate system (26), in which $\sqrt{-g} = r^2 \sin \theta$, such that

$$m = \frac{4}{3} \pi r_0^3 \rho_N = -\frac{8}{3} \pi r_0^3 \rho_v, \quad (64)$$

to which the solution is

$$r_0 = \frac{\hbar}{\left(\frac{2}{3}\right)^{2/3} m^{1/3} M_P^{2/3}}. \quad (65)$$

Thus, r_0 and k' satisfy the inequalities

$$l_P \ll r_0, \quad 1/k' \ll \lambda_C. \quad (66)$$

In this region, gravity can still be regarded as classical, due to the first inequality (66), while the fermion has to be treated quantum mechanically, by means of the Dirac equation, due to the second inequality, thus justifying the method.

One of the most important outstanding problems in elementary particle physics is to explain the origin of the mass spectrum of fermions and bosons. This problem seems no closer to solution in the present approach, simply being transformed into the corresponding one of the spectrum of values of Λ .

Nevertheless, it is interesting that the heterotic superstring theory of Gross *et al.* [21–23], obtained by taking into account the higher-derivative gravitational terms up to order \mathcal{R}^4 in the effective action for the bosonic sector after reduction to four dimensions, gives rise to two possible vacuum states. These vacua are Minkowski space and de Sitter space, with a positive cosmological constant at the Planck scale, however, far larger than any particle mass-squared, given by Eq. (52) of Ref. [24],

$$\Lambda = \left[\frac{18}{337\zeta(3) + 1/2} \right]^{1/3} A_r^{-1} \kappa^{-2}, \quad (67)$$

where $\zeta(3) = 1.202$ is the Riemann zeta function and $A_r \approx 1/g_s^2$ is the modulus of the physical four-space, which also defines the inverse tree-level gauge coupling.

The field-theory limit of the superstring is a supergravity theory preserving $N = 1$ space-time supersymmetry, which we recall, from the classification of Nahm [25], is permitted in anti-de Sitter space, but not in de Sitter space (which permits only $N = 2$ space-time supersymmetry) — see also Weinberg [26].

6. The dimensionality of the spinor space

Returning to the question of the number of components defining a fermionic spinor, we first recall that the Dirac equation necessarily involves a four-component wave function, since the γ_i cannot be represented by matrices of dimension less than 4×4 (see Schweber [27]). This result is equally valid in curved space-time, and consequently the spinorial connection Γ_i cannot be represented by matrices of dimensionality less than 4×4 either, for the γ_i are necessary to relate the curved-space metric g_{ij} to the tangent space η_{ab} .

Therefore, the reduction to a two-component spinor is only possible after imposing some type of symmetry or constraint. If we write the Dirac spinor in the form $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, where ψ_1 and ψ_2 are two-component spinors, we obtain Eqs. (6) and (7) in Minkowski space-time, which separate into Eqs. (8) and (9) for a massless spinor. In this case it is possible to set $\psi_2 = 0$, the two-component spinor ψ_1 then defining a left-handed Weyl fermion given by Eq. (10). Clearly, the wave equation (56) describing a massive spinor is separable in the same way.

From Eqs. (6) and (7), we see, however, that a massive fermion can also be described by a two-component theory if a functional relationship exists between ψ_1 and ψ_2 . Due to the difference between Eqs. (6) and (7), the theory cannot be made chirally symmetric, but it is possible instead to

impose invariance under the charge conjugation operator C , thereby defining a Majorana spinor[28] (see also Racah[29])¹. In the Weyl representation (4), C is given (see Ref. [8], for example) by

$$C = i \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad (68)$$

which satisfies the condition

$$C = C^* = -C^T, \quad (69)$$

as required. Without loss of generality, we can express the wave function in the form [31]

$$\psi = \begin{pmatrix} \chi \\ \sigma_2 \xi^* \end{pmatrix} \quad (70)$$

(see also Ref. [32], for example), and it then follows straightforwardly that the charge conjugate spinor is

$$\psi^C \equiv C\tilde{\psi}^T = -i \begin{pmatrix} \xi \\ \sigma_2 \chi^* \end{pmatrix} \quad (71)$$

The Majorana condition

$$\psi^c = \psi \quad (72)$$

can then be imposed (up to an unimportant phase factor $-i$) by setting

$$\chi = \xi, \quad (73)$$

so that

$$\psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \sigma_2 \psi_1^* \end{pmatrix}. \quad (74)$$

Eq. (7) can now be written entirely in terms of the left-handed Majorana spinor ψ_1 .

For the purposes of the present paper, the important point is that these considerations can be carried over to curved space-time by defining all quantities in the local tetrad frame [5].

This paper was written at the University of Cambridge, Cambridge, England.

¹ More precisely, chiral symmetry would require the existence of an additional mirror sector, which interacts with the visible Universe essentially through gravity alone — see Okun[30] and references therein.

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