TWISTED ACCELERATION-ENLARGED NEWTON–HOOKE HOPF ALGEBRAS

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Ten Abelian twist deformations of acceleration-enlarged Newton–Hooke Hopf algebra are considered. The corresponding quantum space-times are derived as well. It is demonstrated that their contraction limit $\tau \rightarrow \infty$ leads to the new twisted acceleration-enlarged Galilei spaces.

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1. Introduction

The two Newton–Hooke cosmological algebras $\mathrm{NH}_{\pm}$ were introduced in the framework of classification of all kinematical groups [1]. Both algebras contain the characteristic cosmological time scale τ which can be interpreted in terms of the inverse of Hubble's constant for the expanding universe (NH_{+}) or associated with the "period" for the oscillating case (NH_{-}). For time parameter τ approaching infinity we get the Galilei group $\mathcal{G}$ acting on the standard (flat) nonrelativistic space-time.

Recently, there were proposed two acceleration-enlarged Newton–Hooke algebras $\widehat{\mathrm{NH}}_{\pm}$ (see [2, 3]), which contain, apart from rotation (M_{ij}), boost (K_i) and space-time translation (P_i, H) generators, the additional ones denoted by F_i, responsible for constant acceleration. Of course, if all generators F_i are equal zero we obtain the Newton–Hooke algebras $\mathrm{NH}_{\pm}$ [1] (see also [4–6]), while for time parameter τ running to infinity we get the acceleration-enlarged Galilei group $\widehat{\mathcal{G}}$ proposed in [7].

In this article we discuss the role which can be played by the acceleration-enlarged Newton–Hooke symmetries in a context of noncommutative geometry. The suggestion to use noncommutative coordinates goes back to Heisenberg and was formalized by Snyder in [8]. Recently, there were also found formal arguments based mainly on Quantum Gravity [9,10] and String Theory models [11, 12] indicating that space-time at Planck scale should be noncommutative, *i.e.* it should have a quantum nature.

Recently, the Abelian (Reshetikhin) twist deformations (see [13–15]) of the (ordinary[1]) Newton–Hooke Hopf algebras $\mathcal{U}_0(\mathrm{NH}_\pm)$[2] have been proposed in [16]. It has been shown that the corresponding quantum space-times are periodic or expanding in time for $\mathcal{U}_0(\mathrm{NH}_-)$ or $\mathcal{U}_0(\mathrm{NH}_+)$ algebras, respectively. Besides, it was also demonstrated that for cosmological time parameter τ approaching infinity, we get the twisted Galilei quantum groups and the corresponding canonically, Lie-algebraically and quadratically deformed nonrelativistic space-times [17, 18].

In this article we consider ten Abelian twist deformations of the acceleration-enlarged Newton–Hooke Hopf algebras $\mathcal{U}_0(\widehat{\mathrm{NH}}_\pm)$. In such a way we investigate the impact of the cosmological time τ as well as the impact of the additional generators F_i on the structure of quantum space. Particularly, we demonstrate that due to the presence of parameter τ, the corresponding space-times can be periodic or expanding in time for $\mathcal{U}_0(\widehat{\mathrm{NH}}_-)$ or $\mathcal{U}_0(\widehat{\mathrm{NH}}_+)$ Hopf algebras, respectively. Moreover, we also provide twisted acceleration-enlarged Galilei quantum groups and the corresponding space-times, as the $\tau \to \infty$ limit of the considered acceleration-enlarged Newton–Hooke Hopf structures. Surprisingly, the obtained in such a way acceleration-enlarged Galilei quantum spaces provide (due to the presence of generators F_i) the new cubic and quartic type of space-time noncommutativity, *i.e.* they take the form

$$[\, x_\mu, x_\nu \,] = i\alpha_{\mu\nu}^{\rho_1 \ldots \rho_n} x_{\rho_1} \ldots x_{\rho_n}\,, \tag{1}$$

where $x_0 = ct$, with $n = 3$ and 4, respectively[3].

It should be noted that two kinds of such obtained space-times appear to be quite interesting. First of them (see formula (33)) provides the deformation parameter β with dimension $[\,\beta\,] = [\,\text{acceleration} \times \text{acceleration}\,]$, while the second one (see formula (36)) provides the parameter β' with dimension $[\,\beta'\,] = [\,\text{acceleration}\,]$. Such a result indicates that there can appear a direct link between noncommutativity and the intensively studied in the last time, so-called MOND model [19], which assumes that there exists in nature observer independent acceleration parameter a_0[4]. Consequently, it looks sensible to consider the simple classical (Newtonian) models associated

[1] By "ordinary" we mean the Newton–Hooke algebra without additional F_i generators.

[2] The Newton–Hooke Hopf algebras $\mathcal{U}_0(\mathrm{NH}_\pm)$ are given by algebraic commutation relations for $\mathrm{NH}_\pm$ groups, supplemented by the trivial coproduct sector $\Delta_0(a) = a \otimes 1 + 1 \otimes a$.

[3] There was considered in the literature only canonical ($n = 0$), Lie-algebraic ($n = 1$) and quadratic ($n = 2$) type of space-time noncommutativity.

[4] MOND model in a simple way explains the movies of galactic's arms at long distance scale. However, the proper modification of Newton equation as well as the acceleration parameter a_0 are introduced into model without any principal (theoretical) rules.

with the above acceleration-enlarged quantum space-times. In the case of ordinary twisted Galilei symmetry such investigations have been performed in [20] and [21], respectively.

Finally, it should be mentioned that considered acceleration-enlarged Newton–Hooke Hopf algebras and the corresponding quantum space-times play a special role. By the proper contractions limit ($\tau \to \infty$ or/and $F_i \to 0$) of such structures one can derive (or reproduce) the quantum spaces associated with: the twisted acceleration-enlarged Galilei Hopf algebras, the twist-deformed Newton–Hooke quantum groups [16], and twisted Galilei Hopf algebras [17, 18]. For this reason, the considered spaces can be treated as a "source" for other Abelian twist-deformed nonrelativistic space-times.

The paper is organized as follows. In Section 2 ten Abelian classical r-matrices for twisted acceleration-enlarged Newton–Hooke Hopf algebras are considered. The corresponding ten quantum space-times are provided in Section 3, while their $\tau \to \infty$ contractions to the acceleration-enlarged Galilei spaces are discussed in Section 4. Finally, two contractions leading to the well known (ordinary) Newton–Hooke and Galilei space-times are mentioned in Section 5. The final remarks are presented in the last section.

2. Twisted acceleration-enlarged Newton–Hooke Hopf algebras

In accordance with Drinfeld twist procedure [13–15], the algebraic sector of twisted acceleration-enlarged Newton–Hooke Hopf algebra remains undeformed

$$\begin{aligned}
&[M_{ij}, M_{kl}] = i\,(\delta_{il}\,M_{jk} - \delta_{jl}\,M_{ik} + \delta_{jk}M_{il} - \delta_{ik}M_{jl})\,, && [H, P_i] = \pm\frac{i}{\tau^2}K_i\,,\\
&[M_{ij}, K_k] = i\,(\delta_{jk}\,K_i - \delta_{ik}\,K_j)\,, && [M_{ij}, P_k] = i\,(\delta_{jk}\,P_i - \delta_{ik}\,P_j)\,,\\
&[M_{ij}, H] = [K_i, K_j] = [K_i, P_j] = 0\,, && [K_i, H] = -iP_i\,,\;\; [P_i, P_j] = 0\,,\\
&[F_i, F_j] = [F_i, P_j] = [F_i, K_j] = 0\,, && [M_{ij}, F_k] = i\,(\delta_{jk}\,F_i - \delta_{ik}\,F_j)\,,\\
&[H, F_i] = 2iK_i\,, &&
\end{aligned} \tag{2}$$

while the coproducts and antipodes transform as follows

$$\Delta_0(a) \to \Delta_{\cdot}(a) = \mathcal{F}_{\cdot} \circ \Delta_0(a) \circ \mathcal{F}_{\cdot}^{-1}\,, \qquad S_{\cdot}(a) = u_{\cdot}\,S_0(a)\,u_{\cdot}^{-1}\,, \tag{3}$$

with $\Delta_0(a) = a \otimes 1 + 1 \otimes a$, $S_0(a) = -a$ and $u_{\cdot} = \sum f_{(1)} S_0(f_{(2)})$ (we use Sweedler's notation $\mathcal{F}_{\cdot} = \sum f_{(1)} \otimes f_{(2)}$). Present in the commutation relations (2) parameter τ denotes the characteristic for Newton–Hooke algebra cosmological time scale (in the limit $\tau \to \infty$ we get the acceleration-enlarged Galilei Hopf structure $\mathcal{U}_0(\widehat{\mathcal{G}})$). Besides, it should be noted, that the twist factor $\mathcal{F}_{\cdot} \in \mathcal{U}_{\cdot}(\widehat{\rm NH}_{\pm}) \otimes \mathcal{U}_{\cdot}(\widehat{\rm NH}_{\pm})$ satisfies the classical cocycle condition

$$\mathcal{F}_{\cdot 12} \cdot (\Delta_0 \otimes 1)\,\mathcal{F}_{\cdot} = \mathcal{F}_{\cdot 23} \cdot (1 \otimes \Delta_0)\,\mathcal{F}_{\cdot}\,, \tag{4}$$

and the normalization condition

$$(\epsilon \otimes 1)\ \mathcal{F}_{\cdot} = (1 \otimes \epsilon)\ \mathcal{F}_{\cdot} = 1\,, \tag{5}$$

with $\mathcal{F}_{\cdot 12} = \mathcal{F}_{\cdot} \otimes 1$ and $\mathcal{F}_{\cdot 23} = 1 \otimes \mathcal{F}_{\cdot}$.

It is well known, that the twisted algebra $\mathcal{U}_{\cdot}(\widehat{\mathrm{NH}}_{\pm})$ can be described in terms of so-called classical r-matrix $r \in \mathcal{U}_{\cdot}(\widehat{\mathrm{NH}}_{\pm}) \otimes \mathcal{U}_{\cdot}(\widehat{\mathrm{NH}}_{\pm})$, which satisfies the classical Yang–Baxter equation (CYBE)

$$[[\, r_{\cdot}, r_{\cdot}\,]] = [\, r_{\cdot 12}, r_{\cdot 13} + r_{\cdot 23}\,] + [\, r_{\cdot 13}, r_{\cdot 23}\,] = 0\,, \tag{6}$$

where symbol $[[\,\cdot,\cdot\,]]$ denotes the Schouten bracket and for $r = \sum_i a_i \otimes b_i$

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1\,, \qquad r_{13} = \sum_i a_i \otimes 1 \otimes b_i\,, \qquad r_{23} = \sum_i 1 \otimes a_i \otimes b_i\,.$$

In this article we consider ten Abelian twist-deformations of acceleration-enlarged Newton–Hooke Hopf algebra, described by the following r-matrices[5]

$$(i)\quad r_{\beta_1} = \tfrac{1}{2}\beta_1^{kl} F_k \wedge F_l \qquad \left[\ \beta_1^{kl} = -\beta_1^{lk}\ \right], \tag{7}$$

$$(ii)\quad r_{\beta_2} = \tfrac{1}{2}\beta_2^{kl} F_k \wedge P_l \qquad \left[\ \beta_2^{kl} = -\beta_2^{lk}\ \right], \tag{8}$$

$$(iii)\quad r_{\beta_3} = \tfrac{1}{2}\beta_3^{kl} K_k \wedge F_l \qquad \left[\ \beta_3^{kl} = -\beta_3^{lk}\ \right], \tag{9}$$

$$(iv)\quad r_{\beta_4} = \beta_4 F_m \wedge M_{kl} \qquad [\, m,k,l - \text{fixed}\,,\ \ m \neq k,l\,]\,, \tag{10}$$

$$(v)\quad r_{\beta_5} = \tfrac{1}{2}\beta_5^{kl} P_k \wedge P_l \qquad \left[\ \beta_5^{kl} = -\beta_5^{lk}\ \right], \tag{11}$$

$$(vi)\quad r_{\beta_6} = \tfrac{1}{2}\beta_6^{kl} K_k \wedge P_l \qquad \left[\ \beta_6^{kl} = -\beta_6^{lk}\ \right], \tag{12}$$

$$(vii)\quad r_{\beta_7} = \tfrac{1}{2}\beta_7^{kl} K_k \wedge K_l \qquad \left[\ \beta_7^{kl} = -\beta_7^{lk}\ \right], \tag{13}$$

$$(viii)\quad r_{\beta_8} = \beta_8 K_m \wedge M_{kl} \qquad [\, m,k,l - \text{fixed}\,,\ \ m \neq k,l\,]\,, \tag{14}$$

$$(ix)\quad r_{\beta_9} = \beta_9 P_m \wedge M_{kl} \qquad [\, m,k,l - \text{fixed}\,,\ \ m \neq k,l\,]\,, \tag{15}$$

$$(x)\quad r_{\beta_{10}} = \beta_{10} M_{ij} \wedge H\,. \tag{16}$$

[5] $a \wedge b = a \otimes b - b \otimes a$.

Due to Abelian character of the above carriers (all of them arise from the mutually commuting elements of the algebra), the corresponding twist factors can be obtained in a standard way [13–15], *i.e.* they take the form

$$\mathcal{F}_{\beta_k} = \exp\left(ir_{\beta_k}\right) ; \qquad k = 1, 2, ..., 10 . \tag{17}$$

Let us note that first four matrices include acceleration generators F_i, while the next six factors are the same as in the case of Galilei and ordinary Newton–Hooke Hopf algebra, considered in [17] and [16], respectively. Of course, for all deformation parameters β_i approaching zero the discussed above Hopf structures $\mathcal{U}_{\beta_i}(\widehat{\mathrm{NH}}_{\pm})$ become classical, *i.e.* they become undeformed.

3. Quantum acceleration-enlarged Newton–Hooke space-times

Let us now turn to the deformed space-times corresponding to the twist-deformations *(i)–(x)* discussed in pervious section. They are defined as the quantum representation spaces (Hopf modules) for quantum acceleration-enlarged Newton–Hooke algebras, with action of the deformed symmetry generators satisfying suitably deformed Leibnitz rules [22–24].

The action of generators M_{ij}, K_i, P_i, H and F_i on a Hopf module of functions depending on space-time coordinates (t, x_i) is given by

$$H \rhd f(t,\overline{x}) = i\partial_t f(t,\overline{x}) , \qquad P_i \rhd f(t,\overline{x}) = iC_{\pm}\left(\frac{t}{\tau}\right)\partial_i f(t,\overline{x}) , \tag{18}$$

$$M_{ij} \rhd f(t,\overline{x}) = i\left(x_i\partial_j - x_j\partial_i\right) f(t,\overline{x}) , \quad K_i \rhd f(t,\overline{x}) = i\tau\, S_{\pm}\left(\frac{t}{\tau}\right)\partial_i\, f(t,\overline{x}) , \tag{19}$$

and

$$F_i \rhd f(t,\overline{x}) = \pm 2i\tau^2\left(C_{\pm}\left(\frac{t}{\tau}\right) - 1\right)\partial_i f(t,\overline{x}) , \tag{20}$$

with $C_{+}[\frac{t}{\tau}] = \cosh\left[\frac{t}{\tau}\right]$, $C_{-}[\frac{t}{\tau}] = \cos\left[\frac{t}{\tau}\right]$, $S_{+}[\frac{t}{\tau}] = \sinh\left[\frac{t}{\tau}\right]$, $S_{-}[\frac{t}{\tau}] = \sin\left[\frac{t}{\tau}\right]$. Moreover, the $\star$-multiplication of arbitrary two functions is defined as follows

$$f(t,\overline{x}) \star_{\beta_i} g(t,\overline{x}) := \omega \circ \left(\mathcal{F}_{\beta_i}^{-1} \rhd f(t,\overline{x}) \otimes g(t,\overline{x})\right) , \tag{21}$$

where symbol $\mathcal{F}_{\beta_i}$ denotes the twist factor (see (17)) corresponding to the proper acceleration-enlarged Newton–Hooke Hopf algebra and $\omega\circ(a\otimes b) = a\cdot b$.

In such a way we get ten quantum space-times

$$(i) \quad [\, t, x_a\,]_{\star_{\beta_1}} = 0 ,$$

$$[\, x_a, x_b\,]_{\star_{\beta_1}} = 4i\beta_1^{kl}\tau^4\left(C_{\pm}\left(\frac{t}{\tau}\right) - 1\right)^2 \left(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk}\right) , \tag{22}$$

$$(ii)\quad [\,t, x_a\,]_{\star_{\beta_2}} = 0\,,$$
$$[\,x_a, x_b\,]_{\star_{\beta_2}} = \pm i\beta_2^{kl}\tau^2 \left(C_\pm\left(\frac{t}{\tau}\right) - 1\right) C_\pm\left(\frac{t}{\tau}\right)(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk})\,, \quad (23)$$

$$(iii)\quad [\,t, x_a\,]_{\star_{\beta_3}} = 0\,,$$
$$[\,x_a, x_b\,]_{\star_{\beta_3}} = \pm i\beta_3^{kl}\tau^3 \left(C_\pm\left(\frac{t}{\tau}\right) - 1\right) S_\pm\left(\frac{t}{\tau}\right)(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk})\,, \quad (24)$$

$$(iv)\quad [\,t, x_a\,]_{\star_{\beta_4}} = 0\,,$$
$$[\,x_a, x_b\,]_{\star_{\beta_4}} = \pm 4i\beta_4\tau^2 \left(C_\pm\left(\frac{t}{\tau}\right) - 1\right) \times [\,\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ik})\,]\,, \quad (25)$$

$$(v)\quad [\,t, x_a\,]_{\star_{\beta_5}} = 0\,,$$
$$[\,x_a, x_b\,]_{\star_{\beta_5}} = i\beta_5^{kl}\, C_\pm^2\left(\frac{t}{\tau}\right)(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk})\,, \quad (26)$$

$$(vi)\quad [\,t, x_a\,]_{\star_{\beta_6}} = 0\,,$$
$$[\,x_a, x_b\,]_{\star_{\beta_6}} = i\beta_6^{kl}\tau\, C_\pm\left(\frac{t}{\tau}\right) S_\pm\left(\frac{t}{\tau}\right)(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk})\,, \quad (27)$$

$$(vii)\quad [\,t, x_a\,]_{\star_{\beta_7}} = 0\,,$$
$$[\,x_a, x_b\,]_{\star_{\beta_7}} = i\beta_7^{kl}\tau^2\, S_\pm^2\left(\frac{t}{\tau}\right)(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk})\,, \quad (28)$$

$$(viii)\quad [\,t, x_a\,]_{\star_{\beta_8}} = 0\,,$$
$$[\,x_a, x_b\,]_{\star_{\beta_8}} = 2i\beta_8\tau\, S_\pm\left(\frac{t}{\tau}\right) \times [\,\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ak})\,]\,, \quad (29)$$

$$(ix)\quad [\,t, x_a\,]_{\star_{\beta_9}} = 0\,,$$
$$[\,x_a, x_b\,]_{\star_{\beta_9}} = 2i\beta_9\, C_\pm\left(\frac{t}{\tau}\right) \times [\,\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ak})\,]\,, \quad (30)$$

$$(x)\quad [\,t, x_a\,]_{\star_{\beta_{10}}} = 2i\beta_{10}\,[\,\delta_{ia}x_j - x_i\delta_{ja}\,]\,,$$
$$[\,x_a, x_b\,]_{\star_{\beta_{10}}} = 0\,, \quad (31)$$

associated with matrices *(i)–(x)*, respectively.

Let us note that due to the form of functions $C_\pm[\frac{t}{\tau}]$ and $S_\pm[\frac{t}{\tau}]$ the spatial noncommutativities *(i)–(ix)* are expanding or periodic in time respectively. Moreover, all of them introduce classical time and quantum spatial directions. The last type of space-time noncommutativity provides the quantum time and classical spatial variables. It should be also noted that spaces *(i)*, *(ii)*, *(iv)*, *(v)*, *(vii)* and *(ix)* are invariant with respect to time reflection $t \to -t$, while space-times *(i)*, *(iii)*, *(v)*, *(vii)* and *(x)* — with respect to $\vec{x} \to -\vec{x}$ transformation.

Of course, for all deformation parameters β_i approaching zero, the above quantum space-times become commutative.

4. Twisted acceleration-enlarged Galilei Hopf algebras — the $\tau \to \infty$ limit

In this section we provide twisted acceleration-enlarged Galilei Hopf algebras $\mathcal{U}_{\beta_i}(\widehat{\mathcal{G}})$ and corresponding quantum space-times, as the $\tau \to \infty$ limit of Hopf structures discussed in pervious sections. In such a limit the commutation relations (2) become τ-independent, *i.e.* we neglect the impact of the cosmological time scale τ on the structure of the considered Hopf algebras.

First of all, we perform the contraction limit $\tau \to \infty$ of the formulas (2) and (7)–(16). Consequently, the corresponding classical r-matrices remain the same as (7)–(16), while the algebraic sector of all considered $\mathcal{U}_{\beta_i}(\widehat{\mathcal{G}})$ algebras takes the form

$$
\begin{aligned}
&[\, M_{ij}, M_{kl}\,] = i\,(\delta_{il}\, M_{jk} - \delta_{jl}\, M_{ik} + \delta_{jk} M_{il} - \delta_{ik} M_{jl})\;, && [\, H, P_i\,] = 0\,,\\
&[\, M_{ij}, K_k\,] = i\,(\delta_{jk}\, K_i - \delta_{ik}\, K_j)\;, && [\, M_{ij}, P_k\,] = i\,(\delta_{jk}\, P_i - \delta_{ik}\, P_j)\;,\\
&[\, M_{ij}, H\,] = [\, K_i, K_j\,] = [\, K_i, P_j\,] = 0\,, && [\, K_i, H\,] = -iP_i\,,\ [\, P_i, P_j\,] = 0\,,\\
&[\, F_i, F_j\,] = [\, F_i, P_j\,] = [\, F_i, K_j\,] = 0\,, && [\, M_{ij}, F_k\,] = i\,(\delta_{jk}\, F_i - \delta_{ik}\, F_j)\;,\\
&[\, H, F_i\,] = 2iK_i\,. &&
\end{aligned}
\tag{32}
$$

The corresponding coproduct sectors can be obtained by application of the formulas (3) and (17).

Let us now turn to the corresponding quantum nonrelativistic space-times. One can check (see $\tau \to \infty$ limit of the formulas (22)–(31)) that they look as follows[6]

$$(i)\quad [\,t, x_a\,]_{\star_{\beta_1}} = 0\,, \qquad [\,x_a, x_b\,]_{\star_{\beta_1}} = i\beta_1^{kl}\, t^4\, (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk})\,, \tag{33}$$

$$(ii)\quad [\,t, x_a\,]_{\star_{\beta_2}} = 0\,, \qquad [\,x_a, x_b\,]_{\star_{\beta_2}} = \frac{i}{2}\beta_2^{kl}\, t^2\, (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk})\,, \tag{34}$$

$$(iii)\quad [\,t, x_a\,]_{\star_{\beta_3}} = 0\,, \qquad [\,x_a, x_b\,]_{\star_{\beta_3}} = \frac{i}{2}\beta_3^{kl}\, t^3\, (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk})\,, \tag{35}$$

$$(iv)\quad [\,t, x_a\,]_{\star_{\beta_4}} = 0\,, \qquad [\,x_a, x_b\,]_{\star_{\beta_4}} = 2i\beta_4\, t^2 \times \left[\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ak})\right]\,, \tag{36}$$

$$(v)\quad [\,t, x_a\,]_{\star_{\beta_5}} = 0\,, \qquad [\,x_a, x_b\,]_{\star_{\beta_5}} = i\beta_5^{kl}(\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk})\,, \tag{37}$$

$$(vi)\quad [\,t, x_a\,]_{\star_{\beta_6}} = 0\,, \qquad [\,x_a, x_b\,]_{\star_{\beta_6}} = i\beta_6^{kl}\, t\, (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk})\,, \tag{38}$$

$$(vii)\quad [\,t, x_a\,]_{\star_{\beta_7}} = 0\,, \qquad [\,x_a, x_b\,]_{\star_{\beta_7}} = i\beta_7^{kl}\, t^2\, (\delta_{ak}\delta_{bl} - \delta_{al}\delta_{bk})\,, \tag{39}$$

$$(viii)\quad [\,t, x_a\,]_{\star_{\beta_4}} = 0\,, \qquad [\,x_a, x_b\,]_{\star_{\beta_4}} = 2i\beta_4\, t \times \left[\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ak})\right]\,, \tag{40}$$

$$(ix)\quad [\,t, x_a\,]_{\star_{\beta_9}} = 0\,, \qquad [\,x_a, x_b\,]_{\star_{\beta_9}} = 2i\beta_9 \times \left[\delta_{ma}(x_k\delta_{bl} - x_l\delta_{bk}) - \delta_{mb}(x_k\delta_{al} - x_l\delta_{ak})\right]\,, \tag{41}$$

$$(x)\quad [\,t, x_a\,]_{\star_{\beta_{10}}} = 2i\beta_{10}\,[\,\delta_{ia}x_j - x_i\delta_{ja}\,]\,, \qquad [\,x_a, x_b\,]_{\star_{\beta_{10}}} = 0\,, \tag{42}$$

in the case of $\mathcal{U}_{\beta_1}(\widehat{\mathcal{G}}), \ldots, \mathcal{U}_{\beta_{10}}(\widehat{\mathcal{G}})$ Hopf algebras respectively. One can easily see, that space-time *(i)* provides the deformation parameter β with dimension $[\,\beta\,] = [$ acceleration $\times$ acceleration $]$, while the deformation *(iv)* — with $[\,\beta\,] = [$ acceleration $]$. Due to the reasons already mentioned in Introduction both quantum spaces appear to be quite interesting from the physical point of view.

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[6] It should be noted that the commutation relations (33)–(42) can be also derived with use of the formula (21) and differential representation of acceleration-enlarged Galilei generators [7].

Obviously, for all deformation parameters β_i approaching zero the above Hopf algebras become classical, while the corresponding quantum space-times — commutative.

5. Twisted Newton–Hooke (and Galilei) Hopf algebras — the $F_i \to 0$ (and $\tau \to \infty$) limit

It should be noted, that apart from provided in pervious section contraction limit $\tau \to \infty$ of $\mathcal{U}_{\beta_i}(\widehat{\mathrm{NH}}_\pm)$ Hopf algebras, there exist two other contractions. First of them is defined by $F_i \to 0$ limit and leads to the twisted $\mathcal{U}_{\beta_5}(\widehat{\mathrm{NH}}_\pm), \ldots, \mathcal{U}_{\beta_{10}}(\widehat{\mathrm{NH}}_\pm)$ Newton–Hooke Hopf algebras and corresponding quantum space-times, introduced in paper [16]. The second contraction is given by $F_i \to 0$ and $\tau \to \infty$ limit, and provides the twist-deformed $\mathcal{U}_{\beta_5}(\mathcal{G}), \ldots, \mathcal{U}_{\beta_{10}}(\mathcal{G})$ Galilei Hopf algebras, proposed (together with corresponding quantum spaces) in the articles [17] and [18].

6. Final remarks

In this article we consider ten Abelian twist-deformations of acceleration-enlarged Newton–Hooke Hopf algebras. Besides, we demonstrate that as in the case of ordinary twist-deformed Newton–Hooke Hopf algebra, the corresponding spaces can be periodic and expanding in time for $\mathcal{U}_{\beta_i}(\widehat{\mathrm{NH}}_-)$ and $\mathcal{U}_{\beta_i}(\widehat{\mathrm{NH}}_+)$ quantum groups, respectively. In $\tau \to \infty$ limit we also discover new twisted acceleration-enlarged Galilei Hopf algebras and ten quantum space-times (33)–(42).

It should be noted that present studies can be extended in various ways. First of all, one can find the dual Hopf structures $\mathcal{D}_{\beta_i}(\widehat{\mathrm{NH}}_\pm)$ with the use of FRT procedure [25] or by canonical quantization of the corresponding Poisson–Lie structures [26]. Besides, as it was already mentioned in Introduction, one should ask about the basic dynamical models corresponding to the acceleration-enlarged Newton–Hooke and Galilei space-times (22)–(31) and (33)–(42). Finally, one can also consider more complicated (non-Abelian) twist deformations of acceleration-enlarged Newton–Hooke Hopf algebras, *i.e.* one can find the twisted coproducts, corresponding non-commutative space-times and dual Hopf structures. Such problems are now under consideration.

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