UNCOVERING THE DENSITY OF MATTER FROM MULTIPLICITY DISTRIBUTION

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Multiplicity distributions in the form of superposition of Poisson distributions which are observed in multiparticle production are interpreted as reflection of a two-step nature of this process: the creation and evolution of the strongly interacting fluid, followed by its uncorrelated decay into observed hadrons. A method to uncover the density of the fluid from the observed multiplicity distribution is described.

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1. Introduction

In the commonly discussed pictures, multiple particle production is considered as a two-step process. In the first step the matter in created in the form of elementary constituents (*i.e.* quarks and gluons). At this stage it is treated as a continuous medium undergoing a complicated evolution, subject to intense investigations[1]. In the second step ("freeze-out") the medium changes into the observed hadrons.

It is clear that any information one may obtain about the density of matter just before "freeze-out" is of primary importance for understanding the operating mechanisms. This information is, however, inaccessible to direct measurement since one can measure only the final hadrons which are characterized by the discrete multiplicity distribution $P(n)$. Although it is intuitively clear that large amount of matter should produce large hadronic multiplicities, this relation is not straightforward.

[1] Such a picture is particularly well established in heavy ion collisions. For a recent review see, *e.g.* [1].

In the present note we propose a simple idea which allows to relate these two quantities. It is based on the observation that, as shown in many phenomenological analyses of multiparticle production [2], the measured multiplicity distributions can be described as a superposition of Poisson distributions in the form

$$P(n) = \int dt F(t)\, e^{-\bar{n}t} \frac{(t\bar{n})^n}{n!}\,, \tag{1}$$

where $F(t)$ is a non-negative function satisfying the normalization conditions

$$\int dt F(t) = 1\,, \qquad \int dt t F(t) = 1\,. \tag{2}$$

This formula shows that the function $F(t)$ can be interpreted as the distribution of the amount of matter produced in the collision, the Poisson factor being responsible for a random (uncorrelated) emission of particles resulting from its decay.

Once this point of view is accepted, we obtain the direct link between the matter density and multiplicity. Clearly, from the known distribution of the matter density t one can evaluate the multiplicity distribution $P(n)$. A more difficult (and more interesting) question is: what kind of information one can obtain about t from the observed multiplicity distribution $P(n)$.

This problem was solved already many years ago in the limit of very large multiplicities $\bar{n} \to \infty$ and fixed $\bar{t} \equiv n/\bar{n}$ [3]. This is the so-called KNO limit:

$$t \to \frac{n}{\bar{n}}\,, \qquad F\left(\frac{n}{\bar{n}}\right) \to \bar{n} P(n)\,. \tag{3}$$

In this large multiplicity limit there is one-to-one correspondence between n and t.

It is clear that such a simple result cannot hold at finite multiplicities. Indeed, as is seen from (1), each multiplicity n receives contribution from a certain region of t. It is also seen, however, that this region is limited and therefore some information is nevertheless available.

To define more precisely the problem we observe that it separates naturally into two parts:

(i) For any fixed multiplicity n, the maximum $t_s = t_s(n)$ of the integrand in (1) points out to the value of the density t which contributes dominantly to this n. It can thus be identified with the effective value of the density t corresponding to the given multiplicity n.

(ii) The uncertainty of this identification is given by the width of the region in t contributing to a given n, *i.e.* the width $\Delta_s(n)$ of the maximum of the integrand at $t = t_s(n)$. As is well-known, it is related to the second derivative of the integrand at this point.

One sees from these arguments that to uncover the information on the density t which is hidden in the multiplicity distribution $P(n)$, we have to solve the saddle-point problem of the integral in (1).

In the present paper we discuss a method which can be used for this purpose. It allows to estimate $t_s(n)$, $\Delta_s(n)$ and $F(t)$ from a given (measured) multiplicity distribution. As is natural for the saddle-point approximation, the result represents an asymptotic expansion in $1/\bar{n}$.

In the next section we discuss the KNO approximation and its limitations. A method to improve the KNO approximation is shown in Section 3. A specific example is presented in Section 4. Some tests of the procedure are described in Section 5. Our results are summarized in the last section.

2. The KNO approximation

2.1. The saddle-point method

We collect here, for later use, the basic formulae of the saddle-point approximation. Writing

$$P(n) = \frac{\bar{n}^n}{n!} \int_0^\infty dt e^{\Phi(t)} dt \tag{4}$$

we have

$$\Phi(t) = \log F(t) - \bar{n}t + n \log t \tag{5}$$

and the saddle-point condition

$$\Phi'(t_s) = \frac{F'(t_s)}{F(t_s)} - \bar{n} + \frac{n}{t_s} = 0\,. \tag{6}$$

Furthermore,

$$\Phi''(t_s) = \frac{F''(t_s)}{F(t_s)} - \left(\frac{F'(t_s)}{F(t_s)}\right)^2 - \frac{n}{t_s^2}\,, \qquad \Delta_s = 1/\sqrt{\Phi''(t_s)} \tag{7}$$

and the saddle-point formula

$$P(n) \approx F(t_s) e^{-\bar{n}t_s} \frac{(\bar{n}t_s)^n}{n!} \sqrt{\frac{2\pi}{-\Phi''(t_s)}}\,. \tag{8}$$

2.2. Large $\bar{n}$, fixed $n/\bar{n}$ limit

When both $\bar{n}$ and n are very large, the first term in the saddle-point equation (6) can be neglected in the first approximation (because $F(t)$ is independent of n and $\bar{n}$)[2]. Then we have

$$t_{s0} = t_{s0}(n) = \frac{n}{\bar{n}} \equiv \bar{t}\,, \qquad \Phi''(t_{s0}) = -\frac{\bar{n}^2}{n}\,. \tag{9}$$

Introducing (9) into (8) we obtain

$$P(n) = F(t_{s0}) t_{s0} \sqrt{-2\pi/\Phi''(t_{s0})}\ \frac{\bar{n}^n}{n!}\ t_{s0}^n e^{-\bar{n} t_{s0}} \to \frac{F(n/\bar{n})}{\bar{n}} \tag{10}$$

the well-known KNO formula [3].

It should be emphasized that this formula is valid only in the limit $\bar{n} \to \infty$, $n/\bar{n}$ fixed. Indeed, to justify the derivation, both $\bar{n}$ and n must be much larger than F'/F.

To illustrate this point we consider the example of Negative Binomial Distribution (known to be not too far from real data [2]):

$$F(t;k) \;=\; \frac{k^k}{\Gamma(k)} t^{k-1} e^{-kt}\,, \tag{11}$$

$$P(n) \;=\; (1+\bar{n}/k)^{-k} \frac{\Gamma(k+n)}{\Gamma(k) n!} (1+k/\bar{n})^{-n}\,, \tag{12}$$

where k is a parameter ($1/k$ measures the deviation of (12) from the Poisson distribution).

In Figs. 1 and 2 $t_s(n)$ and $F[t_s(n)]$ obtained from (9) and (10) are plotted *versus* $\bar{t} = n/\bar{n}$ and compared to the exact values given by

$$t_s(n) = \frac{n+k-1}{\bar{n}+k}\,, \qquad F(n) \equiv F[t_s(n)] = \frac{k^k}{\Gamma(k)} [t_s(n)]^{k-1} e^{-k t_s(n)}\,. \tag{13}$$

One sees that even for $\bar{n} = 25$ the KNO approximation to the function $t_s = t_s(n)$ obtained from the Negative Binomial Distribution is far from perfect. Also the KNO function itself is not well reproduced. The situation gets worse as k increases since the term F'/F in (6) increases linearly with k.

[2] The values of $\bar{n}$ and n at which this approximation is acceptable depends, of course, on the rate of change of $F(t)$. For each $F(t)$ one can, however, find n and $\bar{n}$ large enough to satisfy this requirement.

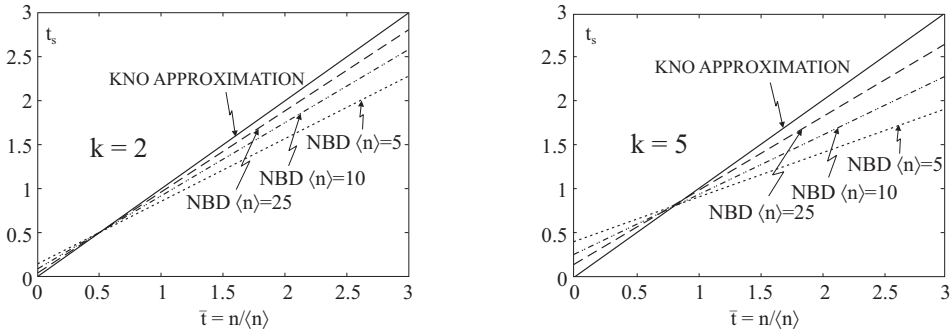

Fig. 1. Saddle-point of the Negative Binomial Distribution *versus* $\bar{t} = \bar{n}/\bar{n}$. Full line: KNO approximation, $t_s = \bar{t}$; other lines: exact values of t_s evaluated from (13), for various $\bar{n}$ (as indicated in the figure). One sees a substantial difference between the KNO approximation and the exact result even at $\bar{n} = 25$.

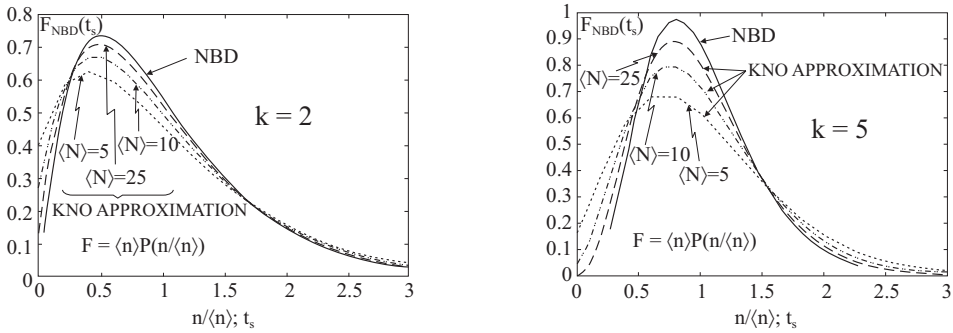

Fig. 2. KNO approximation for the Negative Binomial Distribution. Full line: the function $F[t_s(n)]$ evaluated from (13); other lines: $F[t_s(n)]$ obtained from KNO approximation (10) for various $\bar{n}$ (as indicated in the figure). One sees an apparent violation of the KNO scaling even at $\bar{n} = 25$.

3. Improving KNO

It is clear from the discussion in the previous section that to improve the approximation, it is necessary to reduce, as much as possible, the unknown part of the term F'/F in the saddle-point equation (6). To achieve this goal we propose to write

$$F(t) = F_0(t)G(t)\,, \tag{14}$$

where $F_0(t)$ is a known function, possibly close to $F(t)$ itself. The saddle-point equation reads

$$\frac{G'(t)}{G(t)} + \frac{F_0'(t)}{F_0(t)} + \frac{n}{t} - \bar{n} = 0\,. \tag{15}$$

If the function $F_0(t)$ is sufficiently close to $F(t)$, the first term of (15) can be neglected in the first approximation and one obtains

$$\frac{F_0'(t)}{F_0(t)} + \frac{n}{t} - \bar{n} = 0\,. \tag{16}$$

Since the function $F_0(t)$ is known, solution of (16) is only question of computer time. This allows to construct an improved approximation as follows.

Once the approximate saddle-point $t = t_0$ is determined from (16) one can evaluate $G(n) = G[t_0(n)]$ from the saddle-point formula (8). In this way one obtains the first order approximation to our problem.

This procedure can be iterated as described below.

3.1. Iteration

From the known values of $G(n)$ we can construct the approximate first and second logarithmic derivatives of G:

$$\frac{G'[t_0(n)]}{G(t_0(n)]} \approx \frac{2}{G(t_0(n+1)] + G(t_0(n)]} \frac{G[t_0(n+1)] - G[t_0(n)]}{t_0(n+1) - t_0(n)} \tag{17}$$

and

$$\left[\frac{G''}{G}\right]_0 = \frac{2}{[t_0(n+2) - t_0(n)]G(n+1)} \times \left[\frac{G(n+2) - G(n+1)}{t_0(n+2) - t_0(n+1)} - \frac{G(n+1) - G(n)}{t_0(n+1) - t_0(n)}\right]. \tag{18}$$

Introducing (17) into (15) we find the saddle-point in the next approximation:

$$t_1 = t_1(n) = t_0(n)\left[1 - \frac{G'[t_0(n)]}{G(t_0(n)](\bar{n} + k_0)}\right]^{-1}. \tag{19}$$

Using now (8) we find the next approximation for the function G:

$$G(t_1)F_0(t_1) = \frac{P(n)n!}{(\bar{n}t_1)^n} e^{\bar{n}t_1} \sqrt{\frac{-\Phi''(t_1)}{2\pi}}\,, \tag{20}$$

where $\Phi''(t_1)$ is approximated by

$$\Phi''(t_1) \approx \frac{G''(t_0)}{G(t_0)} - \left[\frac{G'(t_0)}{G(t_0)}\right]^2 + \frac{F_0''(t_1)}{F_0(t_1)} - \left[\frac{F_0'(t_1)}{F_0(t_1)}\right]^2 - \frac{n}{t_1^2}\,. \tag{21}$$

These formulae allow to evaluate $t_1 = t_1(n)$ and $G[t_1(n)]$ and thus represent the second order approximation to our problem. They can serve as a starting point for the next approximation.

4. Negative Binomial Distribution as the first approximation

Selection of the form of $F_0(t)$ is to large extent arbitrary except that it should be possibly close to the real data. The best choice can be obtained by simply fitting (1) to data. To obtain $t_s(n)$ it is then enough to solve the saddle-point equation (6). Since the fit is never ideal, however, it may be worth to evaluate the next approximation, using the formulae of the previous section. As the function $G(t)$ is close to one, the corrections are expected to be small. Nevertheless they may be non-negligible in some regions of n.

As an illustration, we shall now discuss a simple and instructive possibility taking $F_0 = F_0(t, k_0)$ to be that of the Negative Binomial Distribution (11), where the parameter k_0 is chosen in such a way as to minimize the difference between the experimental multiplicity distribution and the distribution (12).

There are two attractive features of this choice. First, it is known that the Negative Binomial Distribution is not very far from the distributions observed hitherto in experiments [2]. This gives a chance to obtain already a reasonable first approximation. Second, the approximate saddle-point equation (16)

$$\frac{k_0 - 1}{t} - k_0 + \frac{n}{t} - \bar{n} = 0 \tag{22}$$

is easy to solve. Using the notation introduced in the previous section we thus obtain

$$t_0 = \frac{n+k_0-1}{\bar{n}+k_0}\,, \qquad \Phi_0''(t_0) = -\frac{n+k-1}{t_0^2}\,, \qquad F(t_0) = \frac{P(n)n!e^{\bar{n}t_0}}{(\bar{n}t_0)^n\sqrt{\frac{-\Phi''(t_0)}{2\pi}}} \tag{23}$$

and

$$G(t_0) = \frac{P(n)n!(\bar{n}+k_0)^{n+k_0}\Gamma(k_0)e^{n+k_0-1}}{\bar{n}^n k_0^{k_0}(n+k_0-1)^{n+k_0-1}\sqrt{2\pi(n+k_0-1)}}\,. \tag{24}$$

Eqs. (23) and (24) give the lowest order estimate of $t_s(n)$ and $F[t_s(n)] = F_0[t_s(n)]G[t_s(n)]$.

Note that consistency requires that if $P(n)$ is itself a Negative Binomial Distribution with the parameter $k = k_0$, we should have $G(t) \approx 1$. To verify this condition we substitute $P(n)$ given by (12) into (24) to obtain

$$G(t_0) = \frac{\Gamma(n+k_0)e^{n+k_0-1}}{(n+k_0-1)^{n+k_0-1}\sqrt{2\pi(n+k_0-1)}}\,. \tag{25}$$

From the Stirling formula one sees that this expression is indeed close to 1 for $n + k_0 - 1 \geq 2$.

Higher order approximations can be obtained starting from (23) and (24) along the lines described in the previous section. In the next section we check how effective is this procedure.

5. Testing the method

To see how the procedure works we applied it to two distributions.

5.1. Derivative Negative Binomial

The first one is defined by the function $F(t)$ in the form[3]

$$F(t) = Ct^{k-1}e^{-\lambda t}[\lambda t - \log \lambda t]\,, \tag{26}$$

where k is a parameter, $\lambda = k\frac{k+1-\psi(k+1)}{k-\psi(k)}$ and $C = \frac{\lambda^k}{\Gamma(k)[k-\psi(k)]}$ [these formulae follow from normalization conditions (2)]. For $P(n)$ one obtains

$$P(n) = C\frac{\Gamma(n+k)}{(\lambda+\bar{n})^{n+k}}\frac{\bar{n}^n}{n!}[k+n-\psi(k+n)]\,. \tag{27}$$

The saddle-point equation is

$$\frac{k+n-1}{t} - (\lambda+\bar{n}) + \frac{\lambda t - 1}{t[\lambda t - \log \lambda t]} = 0\,. \tag{28}$$

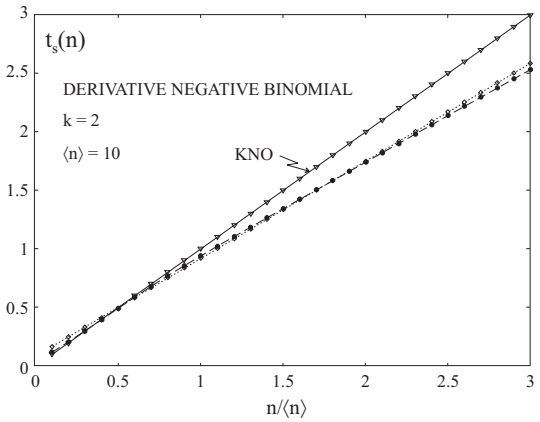

Fig. 3. Derivative Negative Binomial Distribution (27). $t_s(n)$ plotted *versus* $\bar{t} = n/\bar{n}$. One sees that already the first order approximation and the exact value (two lines below the KNO approximation) are almost identical.

[3] Note that $-\frac{d}{dk}[t^{k-1}e^{-kt}] = (t - \log t)t^{k-1}e^{-kt}$. Hence the name.

The ratio of (26) to the auxiliary function (11) is $G(t) \sim [\lambda t - \log \lambda t]$. It varies rather strongly close to $t = 0$ and is fairly gentle at large t. Therefore we expect our approximation to be best at large multiplicities.

In Fig. 3 $t_s(n)$ is plotted *versus* $n/\bar{n}$ for $\bar{n} = 10$. One sees that, already first approximation is pretty close to the exact value, derived (numerically) from (28). The KNO approximation is again very bad.

5.2. Beta distribution

Another distribution considered was the beta distribution, for which the function $F(t)$ is given by

$$F(t) = \frac{C}{T^k} t^{k-1} \left(1 - \frac{t}{T}\right)^{q-1} \Theta(T - t)\,. \tag{29}$$

The normalization conditions (2) imply $T = 1 + q/k$; $C\beta(k, q) = 1$.

The multiplicity distribution following from (29) is

$$P(n)n! = C(T\bar{n})^n \sum_{s=0}^{q-1} \frac{(q-1)!}{s!(q-1-s)!} \frac{(-1)^s}{p} \Phi\left(p - 1; p; -T\bar{n}\right) \tag{30}$$

with $p \equiv k + n + s$ and $\Phi(a; b; z)$ is the confluent hypergeometric function.

The saddle-point equation

$$\frac{n+k-1}{t} - \frac{q-1}{T-t} - \bar{n} = 0 \tag{31}$$

can be explicitely solved, giving

$$t_s = \bar{t} + \frac{k+q-2}{\bar{n}} - \tfrac{1}{2}\left[\sqrt{(\bar{t} - T + (k+q-2)/\bar{n})^2 + 4T(q-1)/\bar{n}} + \bar{t} - T + (k+qv-2)/\bar{n}\right]. \tag{32}$$

One sees that t_s is a nonlinear function of $\bar{t} = n/\bar{n}$ (except for $q = 1$). At large $\bar{n}$ and fixed $\bar{t}$, however, we recover — in accord with the general KNO formula — $t_s \to \bar{t}$.

The β distribution is very close to Negative Binomial at $t \approx 0$ and differs drastically at large t. Therefore, this time we expect that the approximation scheme should be more effective at small multiplicities.

In Fig. 4 the results for $t_s(n)$ are plotted *versus* $n/\bar{n}$ for $k = 5$, $q = 2$ and $\bar{n} = 15$. One sees a great improvement with respect to KNO approximation already in the second approximation (first step of iteration). The agreement is very good at low multiplicities. Some discrepancies remain, however, for multiplicities exceeding $\bar{n}$. Adding more iterations does not improve the situation (which is not surprising for the asymptotic expansion).

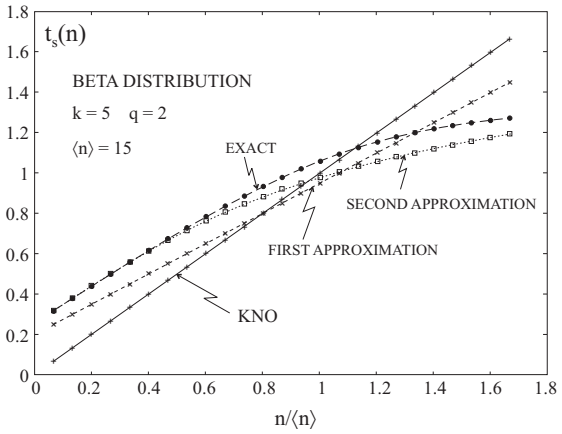

Fig. 4. β distribution (30). $t_s(n)$ is plotted *versus* $\bar{t} = n/\bar{n}$. One sees that the first order approximation (being linear) cannot reproduce the exact result. The second order approximation gives good agreement at $n < \bar{n}$. The discrepancy persists, however, at larger n.

6. Summary and comments

It is proposed that the observed multiplicity distributions which satisfy (1) are interpreted as indication of the two-step character of multiparticle processes: a strongly interacting fluid produced in the first step decays — in the second step — into the observed particles. In this interpretation the parameter t of (1) acquires the physical meaning of the amount of matter in the produced fluid. Thus (1) gives the relation between the observed multiplicity and the density of the fluid. This opens the possibility to investigate various regions of the fluid density by selecting observed particle multiplicities.

Carrying such a program is not straigthforward, however, since (1) does not give a one-to-one correspondence between the particle multiplicity n and the fluid density t. To find t for a given n, the saddle-point problem for the integrand in (1) must be solved. It is shown that the KNO limit does not give good approximation even at moderately high multiplicities. A procedure for improvement is suggested and tested on some simple distributions.

Several comments are in order.

(i) One sees from the arguments given in Section 2 that investigations of KNO scaling [3] must be interpreted with some care. This is illustrated in Fig. 2, where the apparent violation of scaling reflects simply the variation of $\bar{n}$ and is not related to dynamics of the process (which obeys the KNO scaling exactly).

(ii) It is important to remember that the units of the density t in (1) are essentially arbitrary. Therefore, as long as the relation $t = t_s(n)$ for a given distribution is close to the linear one, the KNO approximation may be useful even at relatively small $\bar{n}$.

(iii) It was observed by Praszalowicz [4] that one can improve the method suggested in Section 4 by taking $F_0(t)$ in the form

$$F_0(t) = \frac{(\lambda k)^k}{\Gamma(k)} t^{k-1} e^{-\lambda k t} \,. \tag{33}$$

The approximate saddle-point equation is as easy to solve as (22), and one obtains two parameters to be adjusted for the first approximation, which makes the procedure more flexible.

(iv) The idea to connect multiplicity with the amount of the produced fluid is rather old [5]. It was recently explored by Mc Lerran and Praszalowicz [6] with encouraging results.

(v) It should be emphasized that the procedure advocated in this paper does not rely on a global fit of (1) to the multiplicity distribution (although such a fit may be helpful to obtain a good first approximation). It uses only *local* information from the neighbourhood of a given n. This is of importance if in some regions of n the multiplicity distribution cannot be accurately measured.

(vi) One should remember that the proposed method is based on asymptotic expansion at large $\bar{n}$. Therefore in practice only first and second approximations are really useful: at small $\bar{n}$ higher orders do not improve the result (and may actually spoil it) and at high $\bar{n}$ they are not really necessary.

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