A REMARK ON THE NEGATIVE BINOMIAL DISTRIBUTION

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The concept of clans, emerging in the context of the negative binomial distribution, is generalized. The generalized clans are themselves produced according to negative binomial distribution. This opens new possibilities for interpretation of mechanisms of particle production processes.

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1. Introduction

Experiments show that the multiplicity distributions in varying rapidity intervals are reasonably well described by the negative binomial distribution with the parameter k depending on the size of the interval and on energy of the collision. This observation fuelled a large interest in studies of the negative binomial distribution, first introduced in particle physics by Giovannini already in early seventies [1].

The negative binomial distribution is defined by its generating function which is of the form

$$\Phi(z) \equiv \sum_n P(n) z^n = \left[1 + \frac{\langle n \rangle}{k}(1-z)\right]^{-k}, \tag{1}$$

where $\langle n \rangle$ is the average multiplicity and $1/k$ measure deviation from the Poisson distribution.

Interpretation of this formula went into two different directions.

The first one is based on the identity

$$\left[1+\frac{\langle n\rangle}{k}(1-z)\right]^{-k} = \frac{k^k}{\Gamma(k)}\int dt\, t^{k-1}e^{-kt}e^{-t\langle n\rangle(1-z)} \tag{2}$$

showing that NBD can be represented as a superposition of Poisson distributions with the weight given by the Γ-distribution

$$W(t)\,dt = \frac{k^k}{\Gamma(k)}\,t^{k-1}e^{-kt}dt\,. \tag{3}$$

This formula suggests that the observed particle spectra result from a superposition of "basic" processes in which particles are produced independently. This interpretation finds some justification in various models where the QCD (Lund) strings are the basic building blocks.

The second interpretation stems from the identity

$$\left[1+\frac{\langle n\rangle}{k}(1-z)\right]^{-k} = e^{-k\log\left[1+\frac{\langle n\rangle}{k}(1-z)\right]}\,. \tag{4}$$

Introducing the function $\phi_c(z)$ as

$$\phi_c(z) \equiv 1-\frac{k}{\langle N\rangle}\log\left[1+\frac{\langle n\rangle}{k}(1-z)\right] \tag{5}$$

one can rewrite (4) as

$$\left[1+\frac{\langle n\rangle}{k}(1-z)\right]^{-k} = e^{-\langle N\rangle[1-\phi_c(z)]}\,. \tag{6}$$

This formula shows that NBD can be interpreted as independent production of "clans", decaying into observed particles according to the distribution characterized by the generating function $\phi_c(z)$ ($\langle N\rangle$ is, of course, the average number of produced clans) [2, 3]. One can easily show that the average number of particles in clan decay is $\langle n_c\rangle = \langle n\rangle/\langle N\rangle$.

Demanding that the clan decay products contain at least one particle implies $\phi_c(z=0)=0$, *i.e.*

$$\begin{aligned} \langle N\rangle &= k\log\left[1+\frac{\langle n\rangle}{k}\right]\,, \\ \frac{1}{\langle n_c\rangle} &= \frac{k}{\langle n\rangle}\log\left[1+\frac{\langle n\rangle}{k}\right]\,. \end{aligned} \tag{7}$$

The distribution corresponding to the generating function (5) reads

$$P_c(n_c) = \frac{k}{\langle N\rangle}\frac{1}{n_c}\left[\frac{\langle n\rangle}{\langle n\rangle + k}\right]^{n_c}, \qquad n_c \geq 1\,. \tag{8}$$

The factorial moments of this distribution can be found by expansion of the generating function around $z = 1$ and read:

$$F_p = \langle n_c(n_c-1)\,\ldots\,(n_c-p+1)\rangle = (p-1)!\frac{k}{\langle N\rangle}\left(\frac{\langle n\rangle}{k}\right)^p. \tag{9}$$

Extensive studies of the clan parameters, performed by several groups, are consizely summarized in [4] where also a list of original references can be found. For further applications of the clan concept in particle production, see [5] and references quoted there.

In the present paper we show that these two interpretations can be mixed with each other, thus providing a more flexible possibility of understanding data. The key observation is that, to obtain NBD of the observed particles, the clans need not be produced independently, as postulated in [2,3]. They may be themselves distributed according to NBD. This may allow to remove a serious restriction on production mechanism and to accommodate various ideas in a single picture.

2. Generalized clans

Consider production of N clans distributed according to NBD characterized by the parameter K. The generating function of this distribution is

$$\Phi(z) = \sum_N P(N)z^N = \left[1 + \frac{\langle N\rangle}{K}(1-z)\right]^{-K}. \tag{10}$$

Denoting the generating function of the distribution in clan decay by $\phi_c(z)$ we obtain for the observed distribution

$$\Psi(z) = \sum_n p(n)z^n = \left[1 + \frac{\langle N\rangle}{K}[1-\phi_c(z)]\right]^{-K}. \tag{11}$$

Demanding that this distribution is again NBD we obtain the condition

$$1 + \frac{\langle N\rangle}{K}[1-\phi_c(z)] \equiv \left[1 + \frac{\langle n\rangle}{k}(1-z)\right]^a, \tag{12}$$

where $a = k/K$. It follows that

$$\phi_c(z) = 1 + \frac{K}{\langle N\rangle} - \frac{K}{\langle N\rangle}\left[1 + \frac{\langle n\rangle}{k}(1-z)\right]^a. \tag{13}$$

The condition $\phi_c(z=0)=0$ implies

$$\langle N\rangle = K\left\{\left[1+\frac{\langle n\rangle}{k}\right]^a - 1\right\} . \tag{14}$$

Note that for a fixed k and $K\to\infty$ one recovers the formula (7).

The distribution in the clan decay, following from (13) is obtained by expansion in z and reads

$$P(n_c) = \frac{k}{\langle N\rangle}\left[1+\frac{\langle n\rangle}{k}\right]^a \frac{(1-a)\,\ldots\,(n_c-1-a)}{n_c!}\left[\frac{\langle n\rangle}{k+\langle n\rangle}\right]^{n_c} . \tag{15}$$

The factorial moments are

$$F_p = a(1-a)\,\ldots\,(p-1-a)\frac{K}{\langle N\rangle}\left(\frac{\langle n\rangle}{k}\right)^p . \tag{16}$$

All this has sense only if $a\leq 1$. Again, for a fixed k and $a\to 0$ we recover the formulae (8) and (9).

3. Conclusions

In conclusion, we have shown that interpretation of the negative binomial distribution in terms of clans is not unique. For a given $\langle n\rangle$ and k (which define the observed multiplicity distribution) one can obtain various values of clan parameters, depending on the parameter K describing the degree of correlation in clan production.

This ambiguity allows to consider more realistic particle production mechanisms in which clans' production is considered as a superposition of uncorrelated emission, *e.g.* by mixing various impact parameters. It remains to be seen if the properties of such generalized clans will show interesting regularities when confronted with data.

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