# CALIBRATION OF THE SUBDIFFUSIVE BLACK–SCHOLES MODEL*

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In this paper we discuss subdiffusive mechanism for the description of some stock markets. We analyse the fractional Black–Scholes model in which the price of the underlying instrument evolves according to the subdiffusive geometric Brownian motion. We show how to efficiently estimate the parameters for the subdiffusive Black–Scholes formula *i.e.* parameter $\alpha$ responsible for distribution of length of constant stock prices periods and $\sigma$ — volatility parameter. A simple method how to price subdiffusive European call and put options by using Monte Carlo approach is presented.

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## 1. Introduction

Anomalous diffusion is conveniently described by the Fractional Fokker–Planck Equation (FFPE) with temporal fractional derivatives. The equivalent approach is based on the subordinated Langevin equation. These equations provide useful tools for the description of different types of dynamics in complex systems [1–7]. It turns out that data from financial markets can have subdiffusive or superdiffusive character [8]. Therefore, it is natural to apply the recently developed theory of anomalous diffusion also to different problems in temporary financial modeling. One of them is option pricing on emerging markets, where the number of participants and the number of transactions is rather low. Then the price processes display characteristic for subdiffusion periods in which they stay motionless.

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Option contracts are such financial instruments which play a crucial role in protection of investors against the market risk. Fair price of an option is such a price under which there is not possible to make sure profits by creating any kind of portfolio strategy (no-arbitrage opportunity) [9]. Therefore, option pricing is a challenging problem. First of all, we need to find a so-called martingale measure (see [9] and [10]) for price process describing evolution of underlaying stock. According to such a measure the price process became a martingale. This martingale assumption guarantees that the market game between purchaser and seller of the option be fair for both sides and thus the option price is fair for both players [11].

In many financial markets we observe price processes having many periods of constant values or periods in which they exhibit very small fluctuations [8]. This effect is characteristic for subdiffusive processes since it corresponds to trapping events in which the subdiffusive test particle get immobilized [1–3]. A natural physical way to modeling of such a subdiffusive behavior is the method of subordinated Langevin equations [5, 6].

In this paper we consider the market model which captures periods in which the underlying process stays motionless. It is a generalization of the classical Black–Scholes (BS) model [12] and is based on the assumption that the price of the underlying instrument evolves according to the subdiffusive Geometric Brownian Motion (GBM). Up to our knowledge the first source, where such a martingale model was fully developed is [13], however some ideas of option pricing for subordinated processes were discussed earlier, see [14–16].

In order to develop practical aspects of a market model, which capture the statistical characteristics of the financial market, one needs to know how to calibrate it *i.e.*, how to estimate its parameters and how to calculate the price of derivatives. In this paper we propose a simple and effective method of calibration for the subdiffusive BS model.

In principle, one can investigate subdiffusion processes by using two methods. The first deterministic one is the FFPE, and the second one is the stochastic subordination method (Langevin picture) developed recently [5, 6, 17]. In this paper we focus on the subordinated Langevin equation approach since it permits to apply Monte Carlo methods.

Let us note that subdiffusions with characteristic constant periods (stops) can be obtained from classical diffusion process $X(\tau)$ by subordination [5,6]:

$$Y_\alpha(t) = X\big(S_\alpha(t)\big)\,, \qquad t \in [0,T]\,, \tag{1}$$

where $S_\alpha(t)$ is the inverse $\alpha$-stable subordinator, independent from the diffusion process $X(\tau)$, defined as follows [18–20]:

$$S_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\}\,, \tag{2}$$

and $\{U_\alpha(\tau)\}_{\tau\geq 0}$, $\alpha \in (0,1)$, denotes a strictly increasing $\alpha$-stable Lévy motion [21], with Laplace transform $\langle e^{-kU_\alpha(\tau)}\rangle = e^{-\tau k^\alpha}$. In turn, $X(\tau)$ denotes here the diffusion *i.e.*, corresponding Itô's process given by some Stochastic Differential Equation (SDE) w.r.t. the Brownian motion $B(t)$.

## 2. Subdiffusive Black–Scholes market model

In this section we use subdiffusive mechanism to the description of stock markets and we consider a generalization of the classical BS model, which captures the subdiffusive character of a given financial markets. We assume the same conditions as Black and Scholes in their original paper [12]. The notation using in this section is following: $Y_\alpha(0)$ denotes stock prices at time $0, K > 0$ — strike price, $T$ — exercise date (in years), $\sigma = \text{const.} > 0$ — volatility parameter, $r \geq 0$ — risk free interest rate, and $\alpha$ is the index of subdiffusion.

Our analysis of the subdiffusive market model is based on the assumption that the price of the underlying instrument evolves according to the subordinated GBM (SGBM), which is defined as follows [13]:

$$Y_\alpha(t) = X(S_\alpha(t)) = Y_\alpha(0)e^{\left\{(\mu-\frac{1}{2}\sigma^2)S_\alpha(t)+\sigma B\left(S_\alpha(t)\right)\right\}}, \tag{3}$$

where $X(\tau)$ is the classical GBM [9, 10, 12] and parameter $\mu$ denotes short-term trend. Let us note that the SGBM (3) can be represented by the following subordinated Langevin type SDE:

$$dY_\alpha(t) = \mu Y_\alpha(t)dS_\alpha(t) + \sigma Y_\alpha(t)dB\big(S_\alpha(t)\big)\,. \tag{4}$$

The above representation of SGBM follows from a version of Itô's formula for subdiffusion, see [23]. To describe bond prices we use the same model as in the classical BS model [9, 10, 12]:

$$d\beta_t = r\beta_t\,dt\,, \qquad \beta_0 = 1 \Leftrightarrow \beta_t = e^{rt}\,. \tag{5}$$

First of all, let us note that as shown in [13], the subdiffusive market model (given by Eq. (3), (4) and (5)) is arbitrage-free, because there exists a martingale measure ([22]) $\mathbb{Q}_\alpha$ equivalent to $\mathbb{P}$:

$$\mathbb{Q}_\alpha(A) = \int_A \exp\left\{-\gamma B\big(S_\alpha(T)\big) - \tfrac{1}{2}\gamma^2 S_\alpha(T)\right\} d\mathbb{P}\,, \tag{6}$$

where $\gamma = (r+\mu)/\sigma$ and $A \in \mathcal{F}$ such that $\{e^{-rt}Y_\alpha(t)\}_{t\in[0,T]}$ is $\mathbb{Q}_\alpha$-martingale. Unfortunately, the martingale measure $\mathbb{Q}_\alpha$ equivalent to $\mathbb{P}$ is not unique, so the subdiffusive market model is incomplete [13].

Let us notice that the martingale measure $\mathbb{Q}_\alpha$ defined in (6) is a natural generalization of the martingale measure from the classical Black–Scholes model because if $\alpha \nearrow 1$, then $\mathbb{Q}_\alpha$ reduces to the martingale measure from the classical BS model. Therefore, it is interesting to compare prices in the classical and subdiffusive model. The subdiffusive Black–Scholes formula for European call $C_{\rm BS}^{\rm Sub}(Y_\alpha(0), K, T, \sigma, r, \alpha)$ and put $P_{\rm BS}^{\rm Sub}(Y_\alpha(0), K, T, \sigma, r, \alpha)$ options prices corresponding to the measure $\mathbb{Q}_\alpha$ given by (6) satisfy [13]:

$$C_{\rm BS}^{\rm Sub}(Y_\alpha(0), K, T, \sigma, r, \alpha) = \frac{1}{T^\alpha} \int_0^\infty C_{\rm BS}\big(Y_\alpha(0), K, x, \sigma, r\big) g_\alpha\left(\frac{x}{T^\alpha}\right) dx \,, \quad (7)$$

$$P_{\rm BS}^{\rm Sub}(Y_\alpha(0), K, T, \sigma, r, \alpha) = \frac{1}{T^\alpha} \int_0^\infty P_{\rm BS}\big(Y_\alpha(0), K, x, \sigma, r\big) g_\alpha\left(\frac{x}{T^\alpha}\right) dx \,, \quad (8)$$

where $g_\alpha(z)$ is the p.d.f. of $S_\alpha(1)$ and $C_{\rm BS}$, $P_{\rm BS}$ denote the classical BS prices of the European call and put options [9,10,12]. The p.d.f. $g_\alpha(z)$ occurring in formulas (7) and (8) can be expressed by the special H-Fox function [24,25], so it is possible to calculate analytically the subdiffusive European call and put option prices.

However, it is more convenient to use the Monte Carlo methods to evaluate the expected value of the BS option prices $EC_{\rm BS}$ and $EP_{\rm BS}$ in the subdiffusion model *i.e.*, at $T = S_\alpha(T)$. In order to justify this claim observe, that the right sides of the above pricing formulas can be interpreted as the expected values of the classical BS call and put option prices evaluated at random time $S_\alpha(T)$.

Let us note that for each $t \geq 0$, $S_\alpha(t) \sim \left(\frac{t}{U_\alpha(1)}\right)^\alpha$, where "$\sim$" means equality in distribution. Taking advantage of the above property of the inverse $\alpha$-stable subordinator, the estimator for price of the European call option has the following form:

$$\widehat{C}_{\rm BS}^{\rm Sub}(Y_\alpha(0), K, T, \sigma, r, \alpha) = \frac{1}{n} \sum_{i=1}^{n} C_{\rm BS}(Y_\alpha(0), K, S_i^\alpha(T), \sigma, r) \,, \quad (9)$$

and for price of the European put option:

$$\widehat{P}_{\rm BS}^{\rm Sub}(Y_\alpha(0), K, T, \sigma, r, \alpha) = \frac{1}{n} \sum_{i=1}^{n} P_{\rm BS}(Y_\alpha(0), K, S_i^\alpha(T), \sigma, r) \,, \quad (10)$$

where

$$S_i^\alpha(T) \sim \left(\frac{T}{U_i^\alpha(1)}\right)^\alpha$$

and $U_i^\alpha(1)$ are the independent and identically distributed (i.i.d.), totally skewed positive $\alpha$-stable random variables (therefore $S_i^\alpha(T)$ are also the i.i.d.) To generate $U_i^\alpha(1)$ we can use the following procedure [21]:

$$U_i^\alpha(1) = \frac{\sin\big(\alpha(V+c_1)\big)}{\big(\cos(V)\big)^{1/\alpha}} \left(\frac{\cos(V-\alpha(V+c_1))}{W}\right)^{(1-\alpha)/\alpha},$$

where $c_1 = \pi/2$, $V$ is the uniformly distributed on $(-\pi/2, \pi/2)$ random variable and $W$ has the exponential distribution with mean one.

## 3. Calibration of the subdiffusive market model

In order to apply the subdiffusive BS market model described by Eq. (3), (5) and the subdiffusive Black–Scholes formulas for the European options to the real data, it is important to know how to estimate parameters of the model, Fig. 1 and Fig. 2. Therefore, in this section we describe how to fit the model to the real data.

To model the real-live markets we use the discretization idea *i.e.*, we consider discrete version of the continuous time stochastic processes. The discrete analogon of the classical GBM is given by:

$$X_k = X_{k-1} + X_{k-1}\big(\mu\Delta t + \sigma\sqrt{\Delta t}\varepsilon_k\big)\,, \tag{11}$$

where $\varepsilon_k \sim N(0,1)$ (is a standard normal random variable). The subordinated GBM $Y_\alpha(t) = X\big(S_\alpha(t)\big)$ we get by using algorithms presented in [5] and [17] for discrete analogon of GBM.

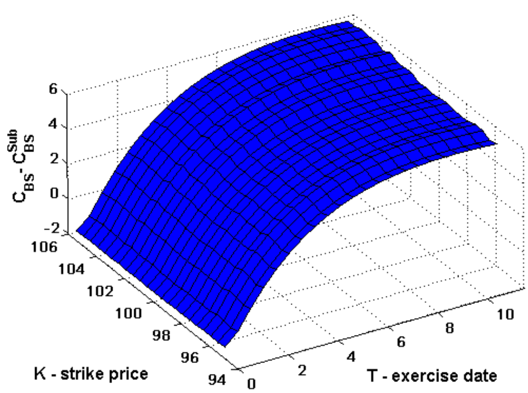


Fig. 1. Difference between classical BS price and subdiffusive BS price for European call option with parameters $Y_\alpha(0) = 100, \sigma = 1, r = 0.05$ and $\alpha = 0.9$. Clearly, for small exercise times $T$ the classical BS price is underestimated, while for larger $T$ it is overestimated in comparison to subdiffusive BS price.

We need only to estimate parameters $\alpha$ and $\sigma$, because the other parameters: $Y_\alpha(0), r$ are known from the market and $K$, $T$ are given from the definition of the option contract. So, we focus here on the $\alpha$ and $\sigma$ estimation. Obviously, subdiffusion $Y_\alpha(t) = X\big(S_\alpha(t)\big)$ combines both properties of the classical Itô's diffusion $X(\tau)$, which is responsible for the volatility ($\sigma$) of the subdiffusive processes, and the inverse $\alpha$-stable subordinator, which is responsible for probability distribution of constant price periods. From a probabilistic interpretation of the subordination mechanism (random change of time is independent from the original diffusion) a natural way for estimation of both parameters follows. It is the idea to decompose initial data vector onto two independent vectors of data. Denote observed data vector by $\mathbb{Y}_\alpha = (Y_0, Y_1, \ldots, Y_n)$, where $Y_i = Y_\alpha(t_i)$ is the price at time $t_i$ and $0 = t_0 < t_1 < \ldots < t_n = T$, $t_k = (kT)/n$ is a discretization of the time period $[0,T]$. The first vector $\mathbb{E}_\alpha = (E_1^\alpha, E_2^\alpha, \ldots, E_M^\alpha)$ contains the sizes of all constant price periods (traps), the second vector $\mathbb{E}_\sigma = (E_1^\sigma, E_2^\sigma, \ldots, E_{N+1}^\sigma)$ contains nonconstant observations from $\mathbb{Y}_\alpha$, it means that if $k = 0, 1, 2, \ldots, n-1$ is the $p$-th index such that $Y_k \neq Y_{k+1}$, then $E_p^\sigma = Y_k$.

The main idea of $\alpha$ parameter estimation for subdiffusions defined by (1) is shown in [8]. This method is based on the fact that the lengths of constant periods (corresponding to jumps of $U_\alpha(\tau)$) of subdiffusion have a totally

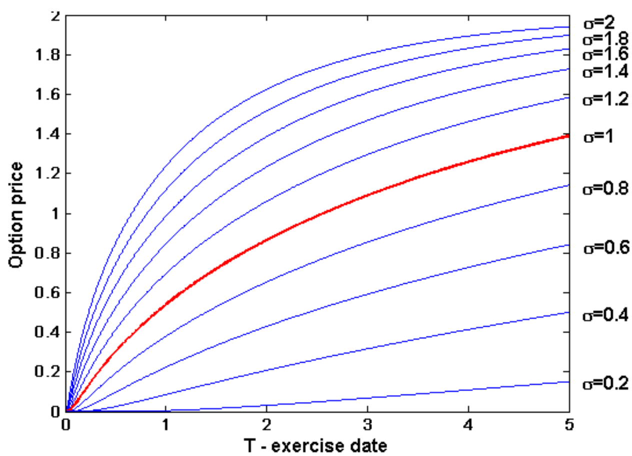


Fig. 2. The price of the classical BS European call option presented as a function of the volatility parameter $\sigma$, for $T \in [0,5]$ and with $Y_\alpha(0) = 2, K = 3, r = 0.02, \alpha = 1$. Clearly, the parameter $\sigma$ has a strong impact on the price of the option. Therefore, a choice of an accurate method for volatility estimation is crucial. For subdiffusive model the problem is even more important.

skewed $\alpha$-stable distribution. In order to estimate parameter $\alpha$ we have to use data from the vector $\mathbb{E}_\alpha$ that is the sizes of all traps and treat them as independent and identically distributed $\alpha$-stable random variables. Then, parameter $\alpha$ can be estimated by using one of the following methods: the Hill estimation method, the Pickands estimate, the EVI estimate (Extreme Value Index), the POT estimate (Peaks Over Treshold), the M–S estimate (Meerschaert and Scheffler method), the PCF (Power Curve Fitting) for the tail index estimation, see [8] and references therein.

To estimate the historical volatility parameter $\sigma$, we use the second vector $\mathbb{E}_\sigma = (E_1^\sigma, E_2^\sigma, \ldots, E_{N+1}^\sigma)$. First, we calculate the natural estimator of the standard deviation for logarithmic returns for the observed data, which in this case is given by:

$$D = \left( \frac{1}{N-1} \sum_{k=1}^{N} \left( R_k^\sigma - \widehat{R}_\sigma \right)^2 \right)^{1/2}, \tag{12}$$

where

$$R_k^\sigma = \ln \left( \frac{E_{k+1}^\sigma}{E_k^\sigma} \right) \quad \text{and} \quad \widehat{R}_\sigma = \frac{1}{N} \sum_{k=1}^{N} R_k^\sigma .$$

In the subdiffusive BS formula it is natural to use the following estimator of the volatility parameter: $\widehat{\sigma} = D\sqrt{L}$, where $L = (n+1)/T$ denotes the number of quotations during the whole year (number of exchange sessions, $n+1$ denotes length of vector $\mathbb{Y}_\alpha$). It should be emphasized here that parameter $L$ depends on the considered data (*i.e.* discretization of period $[0, T]$). For example, if we analyze daily data (closing prices), then the most often $L = 256$. More precisely $L \in [250, 260]$ according to the number of exchange sessions during the given year.

## 4. Conclusions

In this paper, we have sketched a simple algorithm for parameters $(\alpha, \sigma)$ estimation of the subdiffusive Black–Scholes model, which can be implemented efficiently for option pricing. We have demonstrated that in the case of observed subdiffusive character of the market data, the subdiffusive BS model is much better than the classical BS model (see Fig. 1). Therefore, in many cases when the historical financial data have characteristic stops (Fig. 3), it is recommended to use the subdiffusive BS formula instead of the classical one.

Finally, let us observe that the idea to split the observed data (see Fig. 4) into two parts in order to estimate the subdiffusive parameter $\alpha$ and the volatility parameter $\sigma$, is rather general and not restricted only to financial

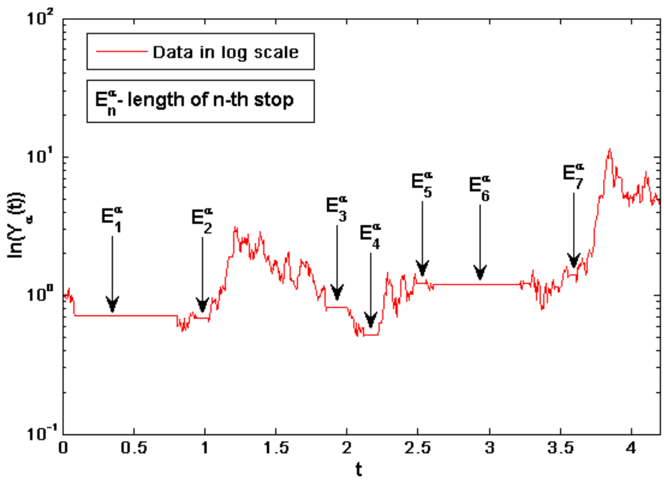


Fig. 3. An example trajectory of the SGBM in the logarithmic scale. The elements of vector $\mathbb{E}_\alpha$ are pointed by arrows.

data. It can be applied also to other models described by the Langevin equations. In particular, it can be used for estimation of the anomalous diffusion coefficient $K_\mathrm{d}$ for process $Y_\alpha(t) = X\big(S_\alpha(t)\big)$, with $X(\tau)$ given by [6]:

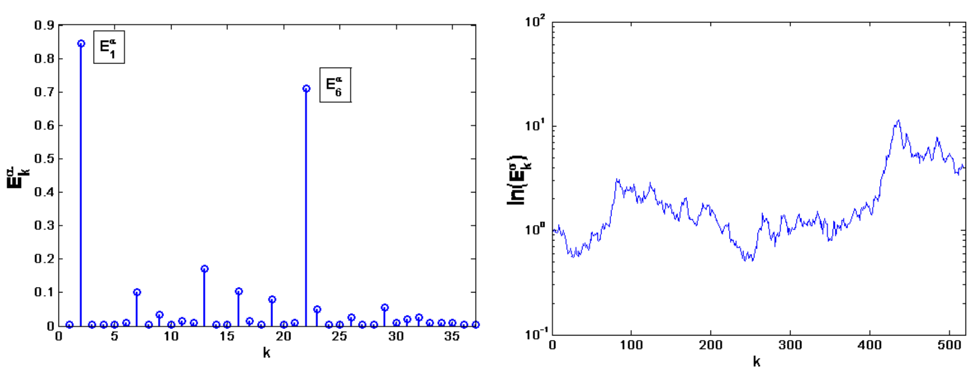


Fig. 4. Vector $\mathbb{E}_\alpha$ used for estimation of parameter $\alpha$ (left panel). Logarithm of data from vector $\mathbb{E}_\sigma$ used for estimation of parameter $\sigma$ (right panel).

$$dX(\tau) = F\big[X(\tau), U_\alpha(\tau)\big]\eta^{-1}d\tau + (2K_{\mathrm{d}})^{\frac{1}{2}}dB(\tau),\ X(0) = X_0\,, \qquad (13)$$

where $F\big[x, U_\alpha(\tau)\big] = -V_x\big[x, U_\alpha(\tau)\big]$ $\big(F\big[x, U_\alpha(\tau)\big]$ is force and $V[x, U_\alpha(\tau)]$ is an external potential$\big)$ and $\eta$ is the generalized friction constant.

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## REFERENCES


[1] R. Metzler, E. Barkai, J. Klafter, *Phys. Rev. Lett.* **82**, 3563 (1999).
[2] E. Barkai, R. Metzler, J. Klafter, *Phys. Rev.* **E61**, 132 (2000).
[3] R. Metzler, J. Klafter, *Phys. Rep.* **339**, 1 (2000).
[4] B. Dybiec, E. Gudowska-Nowak, I.M. Sokolov, *Phys. Rev.* **E76**, 041122 (2007).
[5] M. Magdziarz, A. Weron, K. Weron, *Phys. Rev.* **E75**, 016708 (2007).
[6] M. Magdziarz, A. Weron, K. Weron, *Phys. Rev.* **E77**, 036704 (2008).
[7] B. Dybiec, E. Gudowska-Nowak, *Europhys. Lett.* **88**, 10003 (2009).
[8] J. Janczura, A. Wyłomańska, *Acta Phys. Pol. B* **40**, 1341 (2009).
[9] J.M. Steele, *Stochastic Calculus and Financial Applications*, Springer, Berlin, Heidelberg, New York 2000.
[10] M. Musiela, M. Rutkowski, *Martingale Methods in Financial Modeling*, Springer, 1997.
[11] A. Weron, M. Magdziarz, *Europhys. Lett.* **86**, 60010 (2009).
[12] F. Black, M. Scholes, J. Polit. *Economy* **81**, 637 (1973).
[13] M. Magdziarz, *J. Stat. Phys.* **136**, 553 (2009).
[14] P.K. Clark, *Econometrica* **41**, 135 (1973).
[15] S.R. Hurst, E. Platen, S.T. Rachev, *Financial Engineering and Japanese Markets* **4**, 97 (1997).
[16] A.A. Stanislavsky, *Physica A* **318**, 469 (2003).
[17] M. Magdziarz, A. Weron, J. Klafter, *Phys. Rev. Lett.* **101**, 210601 (2008).
[18] A. Piryatinska, A.I. Saichev, W.A. Woyczynski, *Physica A* **349**, 375 (2005).
[19] M. Magdziarz, K. Weron, *Physica A* **376**, 1 (2006).
[20] M. Magdziarz, K. Weron, *Acta Phys. Pol. B* **37**, 1617 (2006).
[21] A. Janicki, A. Weron, *A Simulation and Chaotic Behavior of α-Stable Stochatic Processes*, Marcel Dekker, New York 1994.
[22] M. Magdziarz, *Stoch. Models* **26**, 2 (2010).
[23] A. Weron, S. Orzeł, *Acta Phys. Pol. B* **40**, 1271 (2009).
[24] C. Fox, Trans. *Am. Math. Soc.* **98**, 395 (1961).
[25] F. Mainardi, G. Pagini, R.K. Saxena, *J. Comput. Appl. Math.* **78**, 321 (2005).