SPHERICAL WAVES OF SPIN-1 PARTICLE IN ANTI DE SITTER SPACE-TIME

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Three possible techniques to deal with a vector particle in the anti de Sitter cosmological model are viewed: the Duffin–Kemmer–Petiau matrix formalism based on the general tetrad recipe, the group theory 5-dimensional approach based on the symmetry group SO(3,2), and the tetrad form of Maxwell equations in complex Riemann–Silberstein–Majorana–Oppenheimer representation. In the first part, a spin-1 massive field is considered in static coordinates of the anti de Sitter space-time in tetrad-based approach. The complete set of spherical solutions with quantum numbers (ϵ, j, m, l) is constructed; angular dependence in wave functions is described in terms of Wigner D-functions. The energy quantization rule has been found. Transition to a massless case of electromagnetic field is specified, and electromagnetic solutions in Lorentz gauge have been constructed. In the second part, the problem of the particle with spin 1 is considered on the base of the 5-dimensional wave equation specified in the same static coordinates. In the third part, an approach, based on complex representation of the Maxwell field is applied in the anti de Sitter model.

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1. Introduction

Examining fundamental fields on the background of de Sitter and anti de Sitter models has a long history. Special value of these geometries consists in their simplicity and high symmetry underlying them which makes us to believe in existence of exact analytical treatment for some fundamental problems of classical and quantum field theory in these curved spaces. In particular, there exist special representations for fundamental wave equations, Dirac's and Maxwell's, which are explicitly invariant under symmetry

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groups SO(4.1) and SO(3.2) for these models. When dealing with a spin-1 field in de Sitter models, mostly the group theory approach is used. Most of the important references are [1–63].

The interest to exact solutions of wave equations for particles with different spins in de Sitter space was greatly increased with discovery of the Hawking radiation [64–66]. In contrast to black hole geometry, de Sitter cosmological model admits exact treatment [67–82,147].

The case of anti de Sitter space seems to be less examined though it is also very interesting because of its topological properties. Any energy spectrum must be discreet, besides this geometry allows for elliptical interpretation and physical manifestation of this in cosmological context is also of special importance. In the context of Hawking effect, the most investigators use the Newman–Penrose formalism [132, 134, 147, 148].

Turning to the case of vector particle, we should note that many years ago a matrix Duffin–Kemmer–Petiau formalism was developed. It has a long and rich history inseparably linked with description of photons and mesons [83–119].

However, this technique till recent time was not used in the curved space-time when constructing explicit solutions, though that possibility was known — see [144, 146]. The situation has been changing now — see [76, 120–129].

In the first part of the present paper, this approach will be applied to the case of spin-1 particle in anti de Sitter space-time; previously, analogous treatment for a particle in de Sitter model was given in [76]. Then we will consider the same problem on the base of the 5-dimensional wave equation. And in the end we will examine the problem by using the tetrad form of the Maxwell equations in complex representation [166, 167], it turns to be the most convenient technique to treat the classical electromagnetic field in curved space-times.

2. Duffin–Kemmer–Petiau equation in Riemannian space

We start with a spin-1 particle in the matrix form

$$\left(i\beta^a \partial_a - \frac{mc}{\hbar} \right) \Phi(x) = 0 \,, \tag{2.1}$$

where (note $g^{na} = \text{diag}(+1, -1, -1, -1)$).

$$\begin{aligned}
\Phi &= (\Phi_0, \Phi_1, \Phi_2, \Phi_3; \Phi_{01}, \Phi_{02}, \Phi_{03}, \Phi_{23}, \Phi_{31}, \Phi_{12}) \,, \\
\beta^a &= \left| \begin{array}{cc} 0 & \kappa^a \\ \lambda^a & 0 \end{array} \right| = \kappa^a \oplus \lambda^a \,, \qquad (\kappa^a)_j^{[mn]} = -i \left(\delta_j^m \, g^{na} - \delta_j^n g^{ma} \right) \,, \\
(\lambda^a)^j_{[mn]} &= -i \left(\delta^a_m \delta^j_n - \delta^a_n \, \delta^j_m \right) = -i \delta^{aj}_{mn} \,.
\end{aligned} \tag{2.2}$$

The basic properties of β^a are

$$\begin{aligned}
\beta^c\beta^a\beta^b &= \left| \begin{array}{cc} 0 & \kappa^c\lambda^a\kappa^b \\ \lambda^c\kappa^a\lambda^b & 0 \end{array} \right| , \\
(\lambda^c\kappa^a\lambda^b)^j_{[mn]} &= i\left(\delta^{cb}_{mn}g^{aj} - \delta^{cj}_{mn}g^{ab}\right) , \\
\left(\kappa^c\lambda^a\kappa^b\right)^{[mn]}_j &= i\left[\delta^a_j\left(g^{cm}g^{bn} - g^{cn}g^{bm}\right) + g^{ac}\left(\delta^n_j g^{mb} - \delta^m_j g^{nb}\right)\right] ,
\end{aligned} \tag{2.3}$$

and most substantial algebraic properties are

$$\begin{aligned}
\beta^c\beta^a\beta^b + \beta^b\beta^a\beta^c &= \beta^c g^{ab} + \beta^b g^{ac} , \\
\left[\beta^c, j^{ab}\right] &= g^{ca}\beta^b - g^{cb}\beta^a , \qquad j^{ab} = \beta^a\beta^b - \beta^b\beta^a , \\
[j^{mn}, j^{ab}] &= \left(g^{na}j^{mb} - g^{nb}j^{ma}\right) - \left(g^{ma}j^{nb} - g^{mb}j^{na}\right) .
\end{aligned} \tag{2.4}$$

In accordance with tetrad recipe one should generalize the above matrix equation (2.1) as follows [168]

$$\begin{aligned}
&\left[i\beta^\alpha(x)(\partial_\alpha + B_\alpha(x)) - \frac{mc}{\hbar}\right]\Phi(x) = 0 , \\
&\beta^\alpha(x) = \beta^a e^\alpha_{(a)}(x) , \qquad B_\alpha(x) = \tfrac{1}{2}\, j^{ab} e^\beta_{(a)} \nabla_\alpha(e_{(b)\beta}) .
\end{aligned} \tag{2.5}$$

This equation contains the tetrad $e^\alpha_{(a)}(x)$ explicitly. Therefore, there must exist possibility to prove the equivalence of such equations associated with various tetrads

$$e^\alpha_{(a)}(x) , \qquad e'^\alpha_{(b)}(x) = L_a{}^b(x) e^\alpha_{(b)}(x) , \tag{2.6}$$

$L_a{}^b(x)$ is a local Lorentz transformation. In can be easily verified that two such equations

$$\begin{aligned}
&\left[i\beta^\alpha(x)(\partial_\alpha + B_\alpha(x)) - \frac{mc}{\hbar}\right]\Phi(x) = 0 , \\
&\left[i\beta'^\alpha(x)(\partial_\alpha + B'_\alpha(x)) - \frac{mc}{\hbar}\right]\Phi'(x) = 0 ,
\end{aligned} \tag{2.7}$$

linked with tetrads $e^\alpha_{(a)}(x)$ and $e'^\alpha_{(b)}(x)$ respectively, can be converted into each other through a local gauge transformation

$$\Phi'(x) = \left| \begin{array}{c} \phi'_a(x) \\ \phi'_{[ab]}(x) \end{array} \right| = \left| \begin{array}{cc} L_a{}^l & 0 \\ 0 & L_a{}^m L_b{}^n \end{array} \right| \; \left| \begin{array}{c} \phi_l(x) \\ \phi_{[mn]}(x) \end{array} \right| . \tag{2.8}$$

At the same time, the wave function from this equation represents scalar quantity relative to general coordinate transformations: if $x^\alpha \to x'^\alpha = f^\alpha(x)$, then $\Phi'(x) = \Phi(x)$. It remains to show that this general covariant matrix formulation can be converted into the Proca formalism in terms of general relativity tensors. To this end, let us take into account the block structure of β^a, J^{ab} and $\Phi(x)$; which gives

$$
\begin{aligned}
i \left[\lambda^c e^\alpha_{(c)} \left(\partial_\alpha + \kappa^a \lambda^b e^\beta_{(a)} \nabla_\alpha e_{(b)\beta} \right) \right]_{[mn]}{}^{l} \Phi_l &= \frac{mc}{\hbar} \Phi_{[mn]} \, , \\
i \left[\kappa^c e^\alpha_{(c)} \left(\partial_\alpha + \lambda^a \kappa^b e^\beta_{(a)} \nabla_\alpha e_{(b)\beta} \right) \right]_{l}{}^{[mn]} \Phi_{[mn]} &= \frac{mc}{\hbar} \Phi_l \, ,
\end{aligned} \tag{2.9}
$$

this is equivalent to

$$
\begin{aligned}
(e^\alpha_{(a)} \partial_\alpha \Phi_b - e^\alpha_{(b)} \partial_\alpha \Phi_a) + (\gamma^c{}_{ab} - \gamma^c{}_{ba}) \Phi_c &= \frac{mc}{\hbar} \Phi_{ab} \, , \\
e^{(b)\alpha} \partial_\alpha \Phi_{ab} + \gamma^{nb}{}_n \Phi_{ab} + \gamma_a{}^{mn} \Phi_{mn} &= \frac{mc}{\hbar} \Phi_a \, ,
\end{aligned} \tag{2.10}
$$

the symbol $\gamma_{abc}(x)$ designates Ricci coefficients $\gamma_{abc}(x) = -e_{(a)\alpha;\beta} e^\alpha_{(b)} e^\beta_{(c)}$. In turn, (2.10) are equivalent to the Proca equations

$$
\nabla_\alpha \Psi_\beta - \nabla_\beta \Psi_\alpha = m \Psi_{\alpha\beta} \, , \qquad \nabla^\beta \Psi_{\alpha\beta} = m \Psi_\alpha \, , \tag{2.11}
$$

converting to the tetrad components $\Phi_a = e^\alpha_{(a)} \Phi_\alpha \, , \Phi_{ab} = e^\alpha_{(a)} e^\beta_{(b)} \Phi_{\alpha\beta}$.

Thus, the manner of introducing interaction between a spin-1 particle and external classical gravitational field can be successfully unified with the approach that occurred for a spin-1/2 particle and was developed by Tetrode, Weyl, Fock, and Ivanenko in 1929. One should attach great significance to that possibility of unification. Moreover, its absence would be a very strange fact. Let us add some more details. The manner of extending the flat space Dirac equation to general relativity case indicates clearly that the Lorentz group underlies equally both these theories. In other words, the Lorentz group retains its importance and significance at changing the Minkowski space model to an arbitrary curved one. In contrast to this, at generalizing the Proca formulation, we automatically destroy any relations to the Lorentz group, though the definition itself for a spin 1 particle as an elementary object was based on this group. Such a gravity sensitiveness to the fermion-boson division might appear rather strange and unattractive asymmetry. Moreover, just this feature has brought about a plenty of speculations about this matter.

3. Separation of variables

Let us start analysis of the spin-1 field on the base of the matrix Duffin–Kemmer–Petiau formalism. With the use of diagonal spherical static tetrad in anti de Sitter space-time $x^\alpha = (t, r, \theta, \phi)$ [145] and corresponding Ricci coefficients

$$
\begin{aligned}
dS^2 &= \left(1+r^2\right) dt^2 - \frac{dr^2}{1+r^2} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad \Phi = 1+r^2\,,\\
e^\alpha_{(0)} &= \left(\frac{1}{\sqrt{\Phi}}, 0, 0, 0\right), \quad e^\alpha_{(3)} = (0, \sqrt{\Phi}, 0, 0)\,,\\
e^\alpha_{(1)} &= \left(0, 0, \frac{1}{r}, 0\right), \quad e^\alpha_{(2)} = \left(1, 0, 0, \frac{1}{r\sin\theta}\right),\\
\gamma_{030} &= \frac{r}{\sqrt{\Phi}}, \qquad \gamma_{311} = \frac{\sqrt{\Phi}}{r}, \qquad \gamma_{322} = \frac{\sqrt{\Phi}}{r}, \qquad \gamma_{122} = \frac{\cos\theta}{r\sin\theta},
\end{aligned}
\tag{3.1}
$$

we get to explicit form of a matrix Duffin–Kemmer equation for a massive spin-1 particle

$$
\begin{aligned}
&\left[i\beta^0\partial_t + i\Phi\left(\beta^3\partial_r + \frac{1}{r}(\beta^1 j^{31} + \beta^2 j^{32}) + \frac{\Phi'}{2\Phi}\beta^0 J^{03}\right) + \frac{\sqrt{\Phi}}{r}\Sigma_{\theta,\phi} - m\sqrt{\Phi}\right]\\
&\times \Phi(x) = 0\,,\\
&\Sigma_{\theta,\phi} = i\beta^1\partial_\theta + \beta^2\frac{i\partial + ij^{12}\cos\theta}{\sin\theta}\,.
\end{aligned}
\tag{3.2}
$$

Spherical waves with angular momentum quantum numbers (j, m) are constructed within the following general substitution (we adhere notation developed in Red'kov [139–142]; before similar techniques were applied by Dray [135, 136], Krolikowski and Turski [137, 138]; many years ago such a tetrad basis was used by Schrödinger [130] and Pauli [131] when looking at the problem of single-valuedness of wave functions in quantum theory — then the case of spin $S = 1/2$ particle was specified; transition to spin-1 case is achieved in (3.2) trough a formal change of Dirac matrices into Duffin–Kemmer ones)

$$
\begin{aligned}
\Phi_{\epsilon jm}(x) &= e^{-i\epsilon t}[f_1(r)D_0, f_2(r)D_{-1}, f_3(r)D_0, f_4(r)D_{+1}, f_5(r)D_{-1},\\
&\qquad f_6(r)D_0, f_7(r)D_{+1}, f_8(r)D_{-1}, f_9(r)D_0, f_{10}(r)D_{+1}]\,,
\end{aligned}
\tag{3.3}
$$

symbol D_σ stands for Wigner [133] D-functions $D^j_{-m,\sigma}(\phi, \theta, 0)$ (we use notation according to the book [143]). In the literature, the techniques of spin-weighted harmonics Goldberg *et al.* [134] (see also in [148]) mostly is used, though equivalence of both approaches is known [135, 136]. Requirement to diagonalize parity operator, $\hat{P}\Phi_{\epsilon jm} = P\Phi_{\epsilon jm}$, gives

$$
\begin{array}{llllll}
P = (-1)^{j+v1}\,, & f_1 = f_3 = f_6 = 0\,, & f_4 = -f_2\,, & f_7 = -f_5\,, & f_{10} = +f_8\,, & \\
P = (-1)^{j}\,, & f_9 = 0\,, & f_4 = +f_2\,, & f_7 = +f_5\,, & f_{10} = -f_8\,. & (3.4)
\end{array}
$$

After separation of the variables (for recursive relations needed see Sec. 10), we arrive to the radial systems (note $\nu = \sqrt{j(j+1)/2}$).

For states with parity $P = (-1)^{j+1}$,

$$
\begin{gathered}
i\epsilon f_5 + i\Phi\left(\frac{d}{dr} + \frac{1}{r} + \frac{\Phi'}{2\Phi}\right) f_8 + i\nu\frac{\sqrt{\Phi}}{r} f_9 - m\sqrt{\Phi} f_2 = 0\,,\\
i\epsilon f_2 - m\sqrt{\Phi} f_5 = 0\,, \qquad -i\Phi\left(\frac{d}{dr} + \frac{1}{r}\right) f_2 - m\sqrt{\Phi} f_8 = 0\,,\\
i2\nu\frac{\sqrt{\Phi}}{r} f_2 - m\sqrt{\Phi} f_9 = 0\,. \qquad (3.5)
\end{gathered}
$$

For states with parity $P = (-1)^{j}$,

$$
\begin{gathered}
\Phi\left(\frac{d}{dr} + \frac{2}{r}\right) F_6 + \frac{2\nu}{r} F_5 + mF_1 = 0\,,\\
i\epsilon F_5 + i\Phi\left(\frac{d}{dr} + \frac{1}{r}\right) F_8 - m\Phi F_2 = 0\,,\\
i\epsilon F_6 - i2\nu r F_8 - mF_3 = 0\,, -i\epsilon F_2 + \frac{\nu}{r} F_1 - mF_5 = 0\,,\\
i\epsilon F_3 + \Phi\frac{d}{dr} F_1 + m\Phi F_6 = 0\,,\\
i\Phi\left(\frac{d}{dr} + \frac{1}{r}\right) F_2 + i\frac{\nu}{r} F_3 + mF_8 = 0\,, \qquad (3.6)
\end{gathered}
$$

in (3.6) we have used substitutions

$$
\begin{array}{lll}
F_1 = \sqrt{\Phi} f_1\,, & F_2 = f_2\,, & F_3 = \sqrt{\Phi} f_3\,,\\
F_5 = \sqrt{\Phi} f_5\,, & F_6 = f_6\,, & F_8 = \sqrt{\Phi} f_8\,.
\end{array}
$$

The case of minimal value $j = 0$ is to be treated separately, because from the very beginning one must use a special substitution

$$
\Phi_{\epsilon jm}(x) = e^{-i\epsilon t}(f_1, 0, f_3, 0, 0, f_6, 0, 0, f_9, 0)\,. \qquad (3.7)
$$

The angular part of the wave operator $\Sigma_{\theta,\phi}$ acts as a zero operator and Eq. (3.2) takes the form

$$
\left[i\beta^0\partial_t + i\Phi\left(\beta^3\partial_r + \frac{1}{r}\left(\beta^1 j^{31} + \beta^2 j^{32}\right) + \frac{\Phi'}{2\Phi}\beta^0 J^{03}\right) - m\sqrt{\Phi}\right]\Phi(x) = 0\,. \qquad (3.8)
$$

Correspondingly, we obtain a very simple radial system

$$-\Phi\left(\frac{d}{dr}+\frac{2}{r}\right)f_6 - m\sqrt{\Phi}f_1 = 0\,, \qquad i\epsilon f_6 - m\sqrt{\Phi}f_3 = 0\,,$$
$$-i\epsilon f_3 - \Phi\left(\frac{d}{dr}+\frac{\Phi'}{2\Phi}\right)f_1 - m\sqrt{\Phi}f_6 = 0\,, \qquad f_9 = 0\,. \tag{3.9}$$

System (3.9) describes states with parity $P=(-1)^0=+1$; when $j=0$, any states with parity $P=(-1)^{0+1}=-1$ do not exist. The system (3.9) reduces to second order differential equation for f_6:

$$\frac{d^2}{dr^2}f_6 + \frac{2(1+2r^2)}{r(1+r^2)}\frac{d}{dr}f_6 + \left[\frac{\epsilon^2}{(1+r^2)^2} - \frac{m^2-2}{1+r^2} - \frac{2}{r^2(1+r^2)}\right]f_6 = 0\,, \tag{3.10}$$

which is solved in hypergeometric functions

$$f_6(r) = r\left(1+r^2\right)^{-\epsilon/2}F\left(\alpha,\beta,\gamma,-r^2\right)\,, \qquad \gamma = 1+3/2\,,$$
$$\alpha = \frac{3/2+1-\epsilon+\sqrt{m^2+1/4}}{2}\,, \qquad \beta = \frac{3/2+1-\epsilon-\sqrt{m^2+1/4}}{2}\,. \tag{3.11}$$

To obtain solutions as polynomials, we impose restriction $\alpha=-n$, $n=0,1,2,\ldots$; so the energy quantization arises

$$\epsilon = N + 3/2 + \sqrt{m^2+1/4}\,, \qquad N = 2n+1 \in \{0,1,2,\ldots\}\,. \tag{3.12}$$

4. Solutions of radial equations at $j > 0$

Let us turn to Eqs. (3.5). Expressing f_5, f_8, f_9 through f_2

$$f_5 = \frac{i}{m\sqrt{\Phi}}\epsilon f_2\,, \quad f_9 = \frac{i}{m}\,\frac{2\nu}{r}f_2\,, \quad f_8 = -\frac{i}{m\sqrt{\Phi}}\,\Phi\left(\frac{d}{dr}+\frac{1}{r}\right)f_2\,, \tag{4.1}$$

for f_2 we get

$$\frac{d^2}{dr^2}f_2 + \frac{2(1+2r^2)}{r(1+r^2)}\frac{d}{dr}f_2 + \left[\frac{\epsilon^2}{(1+r^2)^2} - \frac{m^2-2}{1+r^2} - \frac{j(j+1)}{r^2(1+r^2)}\right]f_2 = 0\,. \tag{4.2}$$

Below solutions of this type will be referred as j-waves. Eq. (4.2) is solved in hypergeometric functions

$$f_2 = U_{\epsilon,j} = (-z)^{j/2}(1-z)^{-\epsilon/2}F(\alpha,\beta,\gamma;z)\,, \qquad \gamma = j+3/2\,,$$
$$\alpha = \frac{3/2+j-\epsilon+\sqrt{m^2+1/4}}{2}\,, \quad \beta = \frac{3/2+j-\epsilon-\sqrt{m^2+1/4}}{2}\,. \tag{4.3}$$

Restriction $\alpha = -n$, $n = 0, 1, 2, \ldots$ makes hypergeometric series polynomials, so we get a quantization rule for energy levels:

$$\epsilon = N + 3/2 + \sqrt{m^2 + 1/4}\,, \qquad N = 2n + j \in \{0, 1, 2, \}\,. \tag{4.4}$$

Now let us turn to Eqs. (3.6). With the use of two different substitutions

$$\begin{aligned} \text{I.}\quad F_1 &= \sqrt{j+1}G_1\,, \quad F_2 = i\sqrt{j/2}G_2\,, \quad F_3 = i\sqrt{J+1}G_3\,, \\ F_5 &= \sqrt{j/2}G_5\,, \quad F_6 = \sqrt{j+1}G_6\,, \quad F_8 = \sqrt{j/2}G_8\,, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \text{II.}\quad F_1 &= \sqrt{j}G_1\,, \quad F_2 = i\sqrt{(j+1)/2}\,G_2\,, \quad F_3 = i\sqrt{j}G_3\,, \\ F_5 &= \sqrt{(j+1)/2}\,G_5\,, \quad F_6 = \sqrt{j}G_6\,, \quad F_8 = \sqrt{(j+1)/2}G_8\,, \end{aligned} \tag{4.6}$$

after substituting expressions for G_5, G_6, G_8 through G_1, G_2, G_3, we arrive at three equations respectively

$$\begin{gathered} \text{I.}\ \left(\frac{j(j+1)}{r^2} + m^2 - \Phi\left(\frac{d}{dr} + \frac{2}{r}\right)\frac{d}{dr}\right)G_1 + \frac{\epsilon j}{r}G_2 + \epsilon\Phi\left(\frac{d}{dr} + \frac{2}{r}\right)\frac{1}{\Phi}G_3 = 0\,, \\ \left(\epsilon^2 - m^2\Phi^2 + \Phi\left(\frac{d}{dr} + \frac{1}{r}\right)\Phi\left(\frac{d}{dr} + \frac{1}{r}\right)\right)G_2 + \frac{\epsilon(j+1)}{r}G_1 + \Phi\frac{j+1}{r}\frac{d}{dr}G_3 = 0\,, \\ \left(\frac{\epsilon^2}{\Phi} - \left(\frac{j(j+1)}{r^2}m^2\right)G_3\right) - \epsilon\frac{d}{dr}G_1 - \frac{j}{r}\Phi\left(\frac{d}{dr} + \frac{1}{r}\right)G_2 = 0\,, \end{gathered} \tag{4.7}$$

$$\begin{gathered} \text{II.}\ \left(\frac{j(j+1)}{r^2} + m^2 - \Phi\left(\frac{d}{dr} + \frac{2}{r}\right)\frac{d}{dr}\right)G_1 + \frac{\epsilon(j+1)}{r}G_2 + \epsilon\Phi\left(\frac{d}{dr} + \frac{2}{r}\right)\frac{1}{\Phi}G_3 = 0\,, \\ \left(\epsilon^2 - m^2\Phi^2 + \Phi\left(\frac{d}{dr} + \frac{1}{r}\right)\Phi\left(\frac{d}{dr} + \frac{1}{r}\right)\right)G_2 + \frac{\epsilon j}{r}G_1 + \Phi\frac{j}{r}\frac{d}{dr}G_3 = 0\,, \\ \left(\frac{\epsilon^2}{\Phi} - \frac{j(j+1)}{r^2} - m^2\right)G_3 - \epsilon\frac{d}{dr}G_1 - \frac{(j+1)}{r}\Phi\left(\frac{d}{dr} + \frac{1}{r}\right)G_2 = 0\,. \end{gathered} \tag{4.8}$$

To solve Eqs. (4.7) and (4.8), one can make use of the Lorentz condition. Its explicit form can be easily found

$$\frac{-i\epsilon}{\sqrt{\Phi}}\,f_1 - \sqrt{\Phi}\left(\frac{d}{dr} + \frac{2}{r} + \frac{\Phi'}{2\Phi}\right)f_3 - \frac{\nu}{r}(f_2 + f_4) = 0\,. \tag{4.9}$$

When $P = (-1)^{j+1}$ Eq. (4.9) holds identically; in the cases I and II it gives, respectively

$$\begin{aligned} \text{I.} \quad -\epsilon\frac{G_1}{\Phi} &= \frac{j}{r}G_2 + \left(\frac{d}{dr} + \frac{2}{r}\right) G_3\,, \\ \text{II.} \quad -\epsilon\frac{G_1}{\Phi} &= \frac{j+1}{r}G_2 + \left(\frac{d}{dr} + \frac{2}{r}\right) G_3\,. \end{aligned} \tag{4.10}$$

Allowing for relations (4.10), let us express G_1 through G_2 and G_3, and substitute the results into 2-nd and 3-rd equations in (4.7) and (4.8). Thus we get respectively

$$\begin{aligned} &\text{I.} \ \left[\frac{d^2}{dr^2} + \left(\frac{2}{r} + \frac{\Phi'}{\Phi}\right)\frac{d}{dr} + \frac{\Phi'}{r\Phi} + \frac{\epsilon^2}{\Phi^2} - \frac{m^2}{\Phi} - \frac{j(j+1)}{\Phi r^2}\right] G_2 - \frac{2(j+1)}{r^2\Phi}G_3 = 0\,, \\ &\left[\frac{d^2}{dr^2} + \left(\frac{2}{r} + \frac{\Phi'}{\Phi}\right)\frac{d}{dr} + \frac{2\Phi'}{r\Phi} - \frac{2}{r^2} + \frac{\epsilon^2}{\Phi^2} - \frac{m^2}{\Phi} - \frac{j(j+1)}{\Phi r^2}\right] G_3 - \frac{2j}{r^2\Phi}G_2 = 0\,, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \text{II.} \quad &\left[\frac{d^2}{dr^2} + \left(\frac{2}{r} + \frac{\Phi'}{\Phi}\right)\frac{d}{dr} + \frac{\Phi'}{r\Phi} + \frac{\epsilon^2}{\Phi^2} - \frac{m^2}{\Phi} - \frac{j(j+1)}{\Phi r^2}\right] G_2 - \frac{2j}{r^2\Phi}G_3 = 0\,, \\ &\left[\frac{d^2}{dr^2} + \left(\frac{2}{r} + \frac{\Phi'}{\Phi}\right)\frac{d}{dr} + \frac{2\Phi'}{r\Phi} - \frac{2}{r^2} + \frac{\epsilon^2}{\Phi^2} - \frac{m^2}{\Phi} - \frac{j(j+1)}{\Phi r^2}\right] \\ &\times G_3 - \frac{2(j+1)}{r^2\Phi}G_2 = 0\,. \end{aligned} \tag{4.12}$$

In the case I, taking $G_3 = +G_2$, from two equations (4.11) we get one the same

$$\begin{aligned} &\text{I.} \ G_3 = +G_2 = U_{\epsilon,j+1}\,, \\ &\left[\frac{d^2}{dr^2} + \frac{2(1+2r^2)}{r(1+r^2)}\frac{d}{dr} + \frac{\epsilon^2}{(1+r^2)^2} - \frac{M^2-2}{1+r^2} - \frac{(j+1)(j+2)}{r^2(1+r^2)}\right] G_2 = 0\,. \end{aligned} \tag{4.13}$$

In the same manner, in the case II, taking $G_3 = -G_2$, we get one the same equation (it differs from previous one by the simple formal changing $(j+1)$ into $(j-1)$:

$$\begin{aligned} &\text{II.} \ G_3 = -G_2 = U_{\epsilon,j-1} \\ &\left[\frac{d^2}{dr^2} + \frac{2(1+2r^2)}{r(1+r^2)}\frac{d}{dr} + \frac{\epsilon^2}{(1+r^2)^2} - \frac{M^2-2}{1+r^2} - \frac{(j-1)j)}{r^2(1+r^2)}\right] G_2 = 0\,. \end{aligned} \tag{4.14}$$

Thus, in addition to the waves of j-type, there exist two other types (all technical details of calculations with hypergeometric functions are omitted)

I. $(j+1)$-type,

$$
\begin{aligned}
G_3 &= G_2 = U_{\epsilon,j+1}\,, \qquad -\epsilon \frac{G_1}{\Phi} = \left(\frac{d}{dr} + \frac{j+2}{r}\right) G_2\,,\\
G_1 &= \sqrt{-z}\, U_{\epsilon,j+1} - \frac{2j+3}{\epsilon}\sqrt{1-z}\, U_{\epsilon-1,j}\,,\\
U_{\epsilon,j+1} &= (-z)^{(j+1)/2}(1-z)^{-\epsilon/2}\, F(\alpha,\beta,\gamma;z)\,,\\
U_{\epsilon-1,j} &= (-z)^{j/2}(1-z)^{-(\epsilon-1)/2}\, F(\alpha,\beta,\gamma-1;z)\,,\\
\alpha &= \frac{3/2 + j + 1 - \epsilon + \sqrt{m^2+1/4}}{2}\,,\\
\beta &= \frac{3/2 + j + 1 - \epsilon - \sqrt{m^2+1/4}}{2}\,,\\
\gamma &= j + 1 + 3/2\,,
\end{aligned}
\tag{4.15}
$$

II. $(j-1)$-type,

$$
\begin{aligned}
-G_3 &= G_2 = U_{\epsilon,j-1}\,, \qquad -\epsilon\, \frac{G_1}{\Phi} = \left(-\frac{d}{dr} + \frac{j-1}{r}\right) G_2\,,\\
G_1 &= -\sqrt{-z}\, U_{\epsilon,j-1} - \frac{2}{\epsilon}\frac{\alpha\beta}{\gamma}\sqrt{1-z}\, U_{\epsilon-1,j}\,,\\
U_{\epsilon,j-1} &= (-z)^{(j-1)/2}(1-z)^{-\epsilon/2} F(\alpha,\beta,\gamma;z)\,,\\
U_{\epsilon-1,j} &= (-z)^{j/2}(1-z)^{-(\epsilon-1)/2} F(\alpha+1,\beta+1,\gamma+1;z)\,,\\
\alpha &= \frac{3/2 + j - 1 - \epsilon + \sqrt{m^2+1/4}}{2}\,,\\
\beta &= \frac{3/2 + j - 1 - \epsilon - \sqrt{m^2+1/4}}{2}\,,\\
\gamma &= j - 1 + 3/2\,.
\end{aligned}
\tag{4.16}
$$

Let us collect results together. There are constructed solutions of three types (below only $f_1,\ldots,f_4$ are specified):

j-wave

$$
f_1 = f_3 = 0\,, \qquad f_2 = -f_4 = U_{\epsilon,j}\,,
$$

$(j+1)$-wave,

$$
\begin{aligned}
f_1 &= \sqrt{j+1}\left[\frac{\sqrt{-z}}{\sqrt{1-z}} U_{\epsilon,j+1} - \frac{2j+3}{\epsilon} U_{\epsilon-1,j}\right],\\
f_2 &= +f_4 = +i\sqrt{\frac{j}{2}}\, U_{\epsilon,j+1}\,, \qquad f_3 = +i\sqrt{j+1}\frac{1}{\sqrt{1-z}}\, U_{\epsilon,j+1}\,,
\end{aligned}
$$

<u>$(j-1)$-wave,</u>

$$
\begin{aligned}
f_1 &= \sqrt{j}\left[-\frac{\sqrt{-z}}{\sqrt{1-z}}\,U_{\epsilon,j-1} - \frac{2}{\epsilon}\frac{\alpha\beta}{\gamma}\,U_{\epsilon-1,j}\right],\\
f_2 &= +f_4 = i\sqrt{\frac{j+1}{2}}U_{\epsilon,j-1}\,, \qquad f_3 = -i\,\sqrt{j}\frac{1}{\sqrt{1-z}}\,U_{\epsilon,j-1}\,.
\end{aligned}
\tag{4.17}
$$

Three types of solutions correspond to three possible values of the orbital angular moment for spin-1 particle at fixed $j : l = j, j+1, j-1$.

5. Massless limit for a spin-1 particle, Maxwell field

Let us shortly detail a massless limit. The Duffin–Kemmer–Petiau equation (3.2) stays much the same with only formal change

$$
m\sqrt{\Phi} \;\;\to\;\; P_6\sqrt{\Phi}\,, \qquad P_6 = \left|\begin{array}{cccc} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I \end{array}\right|,
$$

which produces evident alterations in the radial system

$$
\begin{aligned}
m\sqrt{\phi}f_i \;&\to\; 0\,, \qquad && \text{at } i = 1,2,3,4\,,\\
m\sqrt{\phi}f_i \;&\to\; \sqrt{\phi}f_i\,, \qquad && \text{at } i = 5,\ldots 10\,.
\end{aligned}
\tag{5.1}
$$

In the massless case, the Lorentz condition must be considered as an external gauge restriction. All the other relations remain the same, instead of old parameters of hypergeometric functions now we use new ones

$$
U_{\epsilon,j}\,, \qquad \alpha = \frac{2+j-\epsilon}{2}\,, \qquad \beta = \frac{1+j-\epsilon}{2}\,, \qquad \gamma = j+3/2\,, \tag{5.2}
$$

the energy (frequency) quantization rule is

$$
\epsilon = 2n+j+2 = N+2\,, \qquad N = 2n+j \in \{0,1,2,...\}\,. \tag{5.3}
$$

The case of minimal value $j = 0$ is special, and it should be considered separately. Indeed, the system (3.9) becomes

$$
\begin{aligned}
&-\Phi\left(\frac{d}{dr}+\frac{2}{r}\right)f_6 = 0\,, \qquad f_1 = 0\,, \qquad i\epsilon f_6 - 0\,\sqrt{\Phi}\,f_3 = 0\,,\\
&-i\epsilon f_3 - \Phi\left(\frac{d}{dr}+\frac{\Phi'}{2\Phi}\right)f_1 - \sqrt{\Phi}f_6 = 0\,, \qquad f_9 = 0\,,
\end{aligned}
$$

which is equivalent to

$$g_6 = 0\,, \qquad f_9 = 0\,, \qquad -i\epsilon f_3 - \Phi\left(\frac{d}{dr} + \frac{\Phi'}{2\Phi}\right) f_1 = 0\,. \tag{5.4}$$

Therefore, for all states of the electromagnetic field at $j = 0$, the components of electric and magnetic vectors vanish ($F_{\alpha\beta} = 0$); and non-vanishing $f_1(r), f_3(r)$ correspond to solutions of gradient type $A_\alpha = \nabla_\alpha \Phi$. To have fixed two radial functions in (5.4), one must impose certain gauge condition. In particular, taking the Lorentz condition (see (4.9)), we get (let it be $f_1 = \Phi^{-1/2} F_1, f_3 = \Phi^{-1/2} F_3$):

$$-\frac{i\epsilon}{\Phi} F_3 - \frac{d}{dr} F_1 = 0\,, \qquad -\frac{i\epsilon}{\Phi} F_1 - \left(\frac{d}{dr} + \frac{2}{r}\right) F_3 = 0\,, \tag{5.5}$$

from whence it follows

$$\left[\frac{d^2}{dr^2} + \frac{2(1+2r^2)}{r(1+r^2)}\frac{d}{dr} + \frac{\epsilon^2}{(1+r^2)^2}\right] F_1 = 0\,, \tag{5.6}$$

which represent $j = 0$ spherical solution of the equation $\nabla^\alpha \nabla_\alpha \Phi = 0\,, \; \Phi = e^{-i\epsilon t} f(r)$.

6. 5-dimensional form of the wave equation

It is well known that wave equation for a particle with spin 1 in the de Sitter and anti de Sitter spaces can be presented in 5-dimensional form invariant under the groups SO(4.1) and SO(3.2), respectively. Let us specify the problem of spherical solutions in anti de Sitter model to the 5-dimensional formalism. It is convenient to start with conformal the following coordinates

$$dS^2 = \frac{(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2}{\Phi^2}\,, \qquad \Phi = \frac{(1+x^2)}{2}\,, \tag{6.1}$$

the Proca equations read

$$\partial_\alpha \Psi_\beta - \partial_\beta \Psi_\alpha = m \Psi_{\alpha\beta}\,, \qquad \Phi^2 \partial^\beta \Psi_{\alpha\beta} = m \Psi_\alpha\,. \tag{6.2}$$

Let us introduce five coordinates ξ^a $(a = \alpha, 5)$

$$\xi^\alpha = \frac{x^\alpha}{\Phi}\,, \qquad \xi^5 = \frac{1-x^2}{1+x^2}\,, \qquad x^\alpha = \frac{\xi^\alpha}{1+\xi^5}\,, \qquad \Phi = \frac{1}{1+\xi^5}\,,$$
$$(\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 + (\xi^5)^2 = +1\,,$$
$$dS^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta + (d\xi^5)^2\,. \tag{6.3}$$

This means that the anti de Sitter space-time can be considered as a sphere in 5-dimensional pseudo-Euclidean space; therefore the anti de Sitter geometry possesses 10-parametric symmetry group SO(3.2). Instead $\Psi^{\alpha}(x)$ (in the following designated as $a^{\alpha}(x)$) let us introduce 5-vector $A^{a}(\xi)$

$$A^{\alpha}=\left(\frac{\delta^{\alpha}_{\beta}}{\Phi}-\frac{x^{\alpha}x_{\beta}}{\Phi^{2}}\right)a^{\beta}\,,\quad A^{5}=-\frac{x_{\beta}a^{\beta}}{\Phi^{2}}\,,\quad a^{\alpha}(x)=\Phi\left(A^{\alpha}-x^{\alpha}A^{5}\right)\,. \quad (6.4)$$

These five components $A^{a}(\xi)$ obey additional restriction $A^{a}\xi_{a} = A^{0}\xi^{0} - \vec{A}\vec{\xi} + A^{5}\xi^{5} = 0$. Invariant with respect to SO(3.2) wave equation for vector $A^{a}(\xi)$ should be constructed with the help of the operator $L_{ab} = \xi_{a}(\partial/\partial\xi^{b}) - \xi_{b}(\partial/\partial\xi^{a})$ and looks as follows

$$\left(-\tfrac{1}{2}\,L^{ab}L_{ab}+m^{2}-2\right)A_{c}=0\,,\qquad L_{ab}A^{b}=A_{a}\,,\qquad A^{a}\xi_{a}=0\,. \quad (6.5)$$

7. Spherical waves in 5-dimensional form, separation of variables

Let us consider equations (6.5) in static coordinates of the anti de Sitter space

$$\begin{aligned}
\xi^{1} &= r\sin\theta\cos\phi\,, & \xi^{2} &= r\sin\theta\sin\phi, \xi^{3}=r\cos\theta\,,\\
\xi^{0} &= \sin t\sqrt{1+r^{2}}\,, & \xi^{5} &= \cos t\sqrt{1+r^{2}}\,,\\
t &= \operatorname{arctg}\frac{\xi^{0}}{\xi^{5}}\,, & r &= \sqrt{(\xi^{1})^{2}+(\xi^{2})^{2}+(\xi^{3})^{2}}\,,\\
\theta &= \operatorname{arctg}\frac{\sqrt{(\xi^{1})^{2}+(\xi^{2})^{2}}}{\xi^{3}}\,, & \phi &= \operatorname{arctg}\frac{\xi^{2}}{\xi^{1}}\,.
\end{aligned} \quad (7.1)$$

For any representation of the group SO(3.2) on the functions $\Psi(\xi)$ we have

$$\xi'=S\xi\,,\qquad \Psi'(\xi')=U\Psi(\xi)\Longrightarrow\Psi'(\xi)=U\Psi(S^{-1}\xi)\,.$$

In particular case $U\equiv S$ and $\Psi\equiv A$, for rotation in the plane 0–5

$$\xi^{0'}=\cos\omega\xi^{0}-\sin\omega\xi^{5}\,,\qquad \xi^{5'}=\sin\omega\xi^{0}+\cos\omega\xi^{5}\,,$$

we get

$$\begin{aligned}
A'(\xi) &= (I+\delta\omega J_{50})\,A(\xi)\,, \qquad J_{50}=\mathcal{L}_{50}+\sigma_{50}\,,\\
\mathcal{L}_{50} &= \xi_{5}\,\frac{\partial}{\partial\xi^{0}}-\xi^{0}\frac{\partial}{\partial\xi^{5}}\,, \qquad \sigma_{50}=\left|\begin{array}{ccccc}
0 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
+1 & 0 & 0 & 0 & 0
\end{array}\right|\,.
\end{aligned} \quad (7.2)$$

Generators will be used to construct a 5-dimensional energy and angular momentum operators. From eigen-value equations

$$(+iJ_{50})^{a}{}_{b}A^{b}=\epsilon A^{a}\,, \quad \left(\vec{J}^{\,2}\right)^{a}_{\;\;b}A^{b}=j(j+1)\;A^{a}\,, \quad (J_{3})^{a}{}_{b}A^{b}=mA^{a}\,, \tag{7.3}$$

we obtain the following substitution for 5-vector

$$\begin{aligned}
\vec{A} &= e^{-i\epsilon t}\left[f(r)\vec{Y}^{j+1}_{jm}(\theta,\phi)+g(r)\vec{Y}^{j-1}_{jm}(\theta,\phi)+h(r)\vec{Y}^{j}_{jm}(\theta,\phi)\right]\,,\\
A^{0} &= \left[e^{-i(\epsilon-1)t}F(r)+i\;e^{-i(\epsilon+1)t}G(r)\right]Y_{jm}(\theta,\phi)\,,\\
A^{5} &= \left[i\;e^{-i(\epsilon-1)t}\;F(r)+e^{-i(\epsilon+1)t})\;G(r)\right]Y_{jm}(\theta,\phi)\,.
\end{aligned} \tag{7.4}$$

At given $j=1,2,\dots$ there exist three linearly independent spherical vectors, $\nu=j+1,j,j-1$, when $j=0$ there exists only one that

$$j=0\,, \qquad \vec{A}=e^{-i\epsilon t}f(r)\vec{Y}^{1}_{00}\,, \qquad \tfrac{1}{2}\left(A^{0}\pm iA^{5}\right)=iG(r)e^{-i(\epsilon\pm 1)t}\,. \tag{7.5}$$

Radial functions $f(r)$, $g(r)$, $h(r)$, $F(r)$, $G(r)$ should be determined by Eqs. (6.5). From the first equation in (6.5), taking into account action of $\vec{l}^{\,2}$ on spherical vectors [143]

$$\vec{l}^{2}\vec{Y}^{\nu}_{jm}=\nu(\nu+1)\vec{Y}^{\nu}_{jm}\,, \qquad \vec{l}^{\,2}Y_{jm}=j(j+1)\,Y_{jm}\,, \qquad \nu=j,j+1,j-1\,,$$

for the radial functions $f(r)$, $g(r)$, $h(r)$, $F(r)$ we get equations of one, the same type

$$\begin{aligned}
&\left[\frac{d^{2}}{dr^{2}}+\frac{2(1+2r^{2})}{r(1+r^{2})}\frac{d}{dr}+\frac{\Lambda^{2}}{(1+r^{2})^{2}}-\frac{\nu(\nu+1)}{r^{2}(1+r^{2})}-\frac{(m^{2}-2)}{1+r^{2}}\right]U_{\Lambda,\nu}=0\,,\\
&f=f_{0}U_{\epsilon,j+1}\,, \qquad g=g_{0}U_{\epsilon,j-1}\,, \qquad h=h_{0}U_{\epsilon,j}\,,\\
&F=F_{0}U_{\epsilon-1,j}\,, \qquad G=G_{0}U_{\epsilon+1,j}\,,
\end{aligned} \tag{7.6}$$

f_0, g_0, h_0, F_0, G_0 are unknown numerical constants. Solutions of Eq. (7.6) can be constructed in terms of hypergeometric functions (it suffices to consider in detail only the case $U_{\epsilon,j}$)

$$\begin{aligned}
U_{\epsilon,j} &= (-z)^{j/2}(1-z)^{-\epsilon/2}\,F(\alpha,\beta,\gamma;z)\,, \qquad z=-r^{2}\,, \qquad \gamma=j+3/2\,,\\
\alpha &= \frac{3/2+j-\epsilon+\sqrt{m^{2}+1/4}}{2}\,, \qquad \beta=\frac{3/2+j-\epsilon-\sqrt{m^{2}+1/4}}{2}\,,
\end{aligned} \tag{7.7}$$

we have polynomials when $\alpha=-n\,,n=0,1,2,\dots$; which results in the quantization rule for energy levels

$$\epsilon=N+3/2+\sqrt{m^{2}+1/4}\,, \qquad N=2n+j\in\{\,0,1,2,...\}\,. \tag{7.8}$$

From two remaining equations in (6.5) one can derive relationships between $(G \pm iF)$ and (f, g):

$$G - iF = \frac{1}{\epsilon}\sqrt{1+r^2}\left[\left(\frac{d}{dr} + \frac{j+2}{r}\right) f - \left(\frac{d}{dr} - \frac{j-1}{r}\right) g\right],$$
$$G + iF = \frac{-rf + rg}{\sqrt{1+r^2}}. \qquad (7.9)$$

8. Solutions of the types $(j, j+1, j-1)$

Let us search three linearly independent solutions in the form

$$\begin{array}{llll} j\text{-type}, & f = 0, & g = 0, & h \neq 0, \\ j+1)\text{-type}, & f \neq 0, & g = 0, & h = 0, \\ (j-1)\text{-type}, & f = 0, & g \neq 0, & h = 0. \end{array}$$

In fact, these requirements are equivalent to diagonalizing of the orbital angular operator $\vec{l}^{\,2} = \nu(\nu+1)$,$\nu = j+1, j, j-1$. First, let us consider the wave $(j+1)$:

$$f = \sqrt{\frac{2j+1}{j+1}}\, f_0 U_{\epsilon,j+1}, \qquad F = F_0 U_{\epsilon-1,j}, \qquad G = G_0\, U_{\epsilon+1,j}. \qquad (8.1)$$

Eqs. (7.9) take the form

$$G + iF = -\frac{rf(r)}{\sqrt{1+r^2}}, \qquad G - iF = \frac{1}{\epsilon}\sqrt{1+r^2}\left(\frac{d}{dr} + \frac{j+2}{r}\right) f, \qquad (8.2)$$

or after transition to the variable $z = -r^2$

$$2\frac{G_0}{f_0}U_{\epsilon+1,j} = -\frac{\sqrt{-z}}{\sqrt{1-z}}U_{\epsilon,j+1} + \frac{1}{\epsilon}\sqrt{1-z}\left(-2\sqrt{-z}\frac{d}{dz} + \frac{j+2}{\sqrt{-z}}\right) U_{\epsilon,j+1},$$
$$2i\frac{F_0}{f_0}U_{\epsilon-1,j} = -\frac{\sqrt{-z}}{\sqrt{1-z}}U_{\epsilon,j+1} - \frac{1}{\epsilon}\sqrt{1-z}\left(-2\sqrt{-z}\frac{d}{dz} + \frac{j+2}{\sqrt{-z}}\right) U_{\epsilon,j+1}. \qquad (8.3)$$

Allowing for explicit formulas

$$U_{\epsilon,j+1} = (-z)^{(j+1)/2}(1-z)^{-\epsilon/2}F(\alpha, \beta, \gamma; z),$$
$$U_{\epsilon+1,j} = (-z)^{j/2}(1-z)^{-(\epsilon+1)/2}F(\alpha-1, \beta-1, \gamma-1; z),$$
$$U_{\epsilon-1,j} = (-z)^{j/2}(1-z)^{-(\epsilon-1)/2}F(\alpha, \beta, \gamma-1; z),$$
$$\alpha = \frac{3/2 + j + 1 - \epsilon + \sqrt{m^2 + 1/4}}{2},$$
$$\beta = \frac{3/2 + j + 1 - \epsilon - \sqrt{m^2 + 1/4}}{2},$$
$$\gamma = j + 1 + 3/2, \qquad (8.4)$$

with then help of known formulas for hypergeometric functions we get expressions for G_0, F_0:

$$
\begin{aligned}
G_0 &= \frac{\gamma-1}{\epsilon} f_0 = \frac{j+3/2}{\epsilon} f_0 \,, \\
F_0 &= i\frac{j+3/2}{\epsilon} f_0 = i\frac{1-\gamma}{\alpha+\beta-\gamma} f_0 \,.
\end{aligned} \tag{8.5}
$$

Analogous calculations may be performed for the case of $(j-1)$-waves:

$$
\begin{aligned}
g &= \sqrt{\frac{2j+1}{j}} g_0 U_{\epsilon,j-1}(z) \,, \\
F &= F_0 U_{\epsilon-1,j} \,, \qquad G = G_0 U_{\epsilon+1,j} \,, \\
G+iF &= \frac{rg}{\sqrt{1+r^2}} \,, \qquad G-iF = -\frac{1}{\epsilon}\sqrt{1+r^2}\left(\frac{d}{dr} - \frac{j-1}{r}\right) g \,,
\end{aligned} \tag{8.6}
$$

or in the variable $z=-r^2$

$$
\begin{aligned}
2\frac{G_0}{g_0} U_{\epsilon+1,j} &= \frac{\sqrt{-z}}{\sqrt{1-z}} U_{\epsilon,j-1} - \frac{1}{\epsilon}\sqrt{1-z}\left(-2\sqrt{-z}\frac{d}{dz} - \frac{j-1}{\sqrt{-z}}\right) U_{\epsilon,j-1} \,, \\
2i\frac{F_0}{g_0} U_{\epsilon-1,j} &= \frac{\sqrt{-z}}{\sqrt{1-z}} U_{\epsilon,j-1} + \frac{1}{\epsilon}\sqrt{1-z}\left(-2\sqrt{-z}\frac{d}{dz} - \frac{j-1}{\sqrt{-z}}\right) U_{\epsilon,j-1} \,.
\end{aligned} \tag{8.7}
$$

Using the formulas

$$
\begin{aligned}
U_{\epsilon,j-1} &= (-z)^{(j-1)/2}(1-z)^{-\epsilon/2} F(a,b,c;z) \,, \\
U_{\epsilon+1,j} &= (-z)^{j/2}(1-z)^{-(\epsilon+1)/2} F(a,b,c+1;z) \,, \\
U_{\epsilon-1,j} &= (-z)^{j/2}(1-z)^{-(\epsilon-1)/2} F(a+1,b+1,c+1;z) \,, \\
a &= \frac{3/2+j-1-\epsilon+\sqrt{m^2+1/4}}{2} \,, \\
b &= \frac{3/2+j-1-\epsilon-\sqrt{m^2+1/4}}{2} \,, \\
c &= j-1+3/2 \,,
\end{aligned}
$$

we arrive at

$$
G_0 = \frac{(a-c)(b-c)}{\epsilon c}\, g_0 \,, \qquad F_0 = i\frac{ab}{\epsilon c}\, g_0 \,. \tag{8.8}
$$

Collecting together results we have:

j-wave $\underline{j = 1, 2, 3, \ldots,}$

$$\begin{aligned} \vec{A} &= e^{-i\epsilon t} h_0 \, U_{-i\epsilon, j}(r) \vec{Y}^{j}_{jm}(\theta, \phi) \,, \\ A^0 &= 0 \,, \qquad A^5 = 0 \,, \end{aligned}$$

quantization rule $\epsilon = 2n + j + 3/2 + \sqrt{m^2 + 1/4}$.

$(j-1)$-wave $\underline{j = 1, 2, 3, \ldots,}$

$$\begin{aligned} \vec{A} &= e^{-i\epsilon t} \sqrt{\frac{2j+1}{j}} f(r) \vec{J}^{j-1}_{jm}(\theta, \phi) \,, \\ f(r) &= f_0 U_{\epsilon, j-1} \,, \\ \tfrac{1}{2} \left(A^0 + i A^5 \right) &= i G(r) e^{-i(\epsilon+1)t} Y_{jm} \,, \\ \tfrac{1}{2} \left(A^0 - i A^5 \right) &= F(r) e^{-i(\epsilon-1)t} Y_{jm} \,, \\ G(r) &= \frac{(a-c)(b-c)}{\epsilon c} g_0 U_{\epsilon+1, j} \,, \qquad F(r) = i \frac{ab}{\epsilon c} g_0 U_{\epsilon-1, j} \,, \end{aligned}$$

quantization rule $\epsilon = 2n + j - 1 + 3/2 + \sqrt{m^2 + 1/4}$.

$(j+1)$-wave $\underline{j = 0, 1, 2, 3, \ldots,}$

$$\begin{aligned} \vec{A} &= e^{-i\epsilon t} \sqrt{\frac{2j+1}{j+1}} f(r) \vec{J}^{j+1}_{jm}(\theta, \phi) \,, \\ f(r) &= f_0 U_{\epsilon, j+1} \,, \\ \tfrac{1}{2} \left(A^0 + i A^5 \right) &= i G(r) e^{-i(\epsilon+1)t} Y_{jm} \,, \qquad G(r) = \frac{j+3/2}{\epsilon} f_0 U_{\epsilon+1, j} \,, \\ \tfrac{1}{2} \left(A^0 - i A^5 \right) &= F(r) e^{-i(\epsilon-1)t} Y_{jm} \,, \qquad F(r) = i \frac{j+3/2}{\epsilon} f_0 U_{\epsilon-1, j} \,, \end{aligned}$$

quantization rule $\epsilon = 2n + j + 1 + 3/2 + \sqrt{m^2 + 1/4}$. Degeneration of the energy levels can be clarified by Table I where energy levels are given by (at $j = 0$, we have $\nu = j + 1 = 1$)

$$\begin{aligned} \epsilon &= N + \frac{3}{2} + \sqrt{m^2 + 1/4} \,, \qquad N = 2n + \nu \,, \\ j &= 1, 2, 3, \ldots \,, \qquad \nu = j, j-1, j+1 \,. \end{aligned} \tag{8.9}$$

TABLE I

	j-type	$(j-1)$-type	$(j+1)$-type
$N=1$	$n=0,\ j=1$	$n=0,\ j=2$	$n=0,\ j=0$
$N=2$	$n=0,\ j=2$	$n=0,\ j=3$	$n=0,\ j=1$
	—	$n=0,\ j=1$	—
$N=3$	$n=0,\ j=3$	$n=0,\ j=4$	$n=0,\ j=2$
	$n=1,\ j=1$	$n=1,\ j=2$	$n=1,\ j=0$
$N=4$	$n=0,\ j=4$	$n=0,\ j=5$	$n=0,\ j=3$
	$n=1,\ j=2$	$n=1,\ j=3$	$n=1,\ j=1$
$N=5$	$n=0,\ j=5$	$n=0,\ j=6$	$n=0,\ j=4$
	$n=1,\ j=3$	$n=1,\ j=4$	$n=1,\ j=2$
	$n=2,\ j=1$	$n=2,\ j=2$	$n=2,\ j=0$
$N=6$	$n=0,\ j=6$	$n=0,\ j=7$	$n=0,\ j=5$
	$n=1,\ j=4$	$n=1,\ j=5$	$n=1,\ j=3$
	$n=2,\ j=2$	$n=2,\ j=3$	$n=2,\ j=1$
	—	$n=3,\ j=1$	—
$N=7$	$n=0,\ j=7$	$n=0,\ j=8$	$n=0,\ j=6$
	$n=1,\ j=5$	$n=1,\ j=6$	$n=1,\ j=4$
	$n=2,\ j=3$	$n=2,\ j=4$	$n=2,\ j=2$
	$n=3,\ j=1$	$n=3,\ j=2$	$n=3,\ j=0$
$N=8$	$n=0,\ j=8$	$n=0,\ j=9$	$n=0,\ j=7$
	$n=1,\ j=6$	$n=1,\ j=7$	$n=1,\ j=5$
	$n=2,\ j=4$	$n=2,\ j=5$	$n=2,\ j=3$
	$n=3,\ j=2$	$n=3,\ j=3$	$n=3,\ j=1$
	—	$n=4,\ j=1$	—

9. General covariant Maxwell equations in Riemann–Silberstein–Majorana–Oppenheimer approach

It is well-known that special relativity arose from investigation of symmetry properties of the Maxwell equations with respect to inertial motion of the reference frame — see [149–151]. Naturally, it was electromagnetic field that was the primary object for special relativity: [152, 153, 155, 156]. In 1931 Majorana [158] and Oppenheimer [157] proposed to consider classical Maxwell equations as a quantum photon equations. In this context they introduced 3-vector function obeying Dirac-like massless wave equation. It turned out that much earlier in 1907 the same mathematical form of classical Maxwell theory was performed by Silberstein [153]; besides, he noted himself that the same approach was used earlier by Riemann [154]. This history was much forgotten, and for many years this complex approach to electrodynamics had been connected mainly with Majorana and Oppenheimer. Historical justice was rendered by Bialynicki-Birula [159], see also in [160–166]. Below we use such a complex formalism in Maxwell electrodynamics extended to the case of arbitrary pseudo-Riemannian space–time in accordance with the tetrad recipe of Tetrode–Weyl–Fock–Ivanenko (for more detail, see [167, 168]).

Maxwell equations in arbitrary Riemannian space can be presented in the form

$$\alpha^c \left(e^{\rho}_{(c)} \partial_\rho + \tfrac{1}{2}\, j^{ab} \gamma_{abc} \right) \Psi = J(x)\,,$$
$$\alpha^0 = -iI\,, \qquad \Psi = \left| \begin{array}{c} 0 \\ \boldsymbol{E} + ic\boldsymbol{B} \end{array} \right|, \qquad J = \frac{1}{\epsilon_0} \left| \begin{array}{c} \rho \\ i\boldsymbol{j} \end{array} \right|, \tag{9.1}$$

or

$$-i \left(e^{\rho}_{(0)} \partial_\rho + \tfrac{1}{2}\, j^{ab} \gamma_{ab0} \right) \Psi + \alpha^k \left(e^{\rho}_{(k)} \partial_\rho + \tfrac{1}{2}\, j^{ab} \gamma_{abk} \right) \Psi = J(x)\,. \tag{9.2}$$

Allowing for identities

$$\tfrac{1}{2}\, j^{ab} \gamma_{ab0} = [s_1(\gamma_{230} + i\gamma_{010}) + s_2(\gamma_{310} + i\gamma_{020}) + s_3(\gamma_{120} + i\gamma_{030})]\,,$$
$$\tfrac{1}{2}\, j^{ab} \gamma_{abk} = [s_1(\gamma_{23k} + i\gamma_{01k}) + s_2(\gamma_{31k} + i\gamma_{02k}) + s_3(\gamma_{12k} + i\gamma_{03k})]\,, \tag{9.3}$$

and using the notation

$$e^{\rho}_{(0)} \partial_\rho = \partial_{(0)}\,, \qquad e^{\rho}_{(k)} \partial_\rho = \partial_{(k)}\,, \qquad a = 0, 1, 2, 3\,,$$
$$(\gamma_{01a}, \gamma_{02a}, \gamma_{03a}) = \boldsymbol{v}_a\,, \qquad (\gamma_{23a}, \gamma_{31a}, \gamma_{12a}) = \boldsymbol{p}_a\,, \tag{9.4}$$

Eq. (9.2) in absence of sources reduces to

$$-i\Big[\partial_{(0)} + \boldsymbol{s}(\boldsymbol{p}_0 + i\boldsymbol{v}_0)\Big]\Psi + \alpha^k \Big[\partial_{(k)} + \boldsymbol{s}(\boldsymbol{p}_k + i\boldsymbol{v}_k)\Big]\Psi = 0\,, \tag{9.5}$$

where

$$s_1 = \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau_1 \end{array} \right|, \qquad s_2 = \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau_1 \end{array} \right|, \qquad s_3 = \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau_1 \end{array} \right|,$$
$$\tau_1 = \left| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array} \right|, \quad \tau_2 = \left| \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right|, \quad \tau_3 = \left| \begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right|. \tag{9.6}$$

With the use of spherical tetrad in the anti de Sitter space, the main equation (9.5) is led to the form

$$\left[-\frac{i\partial_t}{\sqrt{\Phi}} + \sqrt{\Phi} \left(\alpha^3 \partial_r + \frac{\alpha^1 s_2 - \alpha^2 s_1}{r} + \frac{r}{\Phi} s_3 \right) + \frac{1}{r} \Sigma_{\theta,\phi} \right] \left| \begin{array}{c} 0 \\ \psi \end{array} \right| = 0\,,$$
$$\Sigma_{\theta,\phi} = \frac{\alpha^1}{r} \partial_\theta + \alpha^2 \frac{\partial_\phi + s_3 \cos\theta}{\sin\theta}\,. \tag{9.7}$$

It is convenient to have the spin matrix s_3 as diagonal, which is reached by a simple transformation to the known cyclic basis

$$\Psi' = U_4 \Psi \,, \qquad U_4 = \left| \begin{array}{cc} 1 & 0 \\ 0 & U \end{array} \right| ,$$

$$U = \left| \begin{array}{ccc} -1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & i/\sqrt{2} & 0 \end{array} \right| , \quad U^{-1} = \left| \begin{array}{ccc} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{array} \right| \tag{9.8}$$

so that

$$U\tau_1 U^{-1} = \frac{1}{\sqrt{2}} \left| \begin{array}{ccc} 0 & -i & 0 \\ -i & 0 & -i \\ 0 & -i & 0 \end{array} \right| = \tau'_1 \,, \qquad j'^{23} = s'_1 = \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau'_1 \end{array} \right| ,$$

$$U\tau_2 U^{-1} = \frac{1}{\sqrt{2}} \left| \begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{array} \right| = \tau'_2 \,, \qquad j'^{31} = s'_2 = \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau'_2 \end{array} \right| ,$$

$$U\tau_3 U^{-1} = -i \left| \begin{array}{ccc} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right| = \tau'_3 \,, \qquad j'^{12} = s'_3 = \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau'_3 \end{array} \right| ,$$

$$\alpha'^1 = \frac{1}{\sqrt{2}} \left| \begin{array}{cccc} 0 & -1 & 0 & 1 \\ 1 & 0 & -i & 0 \\ 0 & -i & 0 & -i \\ -1 & 0 & -i & 0 \end{array} \right| ,$$

$$\alpha'^2 = \frac{1}{\sqrt{2}} \left| \begin{array}{cccc} 0 & -i & 0 & -i \\ -i & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -i & 0 & 1 & 0 \end{array} \right| ,$$

$$\alpha'^3 = \left| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +i \end{array} \right| .$$

Eq. (9.7) becomes

$$\left[-\frac{i\partial_t}{\sqrt{\Phi}} + \sqrt{\Phi} \left(\alpha'^3 \partial_r + \frac{\alpha'^1 s_{\prime 2} - \alpha'^2 s'_1}{r} + \frac{r}{\Phi} s'_3 \right) + \frac{1}{r} \Sigma'_{\theta,\phi} \right] \left| \begin{array}{c} 0 \\ \psi' \end{array} \right| = 0 \,,$$

$$\Sigma'_{\theta,\phi} = \frac{\alpha'^1}{r} \partial_\theta + \alpha'^2 \frac{\partial_\phi + s'_3 \cos\theta}{\sin\theta} \,. \tag{9.9}$$

10. Separating the variables and Wigner functions

Let us diagonalize operators $\boldsymbol{J}^2, J_3$ — corresponding substitution for ψ is

$$\Psi = e^{-i\omega t} \left| \begin{array}{l} 0 \\ f_1(r) D_{-1} \\ f_2(r) D_0 \\ f_3(r) D_{+1} \end{array} \right| , \tag{10.1}$$

where the notation for Wigner D-functions is used: $D_\sigma = D^j_{-m,\sigma}(\phi, \theta, 0)$, $\sigma = -1, 0, +1$; j, m determine angular momentum. With the use the following recursive relations [143]:

$$\begin{array}{rclr}
\partial_\theta D_{-1} & = & \frac{1}{2}\,(a D_{-2} - \nu D_0)\,, & \dfrac{m - \cos\theta}{\sin\theta} D_{-1} = \frac{1}{2}\,(a D_{-2} + \nu D_0)\,, \\
\partial_\theta D_{0} & = & \frac{1}{2}\,(\nu D_{-1} - \nu D_{+1})\,, & \dfrac{m}{\sin\theta} D_{0} = \frac{1}{2}\,(\nu D_{-1} + \nu D_{+1})\,, \\
\partial_\theta D_{+1} & = & \frac{1}{2}\,(\nu D_{0} - a D_{+2})\,, & \dfrac{m + \cos\theta}{\sin\theta} D_{+1} = \frac{1}{2}\,(\nu D_{0} + a D_{+2})\,, \\
\nu & = & \sqrt{j(j+1)}\,, & a = \sqrt{(j-1)(j+2)}\,,
\end{array} \tag{10.2}$$

we get (the factor $e^{-i\omega t}$ is omitted)

$$\Sigma'_{\theta\phi} \Psi' = \frac{\nu}{\sqrt{2}} \left| \begin{array}{r} (f_1 + f_3) D_0 \\ -i f_2 D_{-1} \\ i(f_1 - f_3) D_0 \\ +i f_2 D_{+1} \end{array} \right| . \tag{10.3}$$

Turning back to Maxwell equation (9.9), after simple calculation we arrive at the radial system

$$\begin{array}{ll}
(1) & \sqrt{\Phi} \left(\dfrac{d}{dr} + \dfrac{2}{r} \right) f_2 + \dfrac{1}{r} \dfrac{\nu}{\sqrt{2}} (f_1 + f_3) = 0\,, \\
(2) & \left(-\dfrac{\omega}{\sqrt{\Phi}} - i\,\sqrt{\Phi}\,\dfrac{d}{dr} - i\dfrac{\sqrt{\Phi}}{r} - i\dfrac{r}{\sqrt{\Phi}} \right) f_1 - \dfrac{i}{r} \dfrac{\nu}{\sqrt{2}} f_2 = 0\,, \\
(3) & -\dfrac{\omega}{\sqrt{\Phi}} f_2 + \dfrac{i}{r} \dfrac{\nu}{\sqrt{2}} (f_1 - f_3) = 0\,, \\
(4) & \left(-\dfrac{\omega}{\sqrt{\Phi}} + i\sqrt{\Phi}\dfrac{d}{dr} + i\dfrac{\sqrt{\Phi}}{r} + i\dfrac{r}{\sqrt{\Phi}} \right) f_3 + \dfrac{i}{r} \dfrac{\nu}{\sqrt{2}} f_2 = 0\,.
\end{array} \tag{10.4}$$

Combining equations (2) and (4), instead of (10.4) we get

$$
\begin{aligned}
&(2)+(4), \qquad -\frac{\omega}{\sqrt{\Phi}}(f_1+f_3) - i\left(\sqrt{\Phi}\frac{d}{dr} + \frac{\sqrt{\Phi}}{r} + \frac{r}{\sqrt{\Phi}}\right)(f_1-f_3) = 0\,,\\
&(2)-(4), \qquad -\frac{\omega}{\sqrt{\Phi}}(f_1-f_3) - i\left(\sqrt{\Phi}\frac{d}{dr} + \frac{\sqrt{\Phi}}{r} + \frac{r}{\sqrt{\Phi}}\right)(f_1+f_3)\\
&\qquad\qquad\quad -\frac{2i}{r}\frac{\nu}{\sqrt{2}}f_2 = 0\,,\\
&(3) \qquad\qquad -\frac{\omega}{\sqrt{\Phi}}f_2 + \frac{i}{r}\frac{\nu}{\sqrt{2}}(f_1-f_3) = 0\,,\\
&(1) \qquad\qquad \sqrt{\Phi}\left(\frac{d}{dr}+\frac{2}{r}\right)f_2 + \frac{1}{r}\frac{\nu}{\sqrt{2}}(f_1+f_3) = 0\,.
\end{aligned}
$$

It is easily verified that equation (1) is an identity when allowing for remaining ones. So independent equations are

$$
\begin{gathered}
-\frac{\omega}{\sqrt{\Phi}}f_2 + \frac{i}{r}\frac{\nu}{\sqrt{2}}(f_1-f_3) = 0\,,\\
-\frac{\omega}{\sqrt{\Phi}}(f_1+f_3) - i\left(\sqrt{\Phi}\,\frac{d}{dr} + \frac{\sqrt{\Phi}}{r} + \frac{r}{\sqrt{\Phi}}\right)(f_1-f_3) = 0\,,\\
-\frac{\omega}{\sqrt{\Phi}}(f_1-f_3) - i\left(\sqrt{\Phi}\frac{d}{dr} + \frac{\sqrt{\Phi}}{r} + \frac{r}{\sqrt{\Phi}}\right)(f_1+f_3) - \frac{2i}{r}\frac{\nu}{\sqrt{2}}f_2 = 0\,.
\end{gathered}
\tag{10.5}
$$

Let us introduce new functions $f=(f_1+f_3)/\sqrt{2}\,, g=(f_1-f_3)/\sqrt{2}$, then Eqs. (10.5) look

$$
\begin{aligned}
&f_2 = \frac{i\nu}{\omega}\frac{\sqrt{\Phi}}{r}g = 0\,, \qquad -\frac{\omega}{\Phi}f - i\left(\frac{d}{dr}+\frac{1}{r}+\frac{r}{\Phi}\right)g = 0\,,\\
&\qquad\qquad -\frac{\omega^2}{\Phi}g - i\omega\left(\frac{d}{dr}+\frac{1}{r}+\frac{r}{\Phi}\right)f + \frac{\nu^2}{r^2}g = 0\,,
\end{aligned}
\tag{10.6}
$$

The system (10.6) is simplified by substitutions

$$
\begin{aligned}
g \;&=\; \frac{1}{r\sqrt{1+r^2}}G(r)\,, \qquad f = \frac{1}{r\sqrt{1+r^2}}F(r)\,, \qquad f_2 = \frac{i\nu}{\sqrt{2}\omega}\frac{1}{r^2}G(r) = 0\,,\\
i\omega F \;&=\; \Phi\frac{d}{dr}G\,, \qquad i\omega\frac{d}{dr}F + \frac{\omega^2}{\Phi}G - \frac{\nu^2}{r^2}G = 0\,.
\end{aligned}
\tag{10.7}
$$

Thus we have arrived at a single differential equation for $G(r)$:

$$(1+r^2)\,\frac{d^2G}{dr^2}+2r\frac{dG}{dr}+\left(\frac{\omega^2}{1+r^2}-\frac{\nu^2}{r^2}\right)G=0\,. \tag{10.8}$$

In (10.8) let us introduce a new variable $z=-r^2$, which results in

$$4z(1-z)\frac{d^2G}{dz^2}+2(1-3z)\frac{dG}{dz}-\left(\frac{\omega^2}{1-z}+\frac{\nu^2}{z}\right)G=0\,, \tag{10.9}$$

with the use of substitution $G=z^a(1-z)^bF(z)$, Eq. (10.9) gives

$$\begin{aligned}&4z(1-z)\frac{d^2F}{dz^2}+4\left[2a+\frac{1}{2}-\left(2a+2b+\frac{3}{2}\right)z\right]\frac{dF}{dz}\\&+\left[\frac{4a^2-2a-\nu^2}{z}+\frac{4b^2-\omega^2}{1-z}-4(a+b)\left(a+b+\frac{1}{2}\right)\right]F=0\,.\end{aligned}$$

With requirements

$$\begin{aligned}&4a^2-2a-\nu^2=0\implies\\&a=\frac{1}{4}\pm\frac{1}{4}\sqrt{1+4\nu^2}=\frac{1}{4}\pm\frac{1}{2}\left(j+\frac{1}{2}\right)=-\frac{j}{2},+\frac{j+1}{2}\,,\\&4b^2-\omega^2=0\implies b=\pm\frac{\omega}{2}\,,\qquad\omega>0\,,\end{aligned} \tag{10.10}$$

Eq. (10.10) take the form (to have solutions vanishing at $r=0$ one must take positive values $a=(j+1)/2$)

$$z(1-z)\frac{d^2F}{dz^2}+\left[2a+\frac{1}{2}-\left(2a+2b+\frac{3}{2}\right)z\right]\frac{dF}{dz}-(a+b)\left(a+b+\frac{1}{2}\right)F=0\,, \tag{10.11}$$

which is an equation of hypergeometric type

$$\gamma=2a+\frac{1}{2}\,,\qquad\alpha+\beta=2a+2b+\frac{1}{2}\,,\qquad\alpha\beta=(a+b)\left(a+b+\frac{1}{2}\right)\,,$$

that is

$$\alpha=a+b\,,\qquad\beta=a+b+\frac{1}{2}\,,\qquad\gamma=2a+\frac{1}{2}\,. \tag{10.12}$$

To have polynomials one must take negative value for $b=-\omega/2$. So, the parameters of hypergeometric functions are

$$\alpha=\frac{j+1}{2}-\frac{\omega}{2}\,,\qquad\beta=\frac{j+1}{2}-\frac{\omega}{2}+\frac{1}{2}\,, \tag{10.13}$$

and quantization is given by

$$\alpha = -n\,, \qquad \omega_{n,j} = 2n + j + 1n\,, \qquad (n = 0, 1, 2, ...)\;; \qquad (10.14)$$

or in usual units $\omega = (c/\rho)(2n + j + 1)$; ρ is a curvature radius.

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