DEFORMATION OF NONRELATIVISTIC SPACE-TIME AND FORCES NOTICED BY NONINERTIAL OBSERVER

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We consider the nonrelativistic particle moving on noncommutative space-time in the presence of constant force $\vec{F}$. Further, following the paper M. Daszkiewicz, C.J. Walczyk, *Phys. Rev.* **D77**, 105008 (2008), we recall that the considered noncommutativity generates additional force terms, which appear in the corresponding Newton equation. We demonstrate that the same force terms can be generated by the proper noninertial transformation of classical nonrelativistic space-time.

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The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in [1]. Recently, there were also found formal arguments based mainly on Quantum Gravity [2,3] and String Theory models [4,5], indicating that space-time at Planck scale should be noncommutative, *i.e.* it should have a quantum nature. Consequently, there appeared a lot of papers dealing with noncommutative classical and quantum mechanics (see *e.g.* [6,7]) as well as with field theoretical models (see *e.g.* [8,9]), in which the quantum space-time is employed.

In accordance with the Hopf-algebraic classification of all deformations of relativistic [10] and nonrelativistic [11] symmetries, one can distinguish three basic types of space-time noncommutativity:

1. The canonical (soft) deformation

$$[\, x_\mu, x_\nu \,] = i\theta_{\mu\nu} \,, \tag{1}$$

with constant and antisymmetric tensor $\theta_{\mu\nu}$. The explicit form of corresponding Poincare Hopf algebra has been provided in [12, 13], while its nonrelativistic limit has been proposed in [14].

2. The Lie-algebraic case

$$[\, x_\mu, x_\nu \,] = i\theta^\rho_{\mu\nu} x_\rho \,, \tag{2}$$

with particularly chosen constant coefficients $\theta^\rho_{\mu\nu}$. Particular kind of such space-time modification has been obtained as representations of κ-Poincare [15, 16] and κ-Galilei [17] Hopf algebras. Besides, the Lie-algebraic twist deformations of relativistic and nonrelativistic symmetries have been provided in [18, 19] and [14], respectively.

3. The quadratic deformation

$$[\, x_\mu, x_\nu \,] = i\theta^{\rho\tau}_{\mu\nu} x_\rho x_\tau \,, \tag{3}$$

with constant coefficients $\theta^{\rho\tau}_{\mu\nu}$. Its Hopf-algebraic realization was proposed in [20], [21] and [19] in the case of relativistic symmetry, and in [22] for its nonrelativistic counterpart.

Recently, in paper [23], there has been investigated the impact of mentioned above space-time deformations (with commuting time direction) on a dynamics of simplest classical system — the nonrelativistic particle moving in a field of constant force $\vec{F}$ [1]. Particulary, it has been demonstrated that for hamiltonian function

$$H(\vec{p}, \vec{x}) = \frac{\vec{p}^2}{2m} + V(\vec{x}) \,, \qquad V(\vec{x}) = \sum_{i=1}^{3} F_i x_i \,, \qquad F_i = \text{const.} \,, \tag{4}$$

in the case of canonically deformed phase space[2]

$$\{\, x_i, x_j \,\} = \theta_{ij} \,, \qquad \{\, p_i, p_j \,\} = 0 \,, \qquad \{\, x_i, p_j \,\} = \delta_{ij} \,, \tag{5}$$

the corresponding Newton equation remains undeformed[3]

$$m\ddot{x}_i = F_i \,. \tag{6}$$

[1] We consider in [23] four noncommutative space-times (see (5), (7), (9) and (11)), which represent the mentioned above classes of quantum spaces. All of them correspond to the proper quantum Galilei groups provided in [14] and [22] respectively.

[2] The relation between commutator and classical Poisson bracket is given by $\{\, a, b \,\} \to \frac{1}{i\hbar} [\, \hat{a}, \hat{b} \,]$ ($\hbar = 1$). It should be also noted that all provided below phase spaces satisfy Jacobi identity.

[3] We find all equations of motion (for all deformed phase spaces) following the standard procedure [24]. Due to the linearity of equations (6), (8), (10) and (12) with respect (quantum) spatial (x_i) directions, their form is the same on noncommutative as well as on commutative space-time. Hence, we pass with the above equations of motion to the classical space without any their modification.

In other words, it has been indicated that in such a case the space-time noncommutativity (1) does not generate any additional force term.

More interesting situation appears for the Lie-algebraic modification of nonrelativistic space-time (2). Then, in the case of the following phase space (ρ, τ — fixed, $x_0 = ct$.)

$$\{ x_i, x_j \} = \frac{1}{\kappa} t(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}) \,, \qquad \{ p_i, p_j \} = 0 \,, \qquad \{ x_i, p_j \} = \delta_{ij} \,, \tag{7}$$

with two spatial directions commuting to time direction t, we get

$$\left\{ \begin{array}{lcl} m\ddot{x}_i & = & F_i \,, \\ m\ddot{x}_\rho & = & -\frac{m}{\kappa}F_\tau + F_\rho \,, \\ m\ddot{x}_\tau & = & \frac{m}{\kappa}F_\rho + F_\tau \,, \end{array} \right. \tag{8}$$

with index i different than ρ and τ. Hence, we see that in such a case there appear additional constant force terms associated with deformation parameter κ. Similarly, for the second Lie-algebraically deformed phase space (k, l, γ — fixed and different, $i, j \neq \gamma$)

$$\begin{gathered} \{x_k, x_\gamma\} = \frac{1}{\hat{\kappa}} x_l \,, \quad \{x_l, x_\gamma\} = -\frac{1}{\hat{\kappa}} x_k \,, \quad \{x_k, x_l\} = 0 \,, \quad \{p_k, x_\gamma\} = \frac{1}{\hat{\kappa}} p_l \,, \\ \{p_l, x_\gamma\} = -\frac{1}{\hat{\kappa}} p_k \,, \ \{x_i, p_j\} = \delta_{ij} \,, \ \{x_\gamma, p_\gamma\} = 1 \,, \ \{p_a, p_b\} = 0 \,, \ a, b = 1, 2, 3 \end{gathered} \tag{9}$$

with two spatial directions commuting to space, we get the following modification of Newton equation (6)

$$\left\{ \begin{array}{lcl} m\ddot{x}_\gamma & = & F_\gamma + \frac{m}{\hat{\kappa}} F_k \dot{x}_l - \frac{m}{\hat{\kappa}} F_l \dot{x}_k \,, \\ m\ddot{x}_l & = & F_l + \frac{2m}{\hat{\kappa}} F_\gamma \dot{x}_k + m \left(\frac{F_\gamma}{\hat{\kappa}} \right)^2 x_l \,, \\ m\ddot{x}_k & = & F_k - \frac{2m}{\hat{\kappa}} F_\gamma \dot{x}_l + m \left(\frac{F_\gamma}{\hat{\kappa}} \right)^2 x_k \,. \end{array} \right. \tag{10}$$

The above result means, that the space-time noncommutativity (9) generates additional position and velocity dependent force terms (10).

Finally, it should be noted, that the (last) quadratically deformed phase space considered in [23] (with k, l, γ fixed and different, $i, j \neq \gamma$ and $a, b = 1, 2, 3$)

$$\begin{gathered} \{x_k, x_\gamma\} = \frac{1}{\bar{\kappa}} t x_l \,, \quad \{x_l, x_\gamma\} = -\frac{1}{\bar{\kappa}} t x_k \,, \quad \{ x_k, x_l \} = 0 \,, \quad \{p_k, x_\gamma\} = \frac{1}{\bar{\kappa}} t p_l \,, \\ \{p_l, x_\gamma\} = -\frac{1}{\bar{\kappa}} t p_k \,, \quad \{x_i, p_j\} = \delta_{ij} \,, \qquad \{x_\gamma, p_\gamma\} = 1 \,, \quad \{p_a, p_b\} = 0 \,, \end{gathered} \tag{11}$$

leads to the following equation of motion

$$\left\{\begin{array}{rcl} m\ddot{x}_k & = & F_k - \frac{m}{\bar{\kappa}}F_\gamma\left(t\dot{x}_l + x_l\right) - \frac{m}{\bar{\kappa}}F_\gamma t\left(\dot{x}_l - \frac{1}{\bar{\kappa}}F_\gamma t x_k\right)\,, \\ m\ddot{x}_l & = & F_l + \frac{m}{\bar{\kappa}}F_\gamma\left(t\dot{x}_k + x_k\right) + \frac{m}{\bar{\kappa}}F_\gamma t\left(\dot{x}_k + \frac{1}{\bar{\kappa}}F_\gamma t x_l\right)\,, \\ m\ddot{x}_\gamma & = & F_\gamma + \frac{m}{\bar{\kappa}}F_k\left(t\dot{x}_l + x_l\right) - \frac{m}{\bar{\kappa}}F_l\left(t\dot{x}_k + x_k\right)\,. \end{array}\right. \tag{12}$$

Hence, we see, that in the last case there are generated position and velocity dependent forces as well, but this time, with time dependent coefficients (12).

Obviously, for deformation parameter θ running to zero, and all three parameters κ, $\hat{\kappa}$ and $\bar{\kappa}$ approaching infinity, the above phase spaces and Newton equations become undeformed.

Let us now turn to the more conventional mechanism to generate new force terms in Newton equation (6). First of all, we start with the classical (commutative) phase space

$$\{\,x_i, x_j\,\} = \{\,p_i, p_j\,\} = 0\,, \qquad \{\,x_i, p_j\,\} = \delta_{ij}\,, \tag{13}$$

and the hamiltonian function (4). Obviously, in such a case we get the undeformed equation of motion (6), and identify such a system with the inertial (for example the rest) observer $\mathcal{O}(t, x_1, x_2, x_3)$.

Let us consider the following noninertial transformation from the observer $\mathcal{O}(t, x_1, x_2, x_3)$ to the nonrelativistic observer $\mathcal{O}'(t', x_1', x_2', x_3')$

$$\left\{\begin{array}{rcl} t' & = & t\,, \\ x_i' & = & x_i + v_i t + y_i\,, \\ x_\tau' & = & x_\tau + \frac{m}{2\kappa}F_\rho t^2 + v_\tau t + y_\tau\,, \\ x_\rho' & = & x_\rho - \frac{m}{2\kappa}F_\tau t^2 + v_\rho t + y_\rho\,, \end{array}\right. \tag{14}$$

where v_a and y_a $(a = i, \rho, \tau)$ denote arbitrary constants. As one can easily see, the above transformation connects the inertial observer $\mathcal{O}(t, x_1, x_2, x_3)$ with the uniformly accelerated (in directions ρ and τ) observer $\mathcal{O}'(t', x_1', x_2', x_3')$. By simple calculation one can also check that after transformation (14) the Newton equation (6) takes the form

$$\left\{\begin{array}{rcl} m\ddot{x}_i' & = & F_i\,, \\ m\ddot{x}_\rho' & = & -\frac{m}{\kappa}F_\tau + F_\rho\,, \\ m\ddot{x}_\tau' & = & \frac{m}{\kappa}F_\rho + F_\tau\,, \end{array}\right. \tag{15}$$

i.e. there appear the (additional) inertial force terms which are the same as in equation (8). Hence, we see that from such a point of view, one can identify the dynamical effects of space-time noncommutativity (7) with the ones generated by noninertial transformation (14), while the deformation parameter κ describes the degree of noninertiality.

Similarly, in the case of phase space deformation (9) and (11), one can check that the additional force terms which appear in Newton equations (10) and (12) are generated by the following noninertial transformations of commutative space-time

$$\left\{\begin{array}{rcl} t &=& t'\,, \\ x_\gamma &=& x'_\gamma - \frac{m}{\hat\kappa}F_k\int\limits_0^t x'_l(\tau)d\tau + \frac{m}{\hat\kappa}F_l\int\limits_0^t x'_k(\tau)d\tau\,, \\ x_l &=& x'_l - \frac{2m}{\hat\kappa}F_\gamma\int\limits_0^t x'_k(\tau)d\tau - m\left(\frac{F_\gamma}{\hat\kappa}\right)^2\int\limits_0^t\int\limits_0^{\tau_2} x'_l(\tau_1)d\tau_1 d\tau_2\,, \\ x_k &=& x'_k + \frac{2m}{\hat\kappa}F_\gamma\int\limits_0^t x'_l(\tau)d\tau - m\left(\frac{F_\gamma}{\hat\kappa}\right)^2\int\limits_0^t\int\limits_0^{\tau_2} x'_k(\tau_1)d\tau_1 d\tau_2\,, \end{array}\right. \tag{16}$$

and

$$\left\{\begin{array}{rcl} t &=& t'\,, \\ x_k &=& x'_k + \frac{m}{\bar\kappa}F_\gamma\int\limits_0^t\left(\tau x'_l(\tau)\right)d\tau + \frac{m}{\bar\kappa}F_\gamma t\int\limits_0^t x'_l(\tau)d\tau \\ && -\frac{2m}{\bar\kappa}F_\gamma\int\limits_0^t\int\limits_0^{\tau_2} x'_l(\tau_1)d\tau_1 d\tau_2 + \mathcal{A}_{\bar\kappa,k}(t)\,, \\ x_l &=& x'_l - \frac{m}{\bar\kappa}F_\gamma\int\limits_0^t\left(\tau x'_k(\tau)\right)d\tau - \frac{m}{\bar\kappa}F_\gamma t\int\limits_0^t x'_k(\tau)d\tau \\ && +\frac{2m}{\bar\kappa}F_\gamma\int\limits_0^t\int\limits_0^{\tau_2} x'_k(\tau_1)d\tau_1 d\tau_2 + \mathcal{A}_{\bar\kappa,l}(t)\,, \\ x_\gamma &=& x'_\gamma - \frac{m}{\bar\kappa}F_k\int\limits_0^t\left(\tau x'_l(\tau)\right)d\tau + \frac{m}{\bar\kappa}F_l\int\limits_0^t\left(\tau x'_k(\tau)\right)d\tau\,, \end{array}\right. \tag{17}$$

with $\frac{d^2}{dt^2}\mathcal{A}_{\bar\kappa,k(l)}(t) = -\frac{1}{\bar\kappa^2}F_\gamma^2 t^2 x'_{k(l)}$, respectively. It means, that both effects of deformations can be identified with the consequences of noninertial transformations (16), (17), which connect the observer $\mathcal{O}(t,x_1,x_2,x_3)$ with the observer $\mathcal{O}'(t',x'_1,x'_2,x'_3)$. From such point of view the deformation parameters $\hat\kappa$ and $\bar\kappa$ describe the degree of noninertiality as well.

Of course, for all three parameters κ, $\hat\kappa$ and $\bar\kappa$ approaching infinity, the above transformations become identity.

In this short article we demonstrate, that the additional force terms which appear in Newton equations (8), (10) and (12), can be generated equivalently in two ways — by the presence of space-time noncommutativity

(7), (9) and (11), or by the noninertial transformation of classical space (14), (16) and (17). In such a way we did show that these two approaches lead to the same additional force term (8), (10) and (12).

The above results have been obtained for nonrelativistic particle moving in the presence of constant force $\vec{F}$, but in principle, it can be extended to an arbitrary potential function $V(x)$. However, due to the nonlinear form of the noncommutative space-time function $V(x)$, such a generalization seems to be quite complicated from calculational point of view.

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