THERMODYNAMICAL QUANTITIES AND RELATIVITY*

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An introduction to the old controversy about a relativistic transformation of thermodynamical quantities is presented from a personal point of view. New formulas, derived for an ideal gas, are advocated on the basis of Clausius–Caratheodory axiomatic thermodynamics.

PACS numbers: 03.30.+p, 05.70.Ce

1. Story

When I was a student, some day inspecting the popular exercise-book [1] in hope of finding some solutions to my homework problems, I incidentally came across the following one:

$$\text{``}\mathbf{148}. \quad T' = \frac{\partial E'}{\partial S'} = T\sqrt{1 - \beta^2}\,, \qquad \rho' = \rho(1 - \beta^2)^{-1}.\text{''}$$

Then I went to the text of the exercise (*free translation*):
"**148.** How the thermodynamical quantities — density and temperature — change if another, moving, inertial frame is used?"

My first impression was the second equation is incorrect. Namely, if $\rho = N/\mathcal{V}$ and

$$\mathcal{V}' = \frac{\mathcal{V}}{\gamma}\,, \qquad \gamma = (1 - \beta^2)^{-1/2}\,, \tag{1}$$

changes according to the Lorentz contraction, and N remains invariant, the density changes by $\rho' = \gamma\rho$, rather then by $\rho' = \gamma^2\rho$. But maybe they considered a mass density? Then the mass transformation introduces the remaining γ-factor and the second equation seems indeed correct. What about the first one that $T' = T/\gamma$? Then I realized that although never thinking about

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that I always knew, perhaps due to the pre-established harmony, that the temperature is a Lorentz invariant. In any case, I was surprised that the solution was presented there without any explanation as something absolutely straightforward.

When I returned to the problem many years later I was aware that some other people had already had doubts about these formulas. Particularly, Professor Staruszkiewicz told me that (at time after WW2) Einstein wrote even a letter to his friend von Laue, indicating that some strange and rather incorrect transformational formulas concerning thermodynamical quantities were used in his textbooks without citing the sources. He got the answer that the results came from the well-known old papers by Einstein and by Planck, from the beginning of the century. The discontinuation suggests that Einstein was convinced by the answer. These (so-called Einstein–Planck) formulas [2]

$$T' = \frac{T}{\gamma}, \qquad S' = S, \qquad \mathcal{P}' = \mathcal{P}, \quad \text{(and roughly speaking)} \quad U' = \frac{U}{\gamma}, \quad (2)$$

were explicitly questioned in early sixties by Ott [3], who claimed a different (internal) energy and temperature transformation rule

$$U' = \gamma U, \qquad T' = \gamma T, \tag{3}$$

remaining the entropy and pressure still invariant. Ott's paper begun a storm — "a bomb exploded" [4] — and, together with the important Landsberg [5] claim that the temperature is invariant

$$T' = T, \tag{4}$$

divided the community into a few desperately fighting groups, advocating their "own" formulas against those believed by others [6]. A collection of different approaches may be found in a table, Fig. 1, reprinted from Balescu paper [4]. Omitting details, let us say that the entropy transformation rule — on a basis of that as a logarithm from a number of states it must be a Lorentz invariant — was commonly accepted. Similarly, the invariance of pressure was also accepted, supported by certain hydrodynamical consideration. For instance, the averaged energy-momentum tensor for the ideal gas

$$T^{ik} = N \left\langle \frac{p^i p^k}{\mathcal{V}\varepsilon} \right\rangle = \text{diag} \left\{ \frac{U}{\mathcal{V}}, \mathcal{P}, \mathcal{P}, \mathcal{P} \right\}, \tag{5}$$

in the rest frame suggests to consider both the pressure and the energy-density to be invariant as certain *eigenvalues*, Eqs. (2). Note that a different rule

$$\mathcal{P}' = \gamma^2 \mathcal{P}, \tag{6}$$

was exceptionally mentioned by Sutcliffe [7].

		V	p	S	T	δQ	E	F
I	Planck[1] Einstein[2] Hasenöhrl[3] Jüttner[10] Textbooks[4-9] Pathria[13] Guessous[14] de Broglie[15] Hillion[16] Staruszkiewicz[17] Penney[18] Eberly-Kujawski[41,42]	$V=\dfrac{V_0}{\operatorname{ch} s}$	$p=p_0$	$S=S_0$	$T=\dfrac{T_0}{\operatorname{ch} s}$	$\delta Q=\dfrac{\delta Q_0}{\operatorname{ch} s}$	$E=E_0\,\operatorname{ch} s + p_0 V_0\,\operatorname{sh} s\,\operatorname{th} s$	$F=\dfrac{F_0}{\operatorname{ch} s}$
	Kibble[19] Møller[20]						complicated	
II	Brevik[21]				$T=T_0\,\operatorname{ch} s$	$\delta Q=\delta Q_0\,\operatorname{ch} s$	$E=E_0\,\operatorname{ch} s$	$F=F_0\,\operatorname{ch} s$
	Ott[11] Arzeliés[12] Gamba[22] Børs[23] Souriau[24]							
	Sutcliffe[25]		$p_m=p_0$ $p_{th}=p_0\,\operatorname{ch}^2 s$					
I–II	Rohrlich[26]	$V=V_0/\operatorname{ch} s$ $V'=V_0\,\operatorname{ch} s$			$T=T_0/\operatorname{ch} s$ $T'=T_0\,\operatorname{ch} s$		$E=E_0/\operatorname{ch} s$ $E'=E_0\,\operatorname{ch} s$	
III	Landsberg[27-29]		$p=p_0$		$T=T_0$	$\delta Q=\delta Q_0$	$E=E_0$	$F=F_0$
	Landsberg–Johns[30]	$V=\dfrac{V_0}{\operatorname{ch} s}$				$\delta Q=\delta Q_0/\operatorname{ch} s$	$E=E_0\,\operatorname{ch} s + p_0 V_0\,\operatorname{sh} s\,\operatorname{th} s$	

Letters without subscript: quantities in an arbitrary reference frame
Letters with subscript "0": quantities in the rest frame

$$\operatorname{th} s = v, \qquad \operatorname{ch} s = \frac{1}{(1-v^2)^{\frac{1}{2}}}, \qquad \operatorname{sh} s = \frac{v}{(1-v^2)^{\frac{1}{2}}}, \qquad c = 1$$

Fig. 1. The transformation rules for thermodynamical quantities according to different authors. Reprint from Ref. [4].

It was confusing for me that, despite some preferences concerning particular formulas, all these considerations did not care about the consistency of the basic thermodynamical equation[1]

$$dU' = T'dS' - \mathcal{P}'d\mathcal{V}'. \tag{7}$$

Using for instance Ott's formulas one has a "good looking" energy transformation $dU' = \gamma dU$, a consistent heat term $(TdS)' = \gamma(TdS)$, but an inconsistent work term $(\mathcal{P}d\mathcal{V})' = (\mathcal{P}d\mathcal{V})/\gamma$, *etc.* Under such circumstances I decided to verify personally what is the truth and I found my own

$$T' = T, \qquad S' = \gamma S, \qquad \mathcal{P}' = \gamma^2 \mathcal{P}, \qquad U' = \gamma U. \tag{8}$$

Inspired by:

"*... will ich an alle Wände schreiben, wo es nur Wände gibt, — ich habe Buchstaben, um auch Blinde sehend zu machen ...*" (F. Nietzsche)[2]

I used them as a title of a prepared paper [9] ... My Buchstaben did not.

2. Derivation of equations (8)

The old problem of transformational properties of the thermodynamical quantities, which enter to the basic equation (7), in the moving (along x-axis with a constant velocity V) reference frame, is considered for the ideal relativistic gas.

The description (7) — *thermodynamics* — requires the following:

(i) all three terms of Eq. (7) transform according to the same rule;

(ii) the volume of the box transforms according to Eq. (1);

(iii) the energy transforms like

$$U' = \sum_i \langle \varepsilon_i' \rangle = \sum_i \langle \gamma[\varepsilon_i - (p_x)_i V] \rangle = \sum_i \gamma \langle \varepsilon_i \rangle = \gamma U, \tag{9}$$

where $\langle (p_x)_i \rangle = 0$ in the common resting reference frame of a system (as a whole) and thermal bath;

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[1] In fact the pioneering investigators [2] did use rather a "manifestly incovariant" form $dQ = dU + \mathcal{P}d\mathcal{V} - VdG$, see also [8], leading to a strange energy transformation formula in Fig. 1. For the reasons of this presentation it was intentionally simplified in Eqs. (2).

[2] ... I shall write upon all walls, wherever walls are to be found — I have letters that even the blind will be able to see ... (L.H. Mecken translation).

(iv) the entropy is known (as well as the other thermodynamical quantities) *explicitly* in the resting reference frame from Gibbs–Boltzmann (Juettner) equilibrium distribution, particularly

$$dS = \phi_1(T)dT + \phi_2(\mathcal{V})d\mathcal{V}\,, \tag{10}$$

where $\phi_1[= c_\mathcal{V}/T]$ and $\phi_2[= (\partial \mathcal{P}/\partial T)_\mathcal{V}]$ is a (known in terms of modified Bessel functions) *nonhomogeneous* and *homogeneous* (of the order of -1, which is going to appear *crucial*) function of the single argument, respectively.

The rule for the pressure, Eq. (6), follows from *(i)–(iii)* as a *necessary condition* for the selfconsistency. It is particularly important that the same result may be obtained independently from the mechanical consideration. In fact, considering the point particles we can neglect the collisions between particles. Then each particle undergoes successive collisions with the certain wall of the box $(L_x L_y L_z = \mathcal{V})$ with a certain constant period $\tau_u = 2L_u\varepsilon/p_u$, $u = x,\ y,\ z$, respectively in each direction. In the moving reference frame all τ_u change according to the Lorentz dilatation $\tau_u' = \tau_u/\gamma$. The momentum transfer associated with the single collision in y and z direction is the same $2p_y' = 2p_y$, $2p_z' = 2p_z$ in the moving reference frame and the areas change according to $\sigma_y' = \sigma_y/\gamma$, $\sigma_z' = \sigma_z/\gamma$. The energy-momentum transfer in x-direction $(0,\ 2p_x,\ 0,\ 0)$ corresponds to $(-2V\gamma p_x,\ 2\gamma p_x,\ 0,\ 0)$ in the moving reference frame and $\sigma_x' = \sigma_x$. Consequently, each of the ratios $2p_u/\tau_u\sigma_u$ $[= p_u^2/\mathcal{V}\varepsilon]$ changes by the factor γ^2 and thus $\mathcal{P}' = \gamma^2\mathcal{P}$.
Comparing

$$dU' = T'dS' - \mathcal{P}'d\mathcal{V}' = \gamma(TdS - \mathcal{P}d\mathcal{V}) = \gamma dU \tag{11}$$

and using the pressure transformation formula (6) one obtains

$$\left(\frac{T'}{\gamma T}\right) dS' \left(T', \frac{\mathcal{V}}{\gamma}\right) = dS(T, \mathcal{V})\,. \tag{12}$$

Consequently, see Eq. (10), $(T'/\gamma T)\partial S'/\partial T' \equiv (T'/\gamma T)\psi_1(T')$ cannot depend on $\mathcal{V}$. Because $0 = \partial\psi_1/\partial\mathcal{V}' = \partial^2 S'/\partial\mathcal{V}'\partial T'$ one obtains $\partial S'/\partial\mathcal{V}' \equiv \psi_2(\mathcal{V}')$ depending only on $\mathcal{V}'$. The equality

$$\left(\frac{T'}{\gamma^2 T}\right) \psi_2 \left(\frac{\mathcal{V}}{\gamma}\right) = \phi_2(\mathcal{V}) \tag{13}$$

can only be satisfied (*identically* with respect to $\mathcal{V}$) if

$$T' = \alpha(V)T \tag{14}$$

and both ψ_2 and ϕ_2 are *homogeneous* functions (of the same order) of the argument. Supposing for a moment that $\phi_2 \propto \mathcal{V}^k$ is a homogeneous of the order k [for the relativistic ideal gas $\phi_2(\mathcal{V}) = Nk_{\mathrm{B}}/\mathcal{V}$, so $k = -1$] one obtains

$$\psi_2(\cdot) = \left(\frac{\gamma^{k+2}}{\alpha}\right)\phi_2(\cdot)\,. \tag{15}$$

The resulting, by the use of Eq. (14) and the remaining temperature part of Eq. (12), second equality

$$\left(\frac{\alpha^2}{\gamma}\right)\psi_1(\alpha T) = \phi_1(T)\,, \tag{16}$$

because ϕ_1 for the relativistic ideal gas is *explicitly* a *nonhomogeneous* function, can only be satisfied if *identically*

$$\alpha(V) \equiv 1\,, \tag{17}$$

and

$$\psi_1(\cdot) = \gamma\phi_1(\cdot)\,. \tag{18}$$

Thus the Lorentz invariant temperature (4) is obtained as a *necessary condition* for the selfconsistence of thermodynamics.

Simultaneously, the entropy transforms, like the energy,

$$dS' = \gamma\phi_1\,dT' + \gamma^{k+2}\phi_2\,d\mathcal{V}' \tag{19}$$

and $k = -1$, $\phi_2(\mathcal{V}') = Nk_{\mathrm{B}}/\mathcal{V}'$, are obtained as a *necessary and sufficient* condition for the selfconsistence of thermodynamics.

Consequently, the Gibbs–Boltzmann *thermodynamics* of the ideal relativistic gas is *consistent*, and

$$T' = T\,, \qquad S' = \gamma S\,, \qquad \mathcal{P}' = \gamma^2\mathcal{P}\,, \qquad U' = \gamma U \tag{20}$$

are the resulting Lorentz transformation formulas.

Let us note that still the most popular Einstein–Planck relations (2) are consistent with the condition *(i)*, however they adopt (rather) incorrect energy and (consequently) pressure transformation rule. The Eqs. (9), leading to Ott's rule for energy, cannot be — at least for the ideal gas — questioned, because except the statement that the average is a linear operation, these are the definitions of the successive terms. Similarly, the pressure transformation formula (6) has been proven independently by mechanical consideration. The black body radiation thermodynamics is not of the form (10), but otherwise has the property that *all* functions are *homogeneous*.

This explains why the consideration of the gas of photons cannot lead to unambiguous conclusions. In fact, because $U = a\mathcal{V}T^4$, $\mathcal{P} = aT^4/3$, and $S = 4a\mathcal{V}T^3/3$, then the assumption of arbitrary transformational rules of the form

$$\mathcal{V}' = \mathcal{V}/\gamma, \qquad T' = \gamma^\tau T,$$

$$S'(\mathcal{V}', T') = \gamma^\sigma S(\mathcal{V}, T) = \gamma^{\sigma + 1 - 3\tau} \frac{4a\mathcal{V}'T'^3}{3},$$

$$\mathcal{P}' = \gamma^\pi \mathcal{P} = \gamma^{\pi - 4\tau} \frac{aT'^4}{3},$$

$$U' = \gamma^u U = \gamma^{u + 1 - 4\tau} a\mathcal{V}'T'^4, \tag{21}$$

results with only two equations for the four variables

$$u + 1 = \pi, \qquad u = \sigma + \tau \tag{22}$$

in order to satisfy Eq. (7). Two arbitrary assumptions, $\pi = 0$ and $\sigma = 0$, leading to Eqs. (2) were incorrect. The proper choice, $u = 1$ results with $\pi = 2$, however τ and $\sigma = 1 - \tau$ remain undetermined. The consideration of ideal gas results with $\tau = 0$ and $\sigma = 1$.

3. Final remarks

There is a question to which extent particular results concerning the simple model systems, Eq. (7), may be considered as some general rules. Recall the promise of Clausius–Caratheodory axiomatic thermodynamics that a macroscopic system in equilibrium can be described by a relatively small number of relevant variables in such a way that counting all the "works" related to the quasistatic changes of these variables and subtracting them from the caused internal energy change, an integrable Pfaff form is obtained. By definition, the inverse of temperature $1/T$ is an integrating factor of this differential form and the entropy S is a resulting function of state. From this point of view the transformation rules for volume (1) and pressure (6) are not particularly important — the other systems (magnetics, trapped atoms [10], *etc.*) have their own, different, relevant variables — except one thing. These rules follow directly from basic consideration which constitute our thinking about special relativity, from contraction of measuring rods and dilatation of clocks (note that a free particle moving in a box is a perfect clock, three clocks usually!) and convince us that the natural choice that the internal energy is simply a zeroth component of an energy-momentum of a system as a whole, is just what we generally need. In fact the Lorentz transformation of $(U, 0, 0, 0)$ in the rest frame gives $(\gamma U, -\gamma VU, 0, 0)$ and leads to $U' = \gamma U$, which is consistent with the rule for the product $\mathcal{P}d\mathcal{V}$.

The heat term TdS is universal. Therefore, the derivation presented in Sec. 2 means that, say, if some transformational formulas for thermodynamical quantities exist, then the entropy changes like the energy

$$S' = \gamma S\,, \tag{23}$$

and the temperature is invariant (4). Nowadays there is a growing tendency to treat all thermodynamical concepts to be immanently related to the preferred (rest) reference frame of a considered system. Within this approach the "covariant" formulation of thermodynamical laws is not expected and the thermodynamical functions are considered non-transforming in a sense that the universal rules cannot be established [11–13]. Note that such thinking do not contradict a "weak" conclusion of my paper. Also note that many discrepancies result from possible different definitions of quantities, *e.g.*, temperature — as a parameter of a partition function, entropy — as a Shannon entropy constructed for instance in an explicitly invariant manner, and pressure — as the already mentioned eigenvalue of the energy-momentum tensor, *etc.*

Since Ott article [3] the related papers were mainly focused on temperature. Is a moving body cooler (Einstein–Planck), or hotter (Ott), or is its temperature the same (Landsberg)? There was no controversy about entropy. However our formula, Eq. (23), differs from the previous one. So let us consider the entropy. This is an additive quantity, so it may be considered also for a nonequilibrium case. Imagine a system consisting of several resting (in some common inertial frame) boxes containing gases of a generally different temperatures. Collecting the entropies of successive components we obtain the total entropy of the system $S = \sum_i S_i$. Imagine a similar system, consisting of the same components, however let the boxes have some constant but mutually different velocities V_i (for simplicity, in a CM-frame). What is the total entropy $\bar{S}$ of the second system, compared to the former one? According to the table, Fig. 1, the entropy of each component is still the same, so $\bar{S} = S$. According to Eq. (23) $\bar{S} = \sum_i \gamma(V_i)S_i > S$. Which choice should we prefer?

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