ON THE COSMOLOGY OF WEYL'S GAUGE INVARIANT GRAVITY

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Recently the vector inflation has been proposed as the alternative to inflationary models based on scalar bosons and quintessence scalar fields. In the vector inflationary model, the vector field non-minimally couples to gravity. We should, however, inquire if there exists a relevant fundamental theory which supports the inflationary scenario. We investigate the possibility that Weyl's gauge gravity theory could be such a fundamental theory. That is the reason why the Weyl's gauge invariant vector and scalar fields are naturally introduced. After rescaling the Weyl's gauge invariant Lagrangian to the Einstein frame, we find that in four dimensions the Lagrangian is equivalent to Einstein–Proca theory and does not have the vector field non-minimally coupled to gravity, but has the scalar boson with a polynomial potential which leads to the spontaneously symmetry breakdown.

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1. Introduction

Inflationary models are proposed as some resolutions for the cosmological problems, *e.g.*, the flatness, horizon and monopole problems. These successful models, for example, chaotic inflation [1], k-inflation [2], are based on models of scalar bosons. The chaotic inflationary model has at least a difficulty in which bosonic fields could condensate some domains, *i.e.*, in the early stage, some domains successfully exit but others keep expanding. In k-inflation and the modified modes [3], these inflations are driven by non-minimal and non-canonical kinetic terms, but need some adjustments of conditions of scalar fields and its potentials. In addition, we have detected no such scalar bosons by experiments. From these reasons, recently the vector inflation has been proposed by Ford [4] and some authors have studied the model [5–7]. They consider the following action:

$$S=\int d^4x\sqrt{-g}\left\{\frac{R}{16\pi}-\frac{1}{4}F^2-\frac{1}{2}\left(m^2+\frac{R}{6}\right)A^2\right\}\,, \tag{1}$$

where R is the scalar curvature, F denotes the field strength of vector field A and m is the mass of the vector field. It is worth noting that the massive vector field non-minimally couples to gravity in Eq. (1). The isotropy and the stability of the inflationary model have been discussed [6]. The isotropy of expansion is achieved by N randomly organized vector fields or by a triplet of orthogonal vector fields. However, these discussions have been made to solve the cosmological problems from a aspect of cosmological observations. These models are assumed the bosonic inflatons with potentials that are not completely supported by fundamental physics. It is, therefore, necessary to investigate how the fundamental physical theories support the k-inflationary models.

In the very early stage of our Universe, the gravitational theory is expected to be different from the ordinary Einstein gravity [8], *e.g.*, higher derivative gravity, scalar–tensor gravity. Indeed, quantum gravity or string corrections would affect the cosmological evolution near the Planck scale. In particular, gravitational theory could be speculated to be a scale invariant in this stage as well as other fundamental physics.

In this paper, we study the possibility that the Weyl's gauge gravity is such a fundamental theory. Weyl's gauge theory of gravity is an extension of the Einstein gravity [9–24]. Especially the vector and scalar bosons are naturally introduced in this theory by the scale invariant symmetry. We consider that Weyl's gauge invariant scalar and vector field are expected to play cosmological important role in the very early stage of our Universe. In Sec. 2, Weyl's gauge transformation is introduced as the local scale transformation. Then we construct the minimal Weyl's gauge invariant Lagrangian in arbitrary dimensions in Sec. 3. In Sec. 4, we discuss the cosmology by using the Lagrangian obtained in Sec. 3.

2. Weyl's gauge gravity theory

In this section, we review the Weyl's gauge transformation to construct the gauge invariant Lagrangian.

Consider the scale transformation in D-dimensions

$$ds\to ds'=e^{\Lambda(x)}ds\,, \tag{2}$$

where $\Lambda(x)$ is an arbitrary function of the coordinates x^μ ($\mu = 0, \ldots, D-1$). Then the transformation of metric is realized by

$$g_{\mu\nu} \to g'_{\mu\nu} = e^{2\Lambda(x)} g_{\mu\nu} \,. \tag{3}$$

Thus

$$g^{\mu\nu} \to g'^{\mu\nu} = e^{-2\Lambda(x)} g^{\mu\nu} \,, \tag{4}$$

and

$$\sqrt{-g} \to \sqrt{-g'} = e^{D\Lambda(x)} \sqrt{-g} \,. \tag{5}$$

We can define the field with weight $d = -(D-2)/2$ which transforms as

$$\Phi \to \Phi' = e^{-\frac{D-2}{2}\Lambda(x)} \Phi \,. \tag{6}$$

We consider the covariant derivative of the scalar field

$$\partial_\mu \Phi \Rightarrow \tilde{\partial}_\mu \Phi \equiv \partial_\mu \Phi - \frac{D-2}{2} A_\mu \Phi \,, \tag{7}$$

where A_μ is a Weyl's gauge invariant vector meson and its field strength is given by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{8}$$

Under the Weyl's gauge field transformation

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \Lambda(x) \,, \tag{9}$$

we obtain the transformation of the covariant derivative of the scalar field as

$$\tilde{\partial}_\mu \Phi \to e^{-\frac{D-2}{2}\Lambda(x)} \tilde{\partial}_\mu \Phi \,. \tag{10}$$

Moreover, it is easily seen that

$$F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu} \,. \tag{11}$$

The modified Christoffel symbol is defined as

$$\tilde{\Gamma}^\lambda_{\mu\nu} \equiv \tfrac{1}{2} g^{\lambda\sigma} \left(\tilde{\partial}_\mu g_{\sigma\nu} + \tilde{\partial}_\nu g_{\mu\sigma} - \tilde{\partial}_\sigma g_{\mu\nu} \right) \,, \tag{12}$$

and the modified curvature is given as follows:

$$\tilde{R}^\mu{}_{\nu\rho\sigma} \equiv \partial_\rho \tilde{\Gamma}^\mu_{\nu\sigma} - \partial_\sigma \tilde{\Gamma}^\mu_{\nu\rho} + \tilde{\Gamma}^\mu_{\lambda\rho} \tilde{\Gamma}^\lambda_{\nu\sigma} - \tilde{\Gamma}^\mu_{\lambda\sigma} \tilde{\Gamma}^\lambda_{\nu\rho} \,. \tag{13}$$

In Weyl's gauge theory of gravity, the Lagrangian should be invariant under the scale transformation.

3. Weyl invariant Lagrangian

First, we show the Weyl's gauge invariant sectors of the vector field, the scalar field, the curvature R and R^2 in D-dimensions:

$$\mathcal{L}_A = -\frac{1}{4e^2}\sqrt{-g}\,\Phi^{\frac{2(D-4)}{D-2}} g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \tag{14}$$

$$\mathcal{L}_\Phi = -\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\tilde{\partial}_\mu\Phi\tilde{\partial}_\nu\Phi + \frac{1}{4}\lambda\Phi^{\frac{2D}{D-2}}\right], \tag{15}$$

$$\mathcal{L}_R = \frac{1}{2}\sqrt{-g}\,\epsilon\,\Phi^2\tilde{R}\,, \tag{16}$$

$$\mathcal{L}_{R^2} = \sqrt{-g}\,\alpha\,\Phi^{\frac{2(D-4)}{D-2}}\tilde{R}^2\,, \tag{17}$$

where λ, ϵ, e and α are dimensionless constants and

$$\tilde{R} = R - 2(D-1)\nabla_\mu A^\mu - (D-1)(D-2)A_\mu A^\mu\,. \tag{18}$$

As seen from (18), $\mathcal{L}_{R^2}$ seems to include the term of $RA_\mu A^\mu$.

The simple Lagrangian which consists of A_μ, Φ, $\tilde{R}$ and $\tilde{R}^2$ is the combination of the above sectors. Kao investigated the cosmology of Weyl's gauge gravity in four dimensions [17]. He focused on the higher derivative R^2 and introduced effective scalar potentials. Thus we take the more general higher derivative of R into account, then in general we consider the following Lagrangian including higher order of the curvature $\tilde{R}^n$:

$$\begin{aligned}\frac{\mathcal{L}}{\sqrt{-g}} &= -\frac{1}{4e^2}\,\Phi^{\frac{2(D-4)}{D-2}} g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{2}g^{\mu\nu}\tilde{\partial}_\mu\Phi\tilde{\partial}_\nu\Phi - \frac{1}{4}\lambda\Phi^{\frac{2D}{D-2}} \\ &\quad + \frac{1}{2}\,\epsilon\,\Phi^{\frac{2D}{D-2}}\left(\Phi^{\frac{-4}{D-2}}\tilde{R}\right) + \alpha\,\Phi^{\frac{2D}{D-2}}\left(\Phi^{\frac{-4}{D-2}}\tilde{R}\right)^n\,.\end{aligned} \tag{19}$$

Introducing an auxiliary field χ, we get the equivalent Lagrangian as

$$\begin{aligned}\frac{\mathcal{L}}{\sqrt{-g}} &= -\frac{1}{4\,e^2}\,\Phi^{\frac{2(D-4)}{D-2}} g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{2}g^{\mu\nu}\tilde{\partial}_\mu\Phi\tilde{\partial}_\nu\Phi - \frac{1}{4}\lambda\Phi^{\frac{2D}{D-2}} \\ &\quad + \frac{1}{2}\,\epsilon\,\Phi^{\frac{2D}{D-2}}\chi + \alpha\,\Phi^{\frac{2D}{D-2}}\chi^n \\ &\quad + \left(\frac{1}{2}\,\epsilon\,\Phi^{\frac{2D}{D-2}} + n\alpha\,\Phi^{\frac{2D}{D-2}}\chi^{n-1}\right)\left(\Phi^{\frac{-4}{D-2}}\tilde{R} - \chi\right)\,.\end{aligned} \tag{20}$$

Furthermore, the Lagrangian (20) can be rewritten by the new metric conformally related to the original one and new variables. Here we choose

$$\hat{g}_{\mu\nu} \equiv e^{2\Lambda(x)}g_{\mu\nu}\,, \tag{21}$$

and

$$\hat{A}_\mu \equiv A_\mu - \partial_\mu \Lambda(x)\,, \tag{22}$$

where

$$e^{-\Lambda(x)} = f\left(\Phi^2 + \frac{2n\alpha}{\epsilon}\,\Phi^2\chi\right)^{-\frac{1}{D-2}}\,. \tag{23}$$

Note that a mass scale f was introduced here.

Now we can rewrite Eq. (20) to the following Lagrangian

$$\begin{aligned}\mathcal{L}/\sqrt{-\hat{g}} \;=\; & -\frac{1}{4e^2}\,\phi^{2\frac{D-4}{D-2}}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}\hat{F}_{\mu\nu}\hat{F}_{\rho\sigma} - \frac{1}{2}\left(\partial_\mu\phi - \frac{D-2}{2}\hat{A}_\mu\phi\right)^2 \\ & -\frac{1}{4}\lambda\,\phi^{\frac{2D}{D-2}} - (n-1)\alpha\left(\frac{\epsilon}{2n\alpha}\right)^{\frac{n}{n-1}}\phi^{\frac{2D}{D-2}-\frac{2n}{n-1}}\left(f^{D-2}-\phi^2\right)^{\frac{n}{n-1}} \\ & +\frac{1}{2}\,\epsilon f^{D-2}\left(\hat{R} - 2(D-1)\hat{\nabla}_\mu\hat{A}^\mu - (D-1)(D-2)\hat{A}_\mu\hat{A}^\mu\right)\,,\end{aligned}$$

where

$$\phi \equiv f^{\frac{D-2}{2}}\left(1 + \frac{2n\alpha}{\epsilon}\chi^{n-1}\right)^{-1/2} \tag{24}$$

and "ˆ" indicates the derived quantities from new variables. We should note that $\hat{R}\hat{A}_\mu\hat{A}^\mu$ term and higher terms of the scalar curvature $\hat{R}$ disappear in this expression.

4. Cosmology of Weyl's gauge gravity

Since we obtained the Weyl's gauge invariant Lagrangian in the Einstein frame, we can study the cosmology by using this Lagrangian. Also we consider in four dimensions: $D = 4$ and the order of higher derivative of curvature as $n = 2$.

The Lagrangian (24) reads

$$\begin{aligned}\frac{\mathcal{L}}{\sqrt{-\hat{g}}} \;=\; & \frac{1}{2}\,\epsilon f^2\left(\hat{R} - 6\hat{A}_\mu\hat{A}^\mu\right) - \frac{1}{2}\left(\partial_\mu\phi - \hat{A}_\mu\phi\right)^2 \\ & -\frac{1}{4e^2}\hat{F}^2 - \frac{\epsilon^2}{16\alpha}\left(f^2-\phi^2\right)^2 - \frac{1}{4}\lambda\,\phi^4\,.\end{aligned} \tag{25}$$

This Lagrangian becomes markedly simple for $D = 4$, namely, it consists of a massive vector meson and a canonical scalar sector with a polynomial potential which leads to a spontaneously symmetry breakdown. Hence, the Universe is expected to behave similarly to the well-known inflationary scenario for our minimal Lagrangian (19). From Eq. (25), the conformal vector field could not be candidate of inflaton but could affect the evolution of our Universe. The cosmology of Weyl's gauge gravity has been investigated in four dimensions by Kao [17]. He has focused on the higher curvature term R^2 and introduced an effective potential which the scalar field leads to symmetry-breakdown in the low energy region. Also the vector mesons are not taken into account in contrast to our model. As this result, his model missed the canonical form of the scalar sector.

5. Summary and outlook

In the early stage of the Universe, vector inflations have been discussed as the alternative to the successfully inflationary scenarios based on scalar bosons. However, these inflaton have not been completely supported by relevant fundamental theory of physics. Also the gravity theory is expected to be different from the Einstein gravity in the very early Universe. In particular, gravitational physics is speculated to have a symmetry of scale invariance near Planck scale like other particle physics.

Thus, we study the possibility of the Weyl's gauge invariant theory as a fundamental theory in the early Universe. One of the reasons is that the Weyl's gauge invariant scalar and vector field can be naturally introduced. We construct the Weyl's gauge invariant Lagrangian in arbitrary dimensions that includes an arbitrary higher order of the scalar curvature R^n. This Lagrangian has the $RA^\mu A_\mu$ term. In order to investigate the cosmology, we rewrite this Lagrangian to the the Einstein-like form by using the Weyl's gauge transformation.

Especially, for $D = 4$, the transformed Lagrangian is markedly simple. In this Lagrangian, $\hat{R}$ is not minimally-coupled to the massive vector $\hat{A}^\mu$. Therefore, the Weyl's gauge invariant vector field could not be an inflaton of the vector inflation. However, the Lagrangian has a scalar boson with polynomial potentials, namely, the canonical scalar sector with ϕ^4-potential. Hence the Universe behaves similarly to the ordinary one which has been discussed by many authors.

While the massive the vector field could not be inflaton of the vector inflation, nevertheless, it is expected that the vector meson relates to the dark matter and dark energy. It is worth noting that the study of the cosmology of Weyl's gravity by Kao [17]. In the contrast to our model, he has focused on the higher derivative R^2 and introduced an effective scalar potentials but not taken the vector fields into account. From these reasons, the effective action missed the canonical form of the scalar sector. From the Weyl's gauge gravity point of view, if the inflaton is the Weyl's gauge invariant scalar, the nature seems to select the polynomial potential instead of one in the new inflation.

We need to analyze the behavior of vector field to obtain rigorous behavior of the inflaton. It will be studied in a separate publication. Also we should investigate the generalization to the case of higher and lower dimensions. This will be published in the forthcoming paper.

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