RECURSIVE APPROACH TO SUPERSYMMETRIC QUANTUM MECHANICS FOR ARBITRARY FERMION OCCUPATION NUMBER

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We present in details a numerical approach for solving supersymmetric quantum mechanical systems with a gauge symmetry valid in all fermionic sectors. The method uses a recursive algorithm to calculate matrix elements of any gauge invariant operator in the Fock basis, in particular of the Hamiltonian operator, and can be used for any gauge group. We describe its application to a supersymmetric anharmonic oscillator model with discrete spectrum.

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1. Motivations

For several years a revival of interest in Supersymmetric Yang–Mills Quantum Mechanics (SYMQM) can be observed. These supersymmetric quantum mechanical systems can be obtained, most commonly, by a dimensional reduction of supersymmetric, $D = d + 1$ dimensional, $\mathcal{N} = 1$ Yang–Mills quantum field theories to one point in space. Such procedure reduces the local gauge symmetry of the initial field theory to a global symmetry of the reduced quantum mechanical system. The physical Hilbert space of SYMQM is composed of states invariant under this global symmetry. As well all pertinent operators must be symmetry singlets. These constraints, being the remnants of the Gauss law, make the analytic construction of solutions nontrivial.

The growing interest in these systems have several sources. On one hand, it is due to their conjectured relation with a particular limit of M-theory [1], on the other hand, to the regularized dynamics of relativistic quantum membranes and supermembranes which they describe [2]. Moreover, their bosonic

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sector can be investigated as a zero-volume limit of $D = 4$, Yang–Mills quantum field theory [3–5], providing results which can be compared with lattice calculations. Besides all this, the physically interesting features of SYMQM can be studied on their own. For these reasons much effort has been devoted to the evaluation of spectra of SYMQM.

Among the variety of approaches, many numerical methods have been adapted to investigate supersymmetric Yang–Mills quantum mechanics and recently provided new results. Some of them use path integral picture of quantum mechanics and Monte Carlo integration [6, 7], other the Hamiltonian formulation of quantum mechanics and the Fock space methods [8–10]. Particularly, basics of the algorithm presented in this paper were already described in [11, 12]. Our approach exploits the Fock space formulation, and therefore provides a nonperturbative way to calculate the eigenenergies and eigenstates. It was already applied to many systems uncovering interesting physics [13]. Although early attempts based on this principle proved to be very helpful in obtaining qualitative results, their potential was limited due to the rapid growth of the Fock basis. Comparing to them, the recursive algorithm which we present in this article, not only enables one to obtain numerical results up to very high precision, and thus, permits more quantitative considerations, but also makes the calculations in fermionic sectors possible. The main motivation for it comes from the study of $D = 2$, SYMQM. However, the method is much more general and provides a tool for evaluation of matrix elements of any invariant operator for any gauge group and in any dimension. Due to these multiple possibilities of extensions, this paper is the first of a series of articles presenting the results of studies of the SYMQM systems obtained with our numerical approach. Being the introductory paper, it contains, apart of the detailed discussion of the recursive algorithm itself, the summary of the whole framework which will be needed for the future work. Hence, in the following section we start with an introduction of the basic notions of SYMQM. Next, we construct the Fock space, which is necessary for the numerical calculations, and discuss some of its properties. The main part of the paper, the description of the algorithm, is divided into several sections. First of all, the calculation of the matrix of the scalar products is presented. Then, the orthonormalization procedure and the automatic evaluation of commutators and anticommutators is described. Only at that point the formula used for calculation of matrix elements of any operator can be clearly introduced. Eventually, the full recursive relations will be discussed. We finish by presenting a simple application of our method to a supersymmetric model with discrete spectrum. Conclusions and an outlook of future research directions will be provided in the last section.

2. The framework

In this section we describe several basic concepts constituting our framework. We start by introducing quantum mechanics in the cut Fock basis. Then, we define the supersymmetric Yang–Mills quantum mechanics and derive a particular Hamiltonian which is studied numerically as an illustration of the approach at the end of this article. In the remaining subsections we discuss the construction of the basis of SYMQM and its properties.

2.1. Quantum mechanics in a cut Fock basis

Quantum systems, which are described by a Hamiltonian operator expressed in terms of position and momentum operators, can be analyzed numerically in an eigenbasis of occupation number operators — a Fock basis [8]. Any occupation number operator can be written as $a^\dagger a$, where $a^\dagger$ and a are bosonic creation and annihilation operators respectively, fulfilling the well-known commutation relations

$$[a_p, a_q] = \left[a_p^\dagger, a_q^\dagger\right] = 0\,, \qquad \left[a_p, a_q^\dagger\right] = \delta_{pq}\,. \tag{1}$$

p and q in (1) are indices which label the bosonic degrees of freedom. In the case of supersymmetric system we introduce fermionic creation and annihilation operators, f and $f^\dagger$ respecting the anticommutation relations

$$\{f_p, f_q\} = \left\{f_p^\dagger, f_q^\dagger\right\} = 0\,, \qquad \left\{f_p, f_q^\dagger\right\} = \delta_{pq}\,, \tag{2}$$

where again p and q are indices which describe the fermionic degrees of freedom. Obviously, we also have

$$[f_p, a_q] = \left[f_p^\dagger, a_q\right] = 0\,, \qquad \left[f_p, a_q^\dagger\right] = \left[f_p^\dagger, a_q^\dagger\right] = 0\,. \tag{3}$$

The momentum and position operators are expressed by creation and annihilation operators in the usual way

$$x_q = \frac{1}{\sqrt{2}}\left(a_q + a_q^\dagger\right)\,, \qquad p_q = \frac{1}{i\sqrt{2}}\left(a_q - a_q^\dagger\right)\,, \tag{4}$$

enabling to express the Hamiltonian operator in terms of bosonic and fermionic creation and annihilation operators only.

The construction of the basis starts with the definition of the Fock vacuum $|0\rangle$, as the state fulfilling the conditions

$$a_q|0\rangle = 0\,, \qquad f_q|0\rangle = 0\,, \quad \forall q\,. \tag{5}$$

Any other basis state can be obtained from $|0\rangle$ by a successful action of creation operators.

Eventually, the action of the Hamiltonian operator, which is now an operator function of (1) and (2), is straightforward in such basis. There is no conceptual difficulties in evaluation of its matrix elements, however such calculations may turn to be computationally demanding. The recursive algorithm described in this paper may be a solution to this problem. Once the Hamiltonian matrix is obtained, its eigenvalues correspond simply to the eigenenergies of the quantum system, and its eigenvectors to the eigenstates.

The numerical analysis requires one last step, namely an introduction of a cut-off N_{cut} on the countably infinite Fock basis, since it is impossible to deal with infinite matrices on a computer. There are many ways to introduce such a cut-off depending on the symmetries of the system being investigated. Finally, we have to perform calculations with several increasing N_{cut} and extract the physical results from the limit of infinite cut-off. The properties of such a procedure were analyzed in [8, 14–16].

2.2. Supersymmetric Yang–Mills quantum mechanics

The recursive algorithm will be described in the context of a relatively simple, $D = d + 1 = 2$ supersymmetric Yang–Mills quantum mechanics. In this way we will be able to focus our attention directly on the main features of the algorithm. However, it can be used for many different systems and it is conceptually straightforward to generalize it to the physically more interesting systems like $D = 4$ quantum mechanics [9, 12]. In order to obtain the particular Hamiltonian which will be studied in the last section of this article, we will now derive the general Hamiltonian of $D = 2$, SYMQM. To this end we extend the construction of supersymmetric quantum mechanics [17] via the generalized creation and annihilation operators. We define the latter as

$$A = 1/\sqrt{2}\,\big(-ip + W\big)\,, \tag{6}$$

$$A^\dagger = 1/\sqrt{2}\,\big(ip + W\big)\,, \tag{7}$$

where the superpotential $W = W(x)$ is a function of the position operator x. The supercharges are obtained from A and $A^\dagger$ as $Q = Af$ and $Q^\dagger = A^\dagger f^\dagger$. With the idea of SYMQM in mind, a simple generalization is to include a global SU(N) symmetry. Therefore, we postulate the supercharges to be given by[1]

$$Q = \frac{1}{\sqrt{2}} \sum_{a=1}^{N^2-1} \big(-ip_a + W_a\big) f_a\,, \tag{8}$$

[1] Private communication by J. Wosiek.

$$Q^\dagger = \frac{1}{\sqrt{2}} \sum_{a=1}^{N^2-1} \left(ip_a + W_a\right) f_a^\dagger \,, \tag{9}$$

where a is a color index of the adjoint representation of the SU(N) group, $a = 1, \ldots, N^2-1$, $f_a, f_a^\dagger$ are fermionic operators. Our system contains now N^2-1 bosonic degrees of freedom, described by x_a and p_a, and N^2-1 fermionic degrees of freedom, described by $f_a^\dagger$ and f_a. The reduced Gauss law restricts the physical Hilbert space to only those states which are invariant under the SU(N) group. In order to facilitate the construction of such singlets it is useful to introduce a matrix notation. Thanks to the latter all singlets can be written in terms of traces [19]. Therefore, we define

$$x_{i,j} = \sum_{a=1}^{N^2-1} x_a T_{i,j}^a \,, \qquad p_{i,j} = \sum_{a=1}^{N^2-1} p_a T_{i,j}^a \,,$$
$$f_{i,j}^\dagger = \sum_{a=1}^{N^2-1} f_a^\dagger T_{i,j}^a \,, \qquad f_{i,j} = \sum_{a=1}^{N^2-1} f_a T_{i,j}^a \,,$$

where $T_{i,j}^a$ are the generators of the SU(N) group in the fundamental representation, $i, j = 1, \ldots, N$. Hence, all operators become operator valued matrices. We also introduce a simplified notation for a trace of any matrix, namely, $\mathrm{Tr}(O) \equiv (O)$. However, we will use this notation only when many traces occur and no confusion is induced.

The Hamiltonian of such a system is given by the anticommutator of the supercharges, Eqs. (8) and (9), and its general form reads[2]

$$H = \left\{Q^\dagger, Q\right\} = \frac{1}{2}\left(p_a p_a + W_a W_a\right) + \frac{1}{4}\left(\partial_a W_b + \partial_b W_a\right)\left[f_a^\dagger, f_b\right] . \tag{10}$$

In the simplest case, we choose $W_a = x_a$ and obtain a set of N^2-1 supersymmetric harmonic oscillators. For a slightly more complicated case, let us consider

$$W_a = \frac{g}{2} d_{abc} x_b x_c \,, \tag{11}$$

where g is the coupling constant and d_{abc} is the totally symmetric tensor of the SU(N) group. The Hamiltonian Eq. (10) reduces to,

$$H = \frac{1}{2}\left(p_a p_a + \frac{g^2}{4} d_{abe} d_{ecd} x_a x_b x_c x_d\right) + \frac{g}{2} d_{abc} x_a \left[f_b^\dagger, f_c\right] . \tag{12}$$

[2] We adopt the notation when a repeated index is assumed to be summed over.

One of the interesting features of this model is a nontrivial bosonic potential of fourth order which for the SU(3) group is simply

$$d_{abe}d_{ecd}x_a x_b x_c x_d = \frac{1}{3}\left(\sum_{a=1}^{8}(x_a)^2\right)^2 . \tag{13}$$

Therefore the system of Eq. (12) is expected to have a discrete spectrum of bound states. In order to rewrite this Hamiltonian in terms of traces we use the following identities valid for any SU(N) group,

$$\begin{aligned}
\tfrac{1}{2}\, p_a p_a &= \mathrm{Tr}\ p^2 \,, \\
\tfrac{1}{8}\, x_a x_b x_c x_d\, d_{abe}d_{ecd} &= \mathrm{Tr}\ x^4 - \frac{1}{N}\left(\mathrm{Tr}\ x^2\right)^2 \,, \\
\tfrac{1}{2}\, x_a f_b^\dagger f_c\, d_{abc} &= \mathrm{Tr}\left(x f^\dagger f\right) - \mathrm{Tr}\left(x f f^\dagger\right) \,,
\end{aligned}$$

and eventually obtain

$$H = \mathrm{Tr}\ p^2 + g^2\Big(\mathrm{Tr}\ x^4 - \frac{1}{N}\left(\mathrm{Tr}\ x^2\right)^2\Big) + 2g\,\mathrm{Tr}\left(x\Big[f^\dagger, f\Big]\right) . \tag{14}$$

We suspend the detailed discussion of this Hamiltonian to Section 4 where the numerical results obtained with the recursive algorithm will be presented together with some analytic calculations. In the following subsection we will describe the construction of the Fock basis for SYMQM and some of its properties.

2.3. Elementary bricks and Fock basis of the SU(N) SYMQM

The Fock states are eigenstates of some occupation number operators. In the case of SYMQM models we consider gauge-invariant occupation number operators,

$$\mathrm{Tr}\ a^\dagger a = \sum_{q=1}^{N^2-1} a_q^\dagger a_q \,, \tag{15}$$

$$\mathrm{Tr}\ f^\dagger f = \sum_{q=1}^{N^2-1} f_q^\dagger f_q \,. \tag{16}$$

Most of the Hamiltonians that we have considered so far conserve the fermionic occupation number. Therefore, it is physically motivated to construct the Fock basis independently in each subspace of the physical Hilbert space with a definite fermionic occupation number. Moreover, states containing different total number of quanta are orthogonal, so we can further divide

the fermionic sectors into subspaces with given number of bosonic quanta. However, usually the Hamiltonian does not conserve the bosonic occupation number and thus mixes different bosonic subspaces. We start the construction of the Fock basis in the purely bosonic situation, and then turn to the fermionic sectors.

2.3.1. Bosonic elementary bricks

A general eigenstate of the occupation number operator, having n_B quanta, can be written as [18]

$$|n_B\rangle \;=\; \sum_{i_1,\ldots,i_{n_B}} T_{i_1,i_2,\ldots,i_{n_B}} a^\dagger_{i_1} a^\dagger_{i_2} \ldots a^\dagger_{i_{n_B}} |0\rangle\,, \tag{17}$$

where T is a group invariant tensor. It can be shown [19] that any such invariant tensor can be expressed as linear combination of products of trace tensors. The latter are just traces of products of $T^a_{i,j}$ matrices, of which the simplest ones are $\mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta_{ab}$ and $\mathrm{Tr}\, T^a T^b T^c = \frac{1}{4} d_{abc} + \frac{i}{4} f_{abc}$. Therefore, in matrix notation a state $|n_B\rangle$ can be rewritten as

$$|n_B\rangle \;=\; \sum_{\left\{\sum_{j=2}^{N} j k_j = n_B\right\}} \gamma_{k_2,\ldots,k_{n_B}} (a^{\dagger 2})^{k_2} (a^{\dagger 3})^{k_3} \;\ldots\; (a^{\dagger n_B})^{k_{n_B}} |0\rangle\,, \tag{18}$$

where $\gamma_{k_2,\,\ldots\,,k_{n_B}}$ are arbitrary coefficients and the sum is over all such combinations of exponents k_j that, $\sum_{j=2}^{n_B} j k_j = n_B$, so that the state $|n_B\rangle$ is composed of n_B quanta. Notice that even though a single quantum created by $a^\dagger_{i,j}$ is gauge dependent, total numbers of quanta, n_B, being the eigenvalues of the operators Eqs. (15) and (16), are gauge independent.

Subsequently, $|n_B\rangle$ can be significantly simplified with the use of the Cayley–Hamilton theorem. It states, that any matrix, A, satisfies its own characteristic equation. Therefore, we have for the SU(2), SU(3) and SU(4) groups

$$\begin{aligned}
&\mathrm{SU}(2): \quad A^2 - \tfrac{1}{2}\left(A^2\right)\mathcal{I} = 0\,,\\
&\mathrm{SU}(3): \quad A^3 - \tfrac{1}{2}\left(A^2\right)A - \tfrac{1}{3}\left(A^3\right)\mathcal{I} = 0\,,\\
&\mathrm{SU}(4): \quad A^4 - \tfrac{1}{2}\left(A^2\right)A^2 - \tfrac{1}{3}\left(A^3\right)A + \tfrac{1}{8}\left(A^2\right)^2\mathcal{I} - \tfrac{1}{4}\left(A^4\right)\mathcal{I} = 0\,.
\end{aligned} \tag{19}$$

One can use these equalities to reduce traces containing more than N operators of the same kind to simpler ones. We demonstrate this on an example with $A = a^\dagger$ and some arbitrary operator B, which can be any operator involving bosonic or fermionic creation and annihilation operators. Particularly, B can be again a single bosonic creation operator. Thus, multiplying

Eqs. (19) by B from the right-hand side and taking the trace, we obtain a set of relations, such as

$$\begin{aligned}
\mathrm{SU}(2)(a^\dagger a^\dagger B) &= \tfrac{1}{2}(a^\dagger a^\dagger)(B)\,,\\
\mathrm{SU}(3)(a^\dagger a^\dagger a^\dagger B) &= \tfrac{1}{2}(a^\dagger a^\dagger)(a^\dagger B) + \tfrac{1}{3}(a^\dagger a^\dagger a^\dagger)(B)\,,\\
\mathrm{SU}(4)(a^\dagger a^\dagger a^\dagger a^\dagger B) &= \tfrac{1}{2}(a^\dagger a^\dagger)(a^\dagger a^\dagger B) + \tfrac{1}{3}(a^\dagger a^\dagger a^\dagger)(a^\dagger B)\\
&\quad -\tfrac{1}{8}(a^\dagger a^\dagger)^2(B) + \tfrac{1}{4}(a^\dagger a^\dagger a^\dagger a^\dagger)(B)\,.
\end{aligned}$$

Hence, a general state with n_B quanta for some given N, simplifies to

$$|n_B\rangle_N = \sum_{\left\{\sum_{j=2}^N jk_j = n_B\right\}} \gamma_{k_2,\ldots,k_N}\,(a^{\dagger 2})^{k_2}(a^{\dagger 3})^{k_3}\ldots(a^{\dagger N})^{k_N}|0\rangle\,, \tag{20}$$

where the traces with more than N creation operators were reduced and the highest trace is now $(a^{\dagger N})$.

We are now in position to define the set of *bosonic elementary bricks*, which is the set of $N-1$ linearly independent single traces of creation operators, which cannot be further reduced by the Cayley–Hamilton theorem. Table I contains examples of such sets for $N=2$, $N=3$ and $N=4$. Products of powers of elementary bosonic bricks acting on the Fock vacuum compose the set of states

$$\left\{(a^{\dagger 2})^{k_2}(a^{\dagger 3})^{k_3}\ldots(a^{\dagger N})^{k_N}|0\rangle\right\}_{\sum_{j=2}^N jk_j = n_B} \equiv \left|\{n_B\}\right\rangle, \tag{21}$$

which spans the subspace of the Hilbert space with n_B bosonic quanta. We adopted a generalized notation in which $|\{n_B\}\rangle$ is a vector of all states with n_B quanta. The set of states Eq. (21) is excessively called the Fock basis, though in general it is not orthonormal. Only after the application of an orthonormalization procedure it will be transformed into a basis. By acting on it with additional elementary bricks and orthonormalizing, one can obtain the basis in sectors with yet higher number of bosonic quanta. In this way, starting with the Fock vacuum $|0\rangle$, one can recursively generate the basis for any n_B.

TABLE I

Elementary bosonic bricks for SU(2), SU(3) and SU(4).

SU(2)	SU(3)	SU(4)
$(a^\dagger a^\dagger)$	$(a^\dagger a^\dagger)$	$(a^\dagger a^\dagger)$
	$(a^\dagger a^\dagger a^\dagger)$	$(a^\dagger a^\dagger a^\dagger)$
		$(a^\dagger a^\dagger a^\dagger a^\dagger)$

2.3.2. Fermionic bricks

The definition of fermionic bricks is a bit more complicated. Besides the fermionic bricks which are single-trace operators, as, for example, all fermionic operators in the case of the SU(2) symmetry group (see Table II), we must also take into account bricks which are multiple-trace operators.

TABLE II

SU(2) fermionic bricks.

$n_F = 1$	$n_F = 2$	$n_F = 3$
$(f^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger)$	$(f^\dagger f^\dagger f^\dagger)$

They appear in higher fermionic sectors in the case of symmetry groups with $N > 2$ (see Tables III and IV). In order to make this distinction clear we will now introduce some definitions and appropriate notation.

TABLE III

SU(3) fermionic bricks.

$n_F = 1$	$n_F = 2$	$n_F = 3$	$n_F = 4$
$(f^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger)$	$(f^\dagger f^\dagger f^\dagger)$	$(f^\dagger f^\dagger f^\dagger f^\dagger a^\dagger)$
$(f^\dagger a^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger a^\dagger)$	$(f^\dagger f^\dagger f^\dagger a^\dagger)$	$(f^\dagger a^\dagger)(f^\dagger f^\dagger f^\dagger)$
	$(f^\dagger a^\dagger a^\dagger f^\dagger a^\dagger)$	$(f^\dagger f^\dagger f^\dagger a^\dagger a^\dagger)$	$(f^\dagger f^\dagger f^\dagger f^\dagger a^\dagger a^\dagger)$
	$(f^\dagger a^\dagger)(f^\dagger a^\dagger a^\dagger)$	$(f^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger)$	$(f^\dagger a^\dagger a^\dagger)(f^\dagger f^\dagger f^\dagger)$
		$(f^\dagger a^\dagger f^\dagger f^\dagger a^\dagger a^\dagger)$	$(f^\dagger a^\dagger)(a^\dagger f^\dagger f^\dagger f^\dagger)$
		$(f^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger)$
		$(f^\dagger a^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger)$	$(f^\dagger a^\dagger a^\dagger)(f^\dagger f^\dagger f^\dagger a^\dagger)$
		$(f^\dagger a^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger a^\dagger)$
			$(f^\dagger a^\dagger)(f^\dagger a^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger)$
			$(f^\dagger f^\dagger a^\dagger)(f^\dagger a^\dagger f^\dagger a^\dagger a^\dagger)$

In analogy to the set of elementary bosonic bricks, we define the set of *elementary fermionic bricks*. The latter will contain all single traces with n_F fermionic creation operators, which cannot be further reduced by the Cayley–Hamilton theorem[3]. An algorithmic way to obtain it, is to start with the set of bosonic elementary bricks for a given N. Then, for each trace, one should perform n_F times either one of the following operations: replace one bosonic creation operator by a fermionic one or insert one fermionic

[3] There is no simple counterpart of the Cayley–Hamilton theorem for anticommuting matrices. However, an appropriate choice of the B operator in (20) will produce identities which can be used to simplify or exclude, due to linear dependence, some of the possible composite fermionic bricks [18].

TABLE IV

SU(4) fermionic bricks.

$n_F = 1$	$n_F = 2$
$(f^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger)$
$(f^\dagger a^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger a^\dagger)$
$(f^\dagger a^\dagger a^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger a^\dagger a^\dagger)$
	$(f^\dagger a^\dagger f^\dagger a^\dagger a^\dagger)$
	$(f^\dagger a^\dagger)(f^\dagger a^\dagger a^\dagger)$
	$(f^\dagger a^\dagger f^\dagger a^\dagger a^\dagger a^\dagger)$
	$(f^\dagger a^\dagger)(f^\dagger a^\dagger a^\dagger a^\dagger)$
	$(f^\dagger a^\dagger a^\dagger f^\dagger a^\dagger a^\dagger a^\dagger)$
	$(f^\dagger a^\dagger a^\dagger)(f^\dagger a^\dagger a^\dagger a^\dagger)$

creation operators into the trace. In order to obtain a complete basis, the set of elementary bricks must be enlarged by operators, which are products of fermionic elementary bricks with smaller number of fermionic quanta and contain n_F fermionic creation operators in total. In this way we ensure that all possible invariant contractions of n_F fermionic creation operators with a number of bosonic creation operators are considered. However, the problem of linear dependence appears and one has to pick out only the linearly independent operators. The linear independence can be checked by an explicit calculation of the determinant of the Gram matrix of states constructed with those bricks. Fortunately, there exists also an independent, and more direct way of computing the number of Fock basis states which will be described in Subsection 2.4.

The enlarged set of all linearly independent fermionic bricks will be referred to as the set of *composite fermionic bricks*. Its elements will be labeled by an index α, and denoted by $C^\dagger(n_B, n_F, \alpha)$, where n_B and n_F are the number of bosonic and fermionic creation operators, respectively. Such notation is used in order to treat all bosonic and fermionic bricks in the same way. The additional index α distinguishes the operators with the same n_F and n_B.

Moreover, we will denote:

- the number of composite bricks with n_F fermionic and n_B bosonic quanta by $d(n_F, n_B)$,
- the total number of composite bricks with n_F fermions by $d(n_F)$.

Obviously,

$$d(n_F) = \sum_{n_B} d(n_F, n_B)\,.$$

We extend our notation to the set of bosonic elementary bricks

$$(a^{\dagger n_B}) \equiv C^\dagger(n_B, 0)\,.$$

As an example, Tables II, III and IV contain the sets of composite bricks for $N = 2$, $N = 3$ and $N = 4$ for some fermionic sectors. Let us briefly comment on Table II. In this simple case, there is exactly one fermionic operator in each fermionic sector (see Section 2.4) and they are just elementary fermionic bricks. One can prove, by simple arguments, that other operators are not possible. For example, an operator of the form $(f^\dagger a^\dagger a^\dagger)$ vanishes identically, since it must involve the totaly antisymmetric tensor ϵ^{ijk} summed with a symmetric combination of bosonic operators $a^{\dagger j} a^{\dagger k}$. In analogy, the possible operator $(f^\dagger a^\dagger)^2$ also vanishes, since it is a square of an anticommuting operator. Such reasonings facilitate the explicit construction of the set of composite fermionic bricks for gauge groups with $N > 2$.

Once the set of composite fermionic bricks is constructed, it is easy to write down a general state with n_B bosonic and n_F fermionic quanta for a given gauge group SU(N). One has to take a linear combination of states obtained by applying one of the composite fermionic bricks with n_F fermionic quanta to a general bosonic Fock state Eq. (20). Hence,

$$|n_B, n_F\rangle_N = \sum_{\alpha=1}^{d(n_F)} C^\dagger(n, n_F, \alpha) \times \sum_{\{\sum_{j=2}^N j k_j = n_B - n\}} \gamma_{k_2,\ldots,k_N}(\alpha) C^\dagger(2,0)^{k_2} C^\dagger(3,0)^{k_3} \ldots C^\dagger(N,0)^{k_N} |0\rangle\,, \quad (22)$$

where the coefficients $\gamma_{k_2,\ldots,k_N}(\alpha)$ can depend now on α. In order to ensure that the total number of bosonic quanta is n_B, we have apply the operator $C^\dagger(n, n_F, \alpha)$ containing n bosonic creation operators, to a purely bosonic state with $n_B - n$ quanta. In analogy to the bosonic case, we can define the set of states,

$$\left\{ C^\dagger(n, n_F, \alpha) C^\dagger(2,0)^{k_2} C^\dagger(3,0)^{k_3} \ldots C^\dagger(N,0)^{k_N} |0\rangle \right\}_{\sum_{j=2}^N j k_j + n = n_B} \equiv |\{n_B, n_F\}\rangle \quad (23)$$

which after orthonormalization will give the basis in the subspace of Hilbert space with n_B and n_F bosonic and fermionic quanta, respectively. The linear independence and completeness of the set of composite fermionic bricks ensures that Eq. (23) form indeed a complete set of states in the fermionic sectors.

2.4. Counting the number of SU(N) SYMQM Fock basis states

In the preceding section we have shown, that the Fock basis in any sector is given by Eq. (21) or Eq. (23). As an example, Table V contains few simplest states of the Fock basis with the SU(2) symmetry.

TABLE V

Construction of the basis for the SU(2) gauge group.

n_B	$n_F = 0$	$n_F = 1$	$n_F = 2$	$n_F = 3$
0	$\|0\rangle$	—	—	$(f^\dagger f^\dagger f^\dagger)\|0\rangle$
1	—	$(f^\dagger a^\dagger)\|0\rangle$	$(f^\dagger f^\dagger a^\dagger)\|0\rangle$	—
2	$\|2\rangle \equiv (a^\dagger a^\dagger)\|0\rangle$	—	—	$(f^\dagger f^\dagger f^\dagger)\|2\rangle$
3	—	$(f^\dagger a^\dagger)\|2\rangle$	$(f^\dagger f^\dagger a^\dagger)\|2\rangle$	
⋮				
2n	$\|2n\rangle \equiv (a^\dagger a^\dagger)^n\|0\rangle$	—	—	$(f^\dagger f^\dagger f^\dagger)\|2n\rangle$
2n+1	—	$(f^\dagger a^\dagger)\|2n\rangle$	$(f^\dagger f^\dagger a^\dagger)\|2n\rangle$	—
⋮				

The linear independence of a set of states such as the one in Table V can be checked by explicit calculations of the determinant of the Gram matrix however such computations become cumbersome for larger number of states. Fortunately there exist an alternative, group-theoretical way of calculating the total number of linearly independent gauge-invariant Fock states. It was suggested by Janik and elaborated by Trzetrzelewski [18, 20]. Instead of an explicit construction of basis states, this approach exploits the orthogonality of the characters. Such an alternative method is of great practical value since it may serve as a crosscheck to our recursive algorithm.

2.4.1. Character method

Let $D(n_B, n_F)$ be the number of gauge-invariant, linearly independent states with n_B bosonic and n_F fermionic quanta. $D(n_B, n_F)$ can be obtained from the orthogonality relation of the characters of the SU(N) group.

Each bosonic and fermionic creation operator transforms according to the adjoint representation of the SU(N) group. Hence, the products of creation operators, which are needed for the construction of basis states, transform as products of the adjoint representations. From the representation theory it is known, that the square of any irreducible representation is reducible and can be expressed as a sum of a symmetric and antisymmetric parts.

This statement written in terms of characters reads,

$$\begin{aligned}\chi(R)\times\chi(R) &= [\chi(R)\times\chi(R)]+\{\chi(R)\times\chi(R)\}\\ &= \tfrac{1}{2}\left(\chi^2(R)+\chi\left(R^2\right)\right)+\tfrac{1}{2}\left(\chi^2(R)-\chi\left(R^2\right)\right)\,,\end{aligned}\qquad(24)$$

where the symbols $[\dots]$ and $\{\dots\}$ denote the symmetric and antisymmetric part, respectively, and $\chi(R^2)$ is the trace of the matrix of the representation R squared. A generalization of the Eq. (24) is known as the Fröbenius theorem [21], and has a practical meaning, since the characters of powers of R are often explicitly given. It gives the expressions for the symmetrized and antisymmetrized characters of a product of p representations R,

$$\left[\times_{k=1}^{p}\chi(R)\right] = \sum_{\sum_{k=2}^{p} k i_k = p}\prod_{k=1}^{p}\frac{1}{i_k!}\frac{\chi^{i_k}(R^k)}{k^{i_k}}\,,\qquad(25)$$

$$\left\{\times_{k=1}^{p}\chi(R)\right\} = \sum_{\sum_{k=2}^{p} k i_k = p}(-1)^{\sum_{k=2}^{p} i_k}\prod_{k=1}^{p}\frac{1}{i_k!}\frac{\chi^{i_k}(R^k)}{k^{i_k}}\,,\qquad(26)$$

where the sum is over all partitions of the number p into numbers $2,\dots,p$, i_j being the multiplicity of the j number in a given partition. Thus, the most general product of n_B bosonic and n_F fermionic creation operators will be in the representation, which character is equal to $[\times_{k=1}^{n_B}\chi(R)]\{\times_{k=1}^{n_F}\chi(R)\}$, where R denotes now the adjoint representation of the SU(N) group. From the orthogonality property of the characters we have

$$D(n_B,n_F)=\int d_{\mu_{\mathrm{SU}(N)}}\,1\,\left[\times_{k=1}^{n_B}\chi(R)\right]\left\{\times_{k=1}^{n_F}\chi(R)\right\}\,,\qquad(27)$$

where 1 stands for the character of the trivial representation and $d\mu_{\mathrm{SU}(N)}$ is the group invariant measure on SU(N).

A convenient parametrization of the group manifold is by N^2-1 Euler angles α_i, all defined on $[0,2\pi]$. For example the group elements of SU(3) read [22]

$$U=e^{i\lambda_3\alpha_1}e^{i\lambda_2\alpha_2}e^{i\lambda_3\alpha_3}e^{i\lambda_5\alpha_4}e^{i\lambda_3\alpha_5}e^{i\lambda_2\alpha_6}e^{i\lambda_3\alpha_7}e^{i\lambda_8\alpha_8}\qquad(28)$$

and the generalization of the above equation to SU(N) can be found in [23]. The last element needed to calculate $D(n_B,n_F)$ are the characters $\chi(R^k)$. They are given by the Weyl formula [24]

$$\chi(R)=\sum_{i,j=1}^{N}e^{i(\alpha_i-\alpha_j)}-1\,,\qquad \chi(R^k)=\sum_{i,j=1}^{N}e^{ik(\alpha_i-\alpha_j)}-1\,.\qquad(29)$$

The invariant measure reads [18],

$$d_{\mu_{\mathrm{SU}(N)}} = \frac{1}{N!}\prod_{i=1}^{N}\frac{d\alpha_i}{2\pi}\Big|\prod_{i<j}(e^{i\alpha_i}-e^{i\alpha_j})\Big|^2\delta_{\mathrm{P}}\left(\sum_{i=1}^{N}\alpha_i\right), \tag{30}$$

where $\delta_{\mathrm{P}}(x)$ is a periodic delta-function given by

$$\delta_{\mathrm{P}}\left(\sum_{i=1}^{N}\alpha_i\right) = \sum_{k=-\infty}^{\infty}\delta\left(\sum_{i=1}^{N}\alpha_i - 2\pi k\right), \tag{31}$$

with k integer.

2.4.2. Generating functions for $D(n_B, n_F)$

Eq. (27) is difficult to evaluate for any N, however it was calculated for few simplest groups [18]. In these cases, the numbers $D(n_B, n_F)$ can be encoded in a generating function with two parameters t and s, $G(t,s)$,

$$G(t,s) = \sum_{n_B,n_F} D(n_B,n_F)t^{n_B}(-s)^{n_F}, \tag{32}$$

which is very useful in practical applications. For $N=3$, $G(t,s)$ can be expressed in terms of simple polynomials in t [18], namely

$$G(t,s) = \left(\prod_{k=2}^{N}\frac{1}{1-t^k}\right)\sum_{i=0}^{N^2-1}(-1)^i s^i c_i(t), \tag{33}$$

and the polynomials $c_i(t)$ read

$$\begin{aligned} c_0(t) &= 1, \\ c_1(t) &= t+t^2, \\ c_2(t) &= t+t^2+2t^3, \\ c_3(t) &= 1+t+2t^2+3t^3+t^4, \\ c_4(t) &= 2t+4t^2+2t^3+2t^4, \\ c_{8-i}(t) &= c_i(t). \end{aligned} \tag{34}$$

In this form some information contained in $G(t,s)$ become evident. The term proportional to s^0 is equal to the generating function for the number of partitions into numbers $\{N, N-1, \ldots, 2\}$. Obviously, there are as many states with n_B quanta as there are ways of obtaining n_B from multiples of the numbers of quanta contained in the elementary bosonic bricks.

TABLE VI

Multiplicity of Fock basis states with given number of bosonic and fermionic quanta for the SU(3) gauge group. These numbers can be obtained from the generating function Eq. (34) and from the orthonormalization procedure of the algorithm independently.

n_B	0	1	2	3	4	5	6	7	8
0	1	—	—	1	—	1	—	—	1
1	—	1	1	1	2	1	1	1	—
2	1	1	1	3	4	3	1	1	1
3	1	1	3	5	4	5	3	1	1
4	1	2	2	5	8	5	2	2	1
5	1	2	4	7	8	7	4	2	1
6	2	2	4	9	10	9	4	2	2
7	1	3	5	9	12	9	5	3	1
8	2	3	5	11	14	11	5	3	2
9	2	3	7	13	14	13	7	3	2
10	2	4	6	13	18	13	6	4	2
11	2	4	8	15	18	15	8	4	2
12	3	4	8	17	20	17	8	4	3
13	2	5	9	17	22	17	9	5	2
14	3	5	9	19	24	19	9	5	3
15	3	5	11	21	24	21	11	5	3
16	3	6	10	21	28	21	10	6	3
17	3	6	12	23	28	23	12	6	3
18	4	6	12	25	30	25	12	6	4
19	3	7	13	25	32	25	13	7	3
20	4	7	13	27	34	27	13	7	4
21	4	7	15	29	34	29	15	7	4
22	4	8	14	29	38	29	14	8	4
23	4	8	16	31	38	31	16	8	4
24	5	8	16	33	40	33	16	8	5
25	4	9	17	33	42	33	17	9	4
26	5	9	17	35	44	35	17	9	5
27	5	9	19	37	44	37	19	9	5
28	5	10	18	37	48	37	18	10	5
29	5	10	20	39	48	39	20	10	5
30	6	10	20	41	50	41	20	10	6

Furthermore, from the polynomials $c_i(t)$ the combinatorial interpretation of the multiplicities of states in the fermionic sectors can be read off. Particularly, the number $d(n_F, n_B)$ of composite fermionic bricks with n_B bosonic quanta in a given fermionic sector is simply given by

$$d(n_F, n_B) = \frac{1}{n_B!} \frac{d^{n_B}}{dt^{n_B}} c_{n_F}(t) \bigg|_{t=0} . \tag{35}$$

As an example, let us take the polynomial $c_1(t)$ for the SU(3) group. We have one brick with a single bosonic quantum — the $(a^\dagger f^\dagger)$ brick, and one brick with two bosonic quanta — the $(a^\dagger a^\dagger f^\dagger)$ brick. A less trivial example is given by the $c_2(t)$ polynomial. Apart of the two operators with a single and double bosonic quanta, $(a^\dagger f^\dagger f^\dagger)$ and $(a^\dagger a^\dagger f^\dagger f^\dagger)$, respectively, we now have two operators with three bosonic quanta, namely, $(f^\dagger a^\dagger a^\dagger f^\dagger a^\dagger)$ and $(f^\dagger a^\dagger)(f^\dagger a^\dagger a^\dagger)$.

Thus, we can crosscheck the number of basis states obtained by direct construction and elimination of linearly dependent states with the one computed using the above group-theoretical predictions. Table VI presents the multiplicity of basis states with given n_B and n_F quanta for $N = 3$ up to $n_B = 30$, calculated with both methods. We simply give a single set of numbers since the results agree exactly.

2.5. Symmetries of the SU(N) SYMQM Fock basis

The Fock basis of SYMQM, constructed in the way described above, has several nontrivial symmetries which can be interpreted as announcements of dynamical symmetries of the Hamiltonians of SYMQM. The most important of them are the supersymmetry and particle–hole symmetry. The former can be observed as a matching of eigenenergies from neighboring fermionic sectors, while the latter is defined as a matching of spectra from the sector with p fermions and the sector with $N^2 - 1 - p$ fermions with $0 \leq p \leq N^2 - 1$. Let us now describe three observations [18] of nontrivial relations among the multiplicities $D(n_B, n_F)$ and their interpretations in terms of these symmetries.

Supersymmetry

- For each n_B there are as many bosonic basis states (states with n_F even) as fermionic basis states (states with n_F odd).

$$\forall_{n_B} \qquad \sum_{n_F\text{–even}} D(n_B, n_F) = \sum_{n_F\text{–odd}} D(n_B, n_F)\,.$$

  Its validity can be checked explicitly for SU(3) case, either in Table VI by summing the numbers of states with n_F even and odd in each row separately, or in Eqs. (34) by summing appropriate polynomials. This relation can be also exactly proved for any N using the general form of the generating function [18].

- Summing the number of states along diagonal lines with $n_B + n_F$ or $n_B - n_F$ fixed, for n_F even and n_F odd separately yields the same results,

$$\forall_{n_B} > 0 \qquad \sum_{n_F\text{–even}} D(n_B \pm n_F, n_F) = \sum_{n_F\text{–odd}} D(n_B \pm n_F, n_F)\,,$$

Such diagonal lines correspond to the action of the supersymmetric gauged harmonic oscillator supercharges, $Q = (fa)$, and $Q^\dagger = (f^\dagger a^\dagger)$. They are of special interest since introducing the cut-offs in the consecutive fermionic sectors according to one of these lines allows to obtain an exact supersymmetric degeneracy for finite cut-off. Again, this result can be proved for any N using the generating functions [18].

This is still not *true* supersymmetry. There is no dynamical super multiplets, because at this stage we have not defined any Hamiltonian. However, it is interesting that already at this level we have such matchings.

Particle–hole symmetry

- For any n_B, the number of states in the sector with n_F fermions is equal to the number of states in the sector with $N^2 - 1 - n_F$ fermions,

$$\forall_{n_B} \quad D(n_B, n_F) = D(n_B, N^2 - 1 - n_F)\,, \qquad n_F = 0, \ldots, N^2 - 1\,.$$

If we expect that the spectrum in the sector with n_F fermionic quanta coincides with the spectrum in the sector with $N^2 - 1 - n_F$ then the equality of the multiplicity of basis states in those sectors can be interpreted as a nontrivial announcing of the particle–hole symmetry.

Summarizing, already at kinematic level one can find symptoms of the symmetries of the SYMQM systems.

3. Description of the algorithm

The main idea of the recursive algorithm has been already described in [11, 12]. It relies on the observation that the most efficient way to evaluate a matrix element of an operator is to relate it to simpler matrix elements of some operators, which have been already evaluated at an earlier stage of calculations. In this way the explicit construction of the Fock basis vectors is not necessary. As an input the algorithm needs the commutators/anticommutators of elementary bricks and any other invariant operators, which appear in these resulting commutators/anticommutators.

In order to expose the algorithm in a clear way, we will start by explaining the construction of the Fock basis and then the calculation of the matrix of scalar products. Having such a matrix, one can orthonormalize

the basis vectors. The procedure used to this end will be described in the subsequent subsections. At that point the formula used for calculation of matrix elements of any operator will become evident. Eventually, the full recursive relations will be presented.

3.1. Recursive construction of the Fock basis

Although the Fock states are not explicitly needed for the computation of matrix elements of an operator, they do appear in the labeling of those matrix elements. Hence, we should have a recursive way of obtaining the Fock basis. Let us assume that such basis is already constructed in the sectors with the number of bosonic and fermionic quanta smaller than n_B and n_F respectively. Then, the Fock basis in the sector with n_B bosonic and n_F fermionic quanta can be created as the sum of all states obtained by the action of appropriate bricks on the already generated Fock basis states. In our generalized notation this can be written as

$$|\{n_B, n_F\}\rangle = \sum_{k=2}^{N} C^{\dagger}(k,0)|\{n_B - k, n_F\}\rangle \,.$$

Note that in general such states will not form an orthonormal set of states. Moreover, the same state may appear in several copies, differing in the order of successive bricks used to build it. Those duplicates will be treated as distinct states. The basis is obtained once this redundancy is removed and the remaining states orthonormalized.

3.2. Matrix of scalar products

The Fock basis obtained recursively form a complete set of states which are however not orthonormalized. Thus, one has to calculate the matrix of scalar products. It is sufficient to calculate the scalar products among the states containing a given number of quanta, n_B and n_F, since those having different number of bosonic or fermionic quanta are orthogonal by definition. The matrix of such scalar products will be denoted by $S(n_B, n_F)$.

The definition of $S(n_B, n_F)$ differs from the standard definition of the Gram matrix in few aspects. The traditional Gram matrix contains the scalar products of linearly independent states and for our system it is a $D(n_B, n_F) \times D(n_B, n_F)$ matrix. Contrary, since in our algorithm the Fock basis is defined recursively, the $S(n_B, n_F)$ is the matrix of scalar products of states that have n_B bosonic and n_F fermionic quanta and are obtained by the action of appropriate bricks on states from sectors with smaller number of bosonic and fermionic quanta. Therefore, the matrix $S(n_B, n_F)$ is usually bigger than the Gram matrix since some of the states can be included several

times. Such redundancy is best illustrated by the following example. For the SU(3) gauge group, there is only one Fock state with 5 bosonic quanta, $|\{5,0\}\rangle = C^\dagger(2,0)C^\dagger(3,0)|0\rangle$. However, the calculation of the matrix of scalar products $S(5,0)$ yields a 2×2 matrix instead of a single number, namely,

$$S(5,0) = \langle\{5,0\}|\{5,0\}\rangle \longrightarrow \begin{pmatrix} S(5,0)_{3,3} & S(5,0)_{3,2} \\ S(5,0)_{2,3} & S(5,0)_{2,2} \end{pmatrix},$$

where the different matrix elements $S(5,0)_{i,j}$ correspond to the multiple possibilities of pulling out an elementary brick out of the state $|\{5,0\}\rangle$,

$$|\{5,0\}\rangle = C^\dagger(2,0)|\alpha\rangle = C^\dagger(3,0)|\beta\rangle\,,$$

where $|\alpha\rangle$ and $|\beta\rangle$ are appropriate remaining states. Such doubling must be eliminated. The procedure which we used to achieve this will be described in the next subsection.

The evaluation of $S(n_B,n_F)$ can be divided into two separate cases.

If $n_F = 0$, we pull out one bosonic elementary brick from each of the basis states. Since we have $N-1$ bosonic elementary bricks (see Tables II, III, IV), there is $N-1$ different ways to do this, if only $n_B \geq N$. If $n_B < N$ we can pull out only $n_B - 1$ different bosonic elementary bricks. Thus, in a generic situation, $S(n_B,0)$ will be a $(N-1)\times(N-1)$ matrix of the form,

$$S(n_B,0) = \begin{pmatrix} S(n_B,0)_{2,2} & S(n_B,0)_{2,3} & \dots & S(n_B,0)_{2,N-1} \\ S(n_B,0)_{3,2} & S(n_B,0)_{3,3} & \dots & S(n_B,0)_{3,N-1} \\ & & \vdots & \\ S(n_B,0)_{N-1,2} & S(n_B,0)_{N-1,3} & \dots & S(n_B,0)_{N-1,N-1} \end{pmatrix},$$

where

$$S(n_B,0)_{p,q} \equiv \langle\{n_B-p,0\}|C(p,0)C^\dagger(q,0)|\{n_B-q,0\}\rangle\,.$$

The matrix element, calculated by extracting the $C(p,0)$ elementary brick from the left state and the $C^\dagger(q,0)$ elementary brick from the right state, denoted by $S(n_B,0)_{p,q}$, can be expressed in terms of matrix elements of operators between basis states with lower number of bosonic quanta as,

$$\begin{aligned} S(n_B,0)_{p,q} \;=\; & \langle\{n_B-p,0\}|\big[C(p,0),C^\dagger(q,0)\big]|\{n_B-q,0\}\rangle \\ & + \langle\{n_B-p,0\}\,|C^\dagger(q,0)C(p,0)|\,\{n_B-q,0\}\rangle\,. \end{aligned} \tag{36}$$

Hence, we expressed S from the sector with n_B quanta in terms of matrix elements of operators evaluated in the sector with $n_B - q$ quanta. The procedure to calculate these matrix elements is described in Subsection 3.3.

If $n_F \neq 0$, we first want to anticommute the composite fermionic bricks with n_F fermionic quanta. Again, in a generic case, we treat $S(n_B, n_F)$ as a matrix of size $d(n_F) \times d(n_F)$ and of the form

$$S(n_B, n_F) = \begin{pmatrix} S(n_B,n_F)_{1,1} & S(n_B,n_F)_{1,2} & \dots & S(n_B,n_F)_{1,d(n_F)} \\ S(n_B,n_F)_{2,1} & S(n_B,n_F)_{2,2} & \dots & S(n_B,n_F)_{2,d(n_F)} \\ & & \vdots & \\ S(n_B,n_F)_{d(n_F),1} & S(n_B,n_F)_{d(n_F),2} & \dots & S(n_B,n_F)_{d(n_F),d(n_F)} \end{pmatrix},$$

where

$$S(n_B, n_F)_{p,q} \equiv \langle \{n_B - n_p, 0\} \, | C(n_p, n_F, p) C^\dagger(n_q, n_F, q) | \, \{n_B - n_q, 0\} \rangle \, .$$

Let us consider one of the scalar products, denoted by $S(n_B, n_F)_{p,q}$, and obtained from the scalar product of two basis vectors, first containing $C(n_p, n_F, p)$, second containing $C^\dagger(n_q, n_F, q)$. We have

$$S(n_B, n_F)_{p,q} \; = \; \langle \{n_B - n_p, 0\} \, | \big\{ C(n_p, n_F, p), C^\dagger(n_q, n_F, q) \big\} | \, \{n_B - n_q, 0\} \rangle \, .$$

The anticommutator $\big\{ C(n_p, n_F, p), C^\dagger(n_q, n_F, q) \big\}$ is a normally ordered operator containing only bosonic creation and annihilation operators. If it is not, one can always bring it to such form[4]. The matrix elements of the operator $C^\dagger(n_q, n_F, q) C(n_p, n_F, p)$ vanish since there are fermionic annihilation operators acting on the Fock vacuum. Thus, by anticommuting all fermionic operators in one step, we can express $S(n_B, n_F)$ in terms of matrix elements of operators between states from the bosonic sector exclusively.

Summarizing, the procedure of calculating $S(n_B, n_F)$ consists of three steps. First, we pull out one bosonic (fermionic) brick from the left and right states. There is in general $N - 1$ ($d(n_F)$) ways to do this. Second, we commute (anticommute) these two bricks, and replace the commutator (anticommutator) by a normally ordered operator. Third, we evaluate the matrix elements of this operator between states with a lower number of quanta (see Section 3.3).

[4] Such an anticommutator is equal to a sum of operators involving only bosonic creation and annihilation operators. In general not all of these operators will be normally ordered operators. However, any such operators can be brought to a normally ordered form by appropriately ordering the creation and annihilation operators, which they are composed of, using the commutation rules Eqs. (1). Such ordering will produce additional operators which have to be taken into account and which can also be brought to a normally ordered. Thus, any anticommutator can always be written in a normally ordered form.

However, before we move to the description of the evaluation of matrix elements of operators, we have to tackle the problem of orthonormalization of the basis states and of the evaluation of the commutators of composite bricks. We do this in the following two subsections.

3.2.1. Linear independence and orthonormalization

The recursive approach produces a set of Fock states in which some states may be contained in several copies. Therefore, one has to implement a mechanism to remove such redundancy. Because of the recursive structure, such mechanism will have to deal only with states with a given number of bosonic and fermionic quanta. Hence, its computational effort is small compared to what would it be should the whole Fock basis be considered. Moreover, the remaining states must be orthonormalized. We use a procedure which realizes these two tasks in one step. It is done by a numerical diagonalization of the matrix of scalar products $S(n_B, n_F)$. Subsequently, eigenvectors with corresponding nonzero eigenvalues are retained. Since the Fock states are not normalized, the normalized eigenvectors have to be multiplied by the inverse of the square root of their corresponding eigenvalues. We group such vectors in a matrix denoted by $R(n_B, n_F)$. Note that $R(n_B, n_F)$ is not to a square matrix in general. Then, we can write

$$R^T(n_B, n_F)S(n_B, n_F)R(n_B, n_F) = 1_{D(n_B,n_F)\times D(n_B,n_F)}\,,$$

where $1_{D(n_B,n_F)\times D(n_B,n_F)}$ is the unity matrix which rank is equal to the size of the subspace of the Hilbert space with n_B bosonic and n_F fermionic quanta.

3.2.2. Automatic evaluation of commutators and anticommutators

We can conclude from Eqs. (36) and (37) that, in order to calculate a scalar product or a matrix element of some operator, the set of commutators and anticommutators of all composite bricks must be supplied. These commutators and anticommutators must be brought to a gauge invariant, normally ordered form and should be maximally reduced using the Cayley–Hamilton theorem. In general, there will appear some new single trace operators containing both creation and annihilation bosonic operators. Their commutators and anticommutators with all composite bricks should be evaluated as well and supplied to the algorithm. The number of such relations to be calculated grows rapidly, both, with increasing N, and increasing fermionic occupation number. Already, for the SU(3) gauge group one needs about a thousand of (anti)commutators. Therefore, a computer program was written to evaluate them. It uses standard relations (1), (2) and (3) between the bosonic and fermionic creation and annihilation

operators to move them among and within traces. As an example Eq. (37) presents one of the commutators needed for the calculations,

$$\begin{aligned}[(a^\dagger a^\dagger aa^\dagger a^\dagger aa),(a^\dagger a^\dagger)] \;=\; & 1.5555(a^\dagger a^\dagger a^\dagger)+1.3333(a^\dagger a^\dagger)(a^\dagger a^\dagger a)\\ & +(a^\dagger a^\dagger a^\dagger)(a^\dagger a^\dagger aa)+1.1111(a^\dagger a^\dagger a^\dagger)(a^\dagger a)\\ & +0.1666(a^\dagger a^\dagger)(a^\dagger a^\dagger a^\dagger)(aa)+(a^\dagger a^\dagger)(a^\dagger a^\dagger aa^\dagger a)\\ & +0.25(a^\dagger a^\dagger)(a^\dagger a^\dagger)(a^\dagger aa)\,. \end{aligned} \tag{37}$$

We have found that it is more efficient to work with composite bricks than elementary ones. For example, if we have treated the operator $(a^\dagger a^\dagger f^\dagger)(a^\dagger f^\dagger)(a^\dagger f^\dagger f^\dagger)$, which is one of the composite bricks in the $n_F = 4$ sector for the SU(3) group, as a product of three operators, we would relate the desired matrix element with matrix elements from the $n_F = 3$ sector, which, in turn, would be related to some matrix elements in the $n_F = 2$ sector, and so on. However, one could treat it as a single operator, and jump to the bosonic sector in one step, considerably decreasing the number of commutation and anticommutation relations needed for such computations. On the other hand, the more elementary bricks will be contained in a composite brick, the more complicated the commutation and anticommutation relations will be. Hence, the usefulness of using more complex composite bricks is a question of balancing between the processor time consumed for evaluation of these relations, the memory needed to store them and the processor time gained by using more complex (anti)commutators.

3.3. Matrix elements of gauge invariant operators

In this section we describe the evaluation of matrix elements of operators, such as those appearing in the right hand side of Eqs. (36) and (37).

Let us denote a generic operator by $O(n_B^O, n_F^O)$. The arguments of O, n_B^O and n_F^O have the following meaning

- n_B^O is the difference between the number of bosonic creation and annihilation operators contained in O,
- n_F^O is the difference between the number of fermionic creation and annihilation operators in O.

In general, n_B^O and n_F^O can be any integers. Furthermore, we denote by $O(n_B^O, n_F^O)_{n_B,n_F}$ the matrix element of O between basis states containing n_B bosonic and n_F fermionic quanta on the right hand side, and $n'_B = n_B + n_B^O$ bosonic and $n'_F = n_F + n_F^O$ fermionic quanta on the left-hand side,

$$O\left(n_B^O, n_F^O\right)_{n_B,n_F} \;=\; \langle\{n'_B, n'_F\}|O\left(n_B^O, n_F^O\right)|\{n_B, n_F\}\rangle\,. \tag{38}$$

$O\left(n_B^O, n_F^O\right)_{n_B,n_F}$ is a matrix of sizes $D(n_B, n_F) \times D(n'_B, n'_F)$, where the numbers $D_{n,m}$ denote the multiplicity of basis states with n bosonic and m fermionic quanta, as it was introduced in Section 2.4.1.

We first deal with some "boundary" situations, and then we consider the generic case.

3.3.1. Boundary cases

Let $\#(x)_O$ denote the number of occurrences of the operator x in the operator O. The following observations can be exploited to simplify the computations:

- If a matrix element of an operator for which $\#(f^\dagger)_O > \#(f)_O$ is to be calculated then it is more convenient to evaluate its complex conjugate. Similarly, if we have to compute a matrix element of an operator for which $\#(f^\dagger)_O = \#(f)_O$, but $\#(a^\dagger)_O > \#(a)_O$ we should rather compute its complex conjugate.
- The matrix element of an operator O which has fermionic or bosonic annihilation operators acting on the Fock vacuum vanishes.
- The matrix element of the bosonic elementary brick between states from the bosonic sector can be read off from the appropriate part of the matrix of scalar products.
- The matrix element of an operator which is a product of two trace operators can be calculated by inserting an identity operator between them, evaluating their matrix elements separately, and eventually multiplying and summing the partial results.

3.3.2. Generic case

In the generic case, we can assume that O is normally ordered (see footnote 4). If it is composed exclusively of creation operators, one can express it in terms of bosonic elementary bricks and use the appropriate boundary case, described above.

The strategy to evaluate a matrix element of O is to drag O over the operators constituting the right hand side state so that it annihilates the Fock vacuum. We start by pulling it through the composite fermionic brick,

$$\begin{aligned}
&\left(O\left(n_B^O, n_F^O\right)_{n_B,n_F}\right)_{.,p} = \langle\left\{n'_B, n'_F\right\}|O\left(n_B^O, n_F^O\right) C^\dagger(n_p, n_F, p)|\left\{n_B, n_F\right\}\rangle \\
&= \langle\left\{n'_B, n'_F\right\}|\left[O\left(n_B^O, n_F^O\right), C^\dagger(n_p, n_F, p)\right]|\left\{n_B - n_p, 0\right\}\rangle \\
&+\langle\left\{n'_B, n'_F\right\}|C^\dagger(n_p, n_F, p)O\left(n_B^O, n_F^O\right)|\left\{n_B - n_p, 0\right\}\rangle\,.
\end{aligned} \tag{39}$$

In order to move further, we substitute the relation for the (anti)commutator of $O(n_B^O, n_F^O)$ and $C^\dagger(p, n_F, \alpha)$. For each operator appearing in this relation we evaluate its matrix element, first checking whether conditions for any of the special cases are met. This task should be easier, since these matrix elements must be evaluated between states with smaller number of bosonic and fermionic quanta. The second term in Eq. (39), can be calculated by inserting an identity operator between the operators $C^\dagger(n_p, n_F, p)$ and $O(n_B^O, n_F^O)$. Again, this computation should be easier. On one hand, the matrix element of the elementary brick $C^\dagger(n_p, n_F, p)$ should be known from the matrix of scalar products. On the other hand, the matrix element of $O(n_B^O, n_F^O)$ involves states with smaller number of bosonic and fermionic quanta.

The purely bosonic case can be treated analogously. We have

$$\begin{aligned}\left(O\left(n_B^O,0\right)_{n_B,0}\right)_{.,p} =& \langle\left\{n'_B,0\right\}|\left[O\left(n_B^O,0\right),C(p,0)\right]|\left\{n_B-p,0\right\}\rangle \\ &+\langle\left\{n'_B,0\right\}|C(p,0)O\left(n_B^O,0\right)|\left\{n_B-p,0\right\}\rangle\,. \end{aligned} \tag{40}$$

In principle, we can proceed with those relations until O hits the Fock vacuum. Since O is a normally ordered operator, such matrix element vanishes by definition. Therefore, collecting all the intermediate results, we should be able to evaluate the desired matrix element. Nevertheless, one has to remember that states in $|\left\{n_B,0\right\}\rangle$ are in general not orthonormal, so one has to implement into relations (39) and (40) the orthonormalization procedure.

3.3.3. Recurrence relations

Since the recursively constructed set of basis states can contain degenerate states, the matrices of operators, calculated as described above, will also contain such redundant matrix elements. In order to get rid of them one has to use the $R(n_B, n_F)$ matrix. Incorporating this matrix in the relations for the matrix element of any operator, enables us to formulate the complete and correct recurrence relations. We have two recurrence relations:

- the expression of the matrix element in the fermionic sectors in terms of matrix elements in the bosonic sector,

$$\begin{aligned}&\langle\left\{n'_B,n'_F\right\}|O(n_B^O,n_F^O)|\left\{n_B,n_F\right\}\rangle = \\ &\Big(\left\langle\left\{n'_B,n'_F\right\}|\left[O(n_B^O,n_F^O),C^\dagger(n_p,n_F,p)\right]|\left\{n_B-n_p,0\right\}\right\rangle \\ &+\left\langle\left\{n'_B,n'_F\right\}|C^\dagger(n_p,n_F,p)O(n_B^O,n_F^O)|\left\{n_B-n_p,0\right\}\right\rangle\Big)\cdot R(n_B,n_F)\,, \end{aligned} \tag{41}$$

and

- the expression of the matrix element in the bosonic sector with bigger number of bosonic quanta in terms of matrix elements with smaller number of bosonic quanta,

$$\begin{aligned}&\langle\{n'_B,0\}|O(n_B^O,0)|\{n_B,0\}\rangle = \\ &\Big(\Big\langle\{n'_B,0\}\big|\big[O(n_B^O,0),C^\dagger(p,0)\big]\big|\{n_B-p,0\}\Big\rangle \\ &+\Big\langle\{n'_B,0\}|C^\dagger(p,0)O(n_B^O,0)|\{n_B-p,0\}\Big\rangle\Big)\cdot R(n_B,0)\,. \end{aligned} \tag{42}$$

Thus, we have expressed the desired matrix element in terms of matrix elements of operators between states will lower number of quanta, which should have been already evaluated during some previous calculations.

With these recurrence relations the presentation of the whole algorithm is complete. One can use them to evaluate the matrix of scalar products in the sectors which have not been considered so far in the calculations. This done, the matrix elements of operators, needed for the calculations in sectors with yet bigger number of quanta, can be computed. In this way one can proceed until the cut-off $N_{\rm cut}$ is reached.

Note that the above algorithm is very universal. In principle it can be used to systems defined in space of any dimensionality. Particularly, if the Hamiltonian is invariant under a SO(d) symmetry our method can be generalized in order to calculate the spectra in the channels with given angular momentum [12]. Moreover, it is applicable to systems with bosonic and fermionic polynomial interactions as well as to systems with discrete or continuum spectrum.

In the next section we describe the results for supersymmetric model with discrete spectrum obtained with our approach.

4. Applications

As an application of the above algorithm we present results for a supersymmetric system given by the Hamiltonian constructed in 2.2,

$$H = \mathrm{Tr}\,p^2 + g^2\left(\mathrm{Tr}\;x^4 - \tfrac{1}{3}\left(\mathrm{Tr}\;x^2\right)^2\right) + 2g\;\mathrm{Tr}\;\Big(x\big[f^\dagger,f\big]\Big)\;, \tag{43}$$

where the gauge symmetry group is chosen to be the SU(3) group.

The main motivation for studying this Hamiltonian is that, being a system with a discrete spectrum, it is a good test-ground for our numerical method. Moreover, the bosonic part of its potential is similar to the potential of SYMQM with the tensor f_{abc} replaced by d_{abc} [12]. Therefore, one hopes that some analytic approaches based on the numerical results from

such simple model can be tested for a future application to the more dimensional SYMQM systems. On the other hand, Eq. (43) is interesting by itself. Being a supersymmetric anharmonic oscillator it contains a nontrivial fermionic interaction.

In the following we start by analyzing the convergence of the eigenvalues, then present the spectra calculated numerically. Subsequently, we briefly describe their symmetries, such as supersymmetry and scaling symmetry. Eventually, we calculate numerically the Witten index for this model.

4.1. Numerical spectra

One of the advantages of the cut Fock space approach as a numerical method is that it enables one to judge on the reliability of the results and estimate their errors. To this end, the convergence of eigenenergies or eigentates with increasing cut-off must be investigated. Table VII contains the energies of the lowest eigenstate in fermionic sector with $n_F = 0, \ldots, 4$ obtained for different N_{cut}. The results in sectors with higher fermionic

TABLE VII

The dependence on the cut-off N_{cut} of the lowest eigenenergy in several fermionic sectors. Note the exact degeneracy of the levels in the $n_F = 0$ and $n_F = 1$ sectors due to supersymmetry and the appearance of the supersymmetric vacuum in the $n_F = 3$ sector.

N_{cut}	$n_F = 0$	$n_F = 1$	$n_F = 2$
1	2.833333333333333	3.750000000000124	3.75000000000008
5	2.805137759654418	2.817654396966426	2.41010649311797
10	2.804878933491876	2.804943385906189	2.38393952020263
15	2.804877899477374	2.804878578502977	2.38379874405844
20	2.804877857980324	2.804877869314702	2.38379576457689
25	2.804877857812559	2.804877857890121	2.38379573799721
30	2.804877857802534	2.804877857804384	2.38379573773261
35	2.804877857802529	2.804877857803605	2.38379573772474
40	2.804877857802507	2.804877857803596	2.38379573772458
N_{cut}	$n_F = 3$	$n_F = 4$	
1	1.009109012532963	3.750000000000082	
5	0.017808308382480	2.001903558629864	
10	0.000102896003680	1.978068350121234	
15	0.000002580129746	1.977963562445323	
20	0.000000018188518	1.977960963230507	
25	0.000000000380532	1.977960939859982	
30	0.000000000013940	1.977960939644200	
35	0.000000000004579	1.977960939638051	
40	0.000000000004250	1.977960939637698	

occupation number are related to the ones presented in the table through the particle–hole symmetry. A convergence to more than 10 digits is achieved with $N_{\rm cut} = 40$, which corresponds to a Fock basis of about 150 states in the bosonic sector and about 1300 states in the $n_F = 4$ sector. Therefore, one can safely use the cut-off $N_{\rm cut} = 40$ in order to evaluate the lowest eigenenergies. The uncertainties of those eigenenergies, defined as the difference of the outcomes for consecutive cut-offs, are negligible. Fig. 1 and Fig. 2

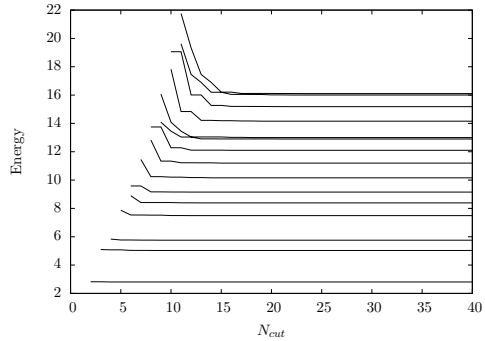

Fig. 1. Convergence of the 15 lowest eigenenergies in the bosonic sector with increasing cut-off. An exponential-like convergence is seen, which is a characteristic feature of bound states.

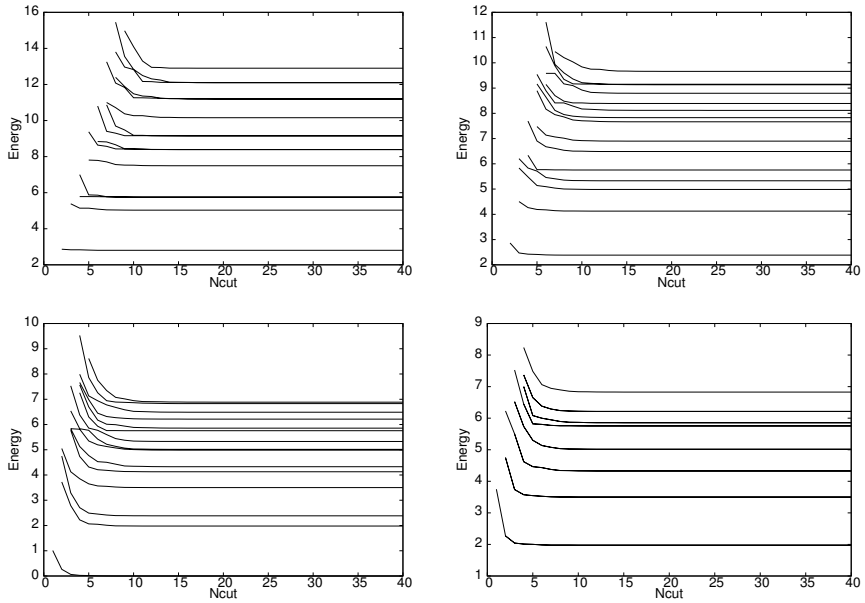

Fig. 2. Convergence of the few lowest eigenenergies in the $n_F = 1, \ldots, 4$ sectors with increasing cut-off.

show the dependence of 15 lowest eigenenergies on the cut-off in different fermionic sectors. All the figures were made for $g = 1.0$. We can clearly see a fast, exponential-like, convergence, which was shown to be a characteristic feature of bound states [14, 15]. Hence, one can conclude that indeed all the spectra are discrete. This should be contrasted with the results for the SYMQM systems. The latter have potentials with flat directions which induce continuum spectra in some sectors [8, 12, 13]. Table VII contains the eigenenergies of lowest states in all 9 fermionic sectors calculated with $g = 1.0$ and a cut-off $N_{\rm cut} = 40$. We comment these results in more details in the following subsection.

4.2. Symmetries

4.2.1. Supersymmetry

Supersymmetry can be seen in Table VIII as an exact degeneracy of the converged eigenenergies in neighboring fermionic sectors. One notices a non-degenerate vacuum state in sector with $n_F = 3$ and its image through the particle–hole symmetry in the sector with $n_F = 5$ signaling an unbroken supersymmetry. The number of supersymmetric vacua was discussed in [27] in the case of free SU(N) model. Accordingly, there are four zero-energy states in sectors with $n_F = 0, 3, 5, 8$. Hence, two of these states disappear when the interaction is turned on. The particle–hole symmetry is responsible for a double degeneracy of eigenenergies in the sector with $n_F = 4$. This sector must contain as many states which are parts of supermultiplets formed with states from the sector with $n_F = 3$ as there are states which are parts of supermultiplets formed with states from the sector with $n_F = 5$. Both these sets must have the same spectra. This is a specific feature of models with SU(N) gauge group with N odd.

Table VII reveals also signatures of supersymmetry. On one hand, the lowest states in sectors with $n_F = 0$ and $n_F = 1$ are exactly degenerate. On the other hand, the zero-energy vacuum states in the sectors with $n_F = 3$ and $n_F = 5$ fermionic quanta are non-degenerate.

4.2.2. Scaling symmetry

The scaling property of the quantum anharmonic oscillator was first noted by Symanzik, and elaborated by Simon [28]. If we consider the transformations

$$x \to \lambda x\,, \qquad p \to \frac{1}{\lambda} p\,, \qquad f \to f\,, \qquad f^\dagger \to f^\dagger\,, \tag{44}$$

then the Hamiltonian is rescaled as

$$H \to H = \frac{1}{\lambda^2}\left(\mathrm{Tr}\, p^2 + g^2\lambda^6\Big(\mathrm{Tr} x^4 - \frac{1}{N}(\mathrm{Tr}\, x^2)^2\Big) + 2g\ \lambda^3 \mathrm{Tr}(x[f^\dagger, f])\right)\,. \tag{45}$$

TABLE VIII

Eigenenergies of few lowest eigenstates with $g = 1.0$. The states in the $n_F = 4$ sector are double degenerate which follows from the particle–hole symmetry. Hence the ^{2} notation. An exact, supersymmetric pairing can be observed among states in adjacent fermionic sectors.

$n_F = 0$	$n_F = 1$	$n_F = 2$	$n_F = 3$	$n_F = 4$	$n_F = 5$	$n_F = 6$	$n_F = 7$	$n_F = 8$
			0		0			
			1.97796	1.97796^2	1.97796			
		2.3838	2.3838		2.3838	2.3838		
2.80488	2.80488						2.80488	2.80488
			3.50147	3.50147^2	3.50147			
		4.12774	4.12774		4.12774	4.12774		
			4.32778	4.32778^2	4.32778			
		4.98444	4.98444		4.98444	4.98444		
			5.01572	5.01572^2	5.01572			
5.02988	5.02988						5.02988	5.02988
		5.32891	5.32891		5.32891	5.32891		
5.75469	5.75469						5.75469	5.75469
	5.75469	5.75469				5.75469	5.75469	
			5.75469	5.75469^2	5.75469			
			5.85572	5.85572^2	5.85572			
			6.21387	6.21387^2	6.21387			
		6.48704	6.48704		6.48704	6.48704		
			6.82769	6.82769^2	6.82769			
			6.88508	6.88508^2	6.88508			
		6.89869	6.89869		6.89869	6.89869		
7.49223	7.49223						7.49223	7.49223
		7.66421	7.66421		7.66421	7.66421		
			7.75149	7.75149^2	7.75149			
			7.81189	7.81189^2	7.81189			
		7.82955	7.82955		7.82955	7.82955		
		8.11747	8.11747		8.11747	8.11747		
8.39012	8.39012						8.39012	8.39012
	8.39012	8.39012				8.39012	8.39012	
			8.39012	8.39012^2	8.39012			

Setting $\lambda = g^{-\frac{1}{3}}$ we obtain the following identity,

$$H(g) \to g^{\frac{2}{3}} H(1)\,. \tag{46}$$

Since, the transformations Eq. (44) can be unitarily implemented, both Hamiltonians in Eq. (46) have identical eigenvalues. Therefore, it is sufficient to calculate the spectrum at $g = 1.0$.

Figure 3 shows the dependence of six lowest eigenenergies from the bosonic sector on the coupling constant. The numerical results are compared with the prediction of Eq. (46). For large g the agreement of both should be noted, whereas the discrepancies for small g are due to finite cut-off effects as discussed below.

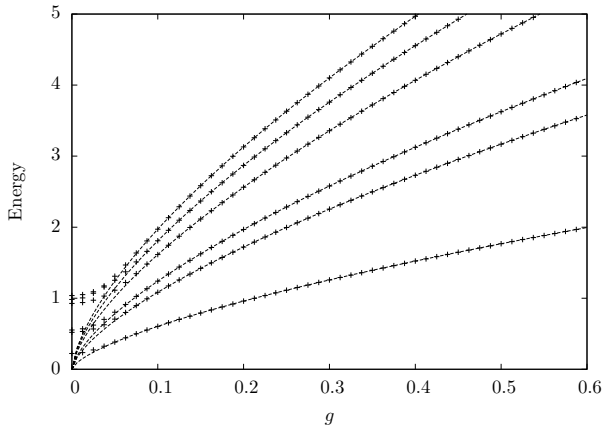

Fig. 3. Dependence of the first six eigenenergies on the coupling constant g in the bosonic sector. Crosses denote the numerical results obtained with cut-off $N_{\mathrm{cut}} = 40$, whereas the dashed lines correspond to the prediction of Eq. (46) with $H(1)$ evaluated numerically. The finite cut-off effects can be seen for small values of the coupling constant.

4.3. Critical slowing down

A critical slowing down can be observed in Fig. 3 in the vicinity of $g = 0$. For a small value of the coupling constant a much higher cut-off is needed in order to obtain converged results. It is because at $g = 0$ the spectrum is free and the eigenenergies corresponding to nonlocalized states calculated by our algorithm do not converge. Rather, they fall off to zero with increasing cut-off in a power like manner, a behavior resulting from approximating a plane wave by a finite set of localized harmonic oscillator eigenstates. Hence, the eigenenergies calculated for small coupling constant at small cutoff cannot follow the curve Eq. (46) of exactly converged energies. Nevertheless, with increasing coupling constant the eigenenergies of bound states converge more and more rapidly. Those energies that have already converged with the cut-off agree with the analytic prediction of Eq. (46). The complete discussion of such critical behavior is out of scope of the present work and will be carried out elsewhere.

4.4. Perturbative expansion

In a cut Fock space the states corresponding to the free states in the continuum are normalizable. Moreover, their exact analytic form can be obtained for a finite cut-off [30, 31]. Therefore, *at fixed finite cut-off* $N_{\rm cut}$ one can perform a perturbative expansion in the coupling constant g. Using well-known formulae for the perturbative corrections to the energy [32] one can obtain the approximate dependence of the eigenenergies on the coupling constant. We present here the results for the lowest eigenstate from the bosonic sector calculated for the cut-off $N_{\rm cut} = 20$,

$$\begin{aligned} E &= E_0 + g^2 V_{E_0,E_0} + g^4 \sum_{E' \neq E_0} \frac{\left|V_{E_0,E'}\right|^2}{E_0 - E'} \\ &\quad + g^6 \left(\sum_{\substack{E' \neq E_0 \\ E'' \neq E_0}} \frac{V_{E_0,E'} V_{E',E''} V_{E'',E_0}}{(E_0 - E')(E_0 - E'')} - V_{E_0,E_0} \sum_{E' \neq E_0} \frac{\left|V_{E_0,E'}\right|^2}{(E_0 - E')^2} \right) \\ &= 0.788363 + 53.6563 g^2 - 986.556 g^4 + 977.818 g^6 \,, \end{aligned} \tag{47}$$

where $V_{E,E'}$ is the matrix element of the potential between states with energies E and E' respectively. The comparison of this formula with the numerical results and the prediction of Eq. (46) is presented on Fig. 4. It is remarkable that the perturbative expansion attains the values of the

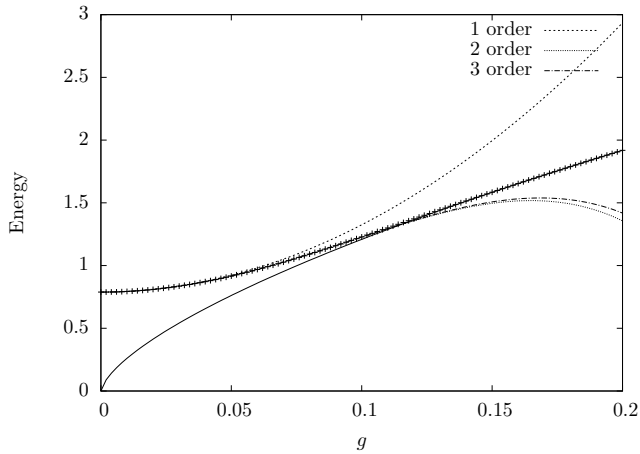

Fig. 4. Comparison of the dependence of the energy of the lowest bosonic eigenstate on the coupling constant g. Numerical data (crosses) are plot together with the prediction of Eq. (46) (solid line), and the perturbative expansion Eq. (47) (dashed lines).

coupling constant where the numerical results have converged for cut-off N_{cut} and agree with Eq. (46). This gives us hope that with an improved perturbative expansion the whole spectrum may be obtained analytically. This issue is being investigated in more details.

4.5. Witten's index

The Witten index is defined as [33, 34]

$$I_{\text{W}}(T) = \sum_{b \in \text{bosonic states}} e^{-E_b T} - \sum_{f \in \text{fermionic states}} e^{-E_f T}. \tag{48}$$

It is a commonly used quantity to study supersymmetry in quantum mechanics. In our set up, the sums over the bosonic and fermionic states are finite due to the cut-off. For a given N_{cut} one can just plug the eigenenergies obtained from each fermionic sector into Eq. (48). Our numerical results are shown on Fig. 5, where the dependence of $I_{\text{W}}(T)$ on the euclidean time T is presented. One notices a rapid convergence of $I_{\text{W}}(T)$ to the value -2. This confirms the fact that the model has two supersymmetric vacua.

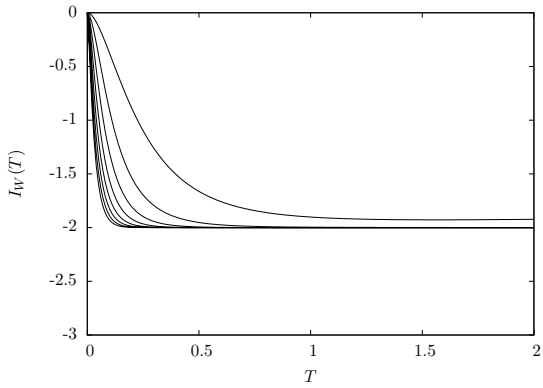

Fig. 5. Witten index. Different curves correspond to increasing cut-offs: 5, 10, 15, ..., 40.

The collapse of the Witten index at $T = 0$ to zero can be explained in the following way. At $T = 0$ the $I_{\text{W}}(T)$ is just the difference in the numbers of bosonic and fermionic states. However, the cut Fock basis has an equal number of bosonic and fermionic states (see Subsection 2.5). Thus, the value of the Witten index at $T = 0$ is zero for any cut-off. This is no longer true at $T \neq 0$. Especially, for $T \to \infty$, contributions from the states with nonzero energies cancel, and $I_{\text{W}}(T)$ counts the numbers of supersymmetric vacua, equal -2 in the present case.

5. Conclusions

In this paper we have described in a very detailed manner a recursive algorithm for evaluation of matrix elements of any gauge invariant operator in a Fock basis of Hilbert space. It can be applied to systems with any SU(N) gauge group and allows the evaluation of spectra in all fermionic sectors. We demonstrated the calculations on an example of anharmonic oscillator with supersymmetric interactions for the SU(3) group.

We started by describing the idea of the numerical approach to quantum mechanics in the Hamiltonian formulation using the cut Fock space method. Then, we presented the construction of gauge-invariant Fock basis, and particularly we introduced the concepts of elementary bosonic bricks and composite fermionic bricks. Such approach provided us a systematic, recursive description of the Fock states with increasing number of quanta. We discussed the properties and symmetries of such basis. Next, we concentrated on the numerical algorithm. We described the calculation of the matrix of scalar products, emphasizing the main ideas of the recursion relations. Then, other parts of the algorithm were presented: the procedure which removes redundant basis vectors and orthonormalizes the remaining ones as well as the program which automatically calculates the commutators and anticommutators of given operators. Finally, the expressions for the evaluation of matrix elements of any operator were outlined, and eventually, the full recursion relations were presented. In the third part of this article we applied our algorithm to a supersymmetric system with SU(3) gauge group and a discrete spectrum. We used it as a particularly well suited test-ground for our approach. We calculated the eigenenergies in all 9 fermionic sectors and discussed their symmetries. Eventually, we were also able to obtain the Witten index for this system.

The main advantage of this algorithm is that it treats bosons and fermions on an equal footing, and thus, enables calculations in any fermionic sector of the Hilbert space. This should be contrasted with the sign problems encountered in lattice field theories. As a result an exact supersymmetric degeneracy can be obtained even for finite cut-off. Moreover, the approach can be applied to systems with discrete and continuous spectra as well as possessing any kind of gauge symmetry. Particularly, the supersymmetric anharmonic oscillator presented in this article with gauge groups with $N \geq 3$ is currently investigated. Similarly, the $D = 2$, supersymmetric Yang–Mills quantum mechanics with several gauge groups, such as SU(3), SU(4) and SU(5) are studied analytically and numerically [30, 31]. The flexibility of the algorithm enables also an generalization to higher dimensions, with the ultimate $D = 10$, SYMQM case in mind. Results of the work is this direction are promising.

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