AMBIGUITIES IN THE ASSOCIATION BETWEEN SYMMETRIES AND CONSERVATION LAWS IN THE PRESENCE OF ALTERNATIVE LAGRANGIAN REPRESENTATIONS

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We identify two alternative Lagrangian representations for the damped harmonic oscillator characterised by a frictional coefficient γ. The first one is explicitly time independent while the second one involves time parameter explicitly. With separate attention to both Lagrangians we make use of the Noether theorem to compute the variational symmetries and conservation laws in order to study how association between them changes as one goes from one representation to the other. In the case of time-independent representation squeezing symmetry leads to conservation of angular momentum for $\gamma = 0$, while for the time-dependent Lagrangian the same conserved quantity results from rotational invariance. The Lie algebra (g) of the symmetry vectors that leaves the action corresponding to the time-independent Lagrangian invariant is semi-simple. On the other hand, g is only a simple Lie algebra for the action characterised by the time-dependent Lagrangian.

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1. Introduction

It is well known that the formal description for the connection between symmetry properties and conserved quantities of a dynamical system is provided by Noether's theorem [1]. This theorem states that if a given differential equation representing the time evolution of some physical system follows from the variational principle, then a continuous symmetry transformation (point, contract or higher-order) that leaves the action functional invariant yields a conservation law. Thus studies in symmetries and conservation laws of a physical system using this theorem require that the equation of motion must follow from the action principle [2]. The object of the present work is to apply Noether's theorem to study the symmetries and conservation laws of dissipative systems. One of the simplest examples of dissipative systems is provided by a one-dimensional harmonic oscillator of natural frequency ω embedded in a viscous medium characterised by frictional coefficient γ. In this case the equation of motion is given by

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = 0\,, \qquad x = x(t)\,. \tag{1}$$

Lanczos [3] observed that the frictional forces which are not derivable from a scalar quantity known as the work function or potential are outside the realm of variational principle although the Newtonian scheme has no difficulty to accommodate them. From this viewpoint it appears that the time reversal symmetry violating equation in (1) does not have a Lagrangian representation to follow from the action principle. Fortunately, it is possible to proceed in two different directions to derive Lagrangians for (1). For instance, one can double the phase-space dimension to include the environment [4], or one may use an explicitly time-dependent Lagrangian [5] to account for irreversibility in time. The Lagrangians thus obtained will permit us to study symmetries of (1) by the use of Noether's theorem.

As early as 1931, Bateman [6] appending an additional degree of freedom, could bring (1) within the framework of action principle. The Bateman Lagrangian

$$L^{\mathrm{B}} = \dot{x}\dot{y} + \frac{\gamma}{2}\left(x\dot{y} - y\dot{x}\right) - \omega^2 xy \tag{2}$$

has a 'mirror image' equation

$$\ddot{y} - \gamma \dot{y} + \omega^2 y = 0 \tag{3}$$

in the associated generalised coordinate $y(t)$. Understandably, the complementary equation in (3) represents a physical system which absorbs energy dissipated in the first. Thus the oscillators in (1) and (3) taken together represent a conservative system such that L^{B} in (2) is time independent. The equation in (1) together with its mirror image (3) goes by the name Bateman dual system.

Both (1) and (3) are non self-adjoint. As a result neither of them satisfy the Helmholtz criterion [7] to have a Lagrangian representation. However, multiplying (1) by $e^{\gamma t}$ and (3) by $e^{-\gamma t}$ we can arrive at a system of self-adjoint differential equations such that

$$L^{\mathrm{S}} = e^{\gamma t}\left(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2\right) + e^{-\gamma t}\left(\tfrac{1}{2}\dot{y}^2 - \tfrac{1}{2}\omega^2 y^2\right) \tag{4}$$

is an admissible Lagrangian for the damped harmonic oscillator and its mirror image. The superscripts B and S on L are used to indicate that the Lagrangian in (2) is due to Bateman and that in (4) has been obtained after converting (1) and (3) into self-adjoint forms.

As opposed to L^{B}, L^{S} is an explicitly time-dependent Lagrangian. Moreover, these two Lagrangians are not connected by a gauge term. This implies that L^{S} and L^{B} represent alternative Lagrangian representations for the damped harmonic oscillator. Description of a physical system using alternative Lagrangians can have deep consequences on the further development of the theory. For example, one can come across ambiguities in the association of symmetries with constants of the motion [8]. We shall make use of the Lagrangians in (2) and (4) separately to study the Noether's symmetries of (1) and (3). This will give us an opportunity to demonstrate explicitly the ambiguities that arise for the association between symmetries and conservation laws in the presence of alternative Lagrangian representations for the damped harmonic oscillator.

In the next section we provide two slightly different formulations for the use of Noether's theorem to investigate the association between symmetries and conservation laws using L^{B} and L^{S}. In Section 3 we use results for our symmetry analysis. Finally, in Section 4 we summarise our outlook on the present work.

2. Noether's theorem

It is well known that the number of symmetries of a Lagrangian is fewer than the number of symmetries of the equation of motion. However, it is possible to identify the Lagrangian or variational symmetries from the Lie symmetries of the Newtonian equation [9]. This identification provides a useful route for the use of Noether's theorem to construct conserved quantities associated with the variational symmetries. On the other hand, one can perform an *ab initio* calculation for the Noetherian conserved quantities by dealing with the infinitesimal criterion for invariance of a variational problem under a group of transformation that map points in configuration space into their infinitesimal neighbourhood [10]. In this section, we give a brief outline of these approaches for the use of Noether's theorem.

2.1. Lie symmetries and conserved quantities

Let the differential equations in (1) and (3) admit a continuous group G with a generator

$$U = \xi \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y}\,, \tag{5}$$

where ξ and η_i $(i = 1, 2)$ are differentiable functions of x, y and t. For the present case the statement of Noether's theorem will be as follows.

If the variational integral $\int L(x,y,\dot{x},\dot{y},t)dt$ is invariant under the group G then

$$I=\xi L+(\eta_1-\xi\dot{x})\,\frac{\partial L}{\partial\dot{x}}+(\eta_2-\xi\dot{y})\,\frac{\partial L}{\partial\dot{y}} \tag{6}$$

provides a conserved quantity for the Euler–Lagrange equations as represented by the Bateman dual system. To apply this theorem one requires to make useful checks on the infinitesimal criterion

$$\dot{\xi}L+U^{(1)}L=0 \tag{7}$$

for the invariance of the variational integral. Here

$$U^{(1)}=U+\left(\dot{\eta}_1-\dot{\xi}\dot{x}\right)\frac{\partial}{\partial\dot{x}}+\left(\dot{\eta}_2-\dot{\xi}\dot{y}\right)\frac{\partial}{\partial\dot{y}} \tag{8}$$

stands for the first prolongation of U. The invariance condition in (7) can be replaced by a divergence condition

$$\dot{\xi}L+U^{(1)}L=\dot{f}\,,\qquad f=f(x,y,t)\,. \tag{9}$$

In this case, the Bateman dual system following from the action principle have, instead of the result given in (6), a conserved quantity [9]

$$I=\xi L+(\eta_1-\xi\dot{x})\,\frac{\partial L}{\partial\dot{x}}+(\eta_2-\xi\dot{y})\,\frac{\partial L}{\partial\dot{y}}-f\,. \tag{10}$$

Symmetries associated with the invariant quantities in (6) are often called (pure) variational symmetries while those corresponding to I in (10) are referred to as divergence symmetries.

2.2. Infinitesimal criterion for invariance and conserved quantities

The key element for the Noether symmetry analysis consists in studying the infinitesimal criterion for the invariance of a variational problem under a group of transformations that map 'points' in configuration space $(\vec{q},t)$ into their infinitesimal neighbourhood $(\vec{q}\,',t')$. Here $\vec{q}=\{q_i\},\ i=1,\ldots n$, stands for the set of generalised coordinates representing the dynamical system under consideration and, as usual, t is the time parameter. Formally, such point transformations are represented as

$$t' \;=\; t+\delta t\,,\qquad \delta t=\epsilon\xi(\vec{q},t)\,, \tag{11a}$$

$$q_i{}' \;=\; q_i+\delta q_i\,,\qquad \delta q_i=\epsilon\eta_i(\vec{q},t) \tag{11b}$$

with ϵ, an infinitesimal parameter. Given the transformation rule for q_i, the corresponding results for $\dot{q}_i$ and $\ddot{q}_i$ are given by [10]

$$\delta\dot{q}_i = \epsilon\left[\dot{\eta}_i(\vec{q},t) - \dot{\xi}(\vec{q},t)\dot{q}_i\right] \tag{12}$$

and

$$\delta\ddot{q}_i = \epsilon\left[\ddot{\eta}_i\left(\vec{q},t\right) - 2\dot{\xi}\left(\vec{q},t\right)\ddot{q}_i - \ddot{\xi}\left(\vec{q},t\right)\dot{q}_i\right]. \tag{13}$$

Considering the variation of an arbitrary analytic function $u(\vec{q},t)$ it is easy to prove that

$$\delta u = \epsilon U u(\vec{q},t) \tag{14}$$

with

$$U = \xi\left(\vec{q},t\right)\frac{\partial}{\partial t} + \sum_{i=1}^{n}\eta_i\left(\vec{q},t\right)\frac{\partial}{\partial q_i}. \tag{15}$$

The operator U is the generator of the infinitesimal point transformations in (11) and represents a vector field on $(\vec{q},t)$ since it assigns a tangent vector to each points within $(\vec{q},t)$. A similar consideration when applied to $v(\vec{q},\dot{\vec{q}},t)$ gives

$$\delta v = \epsilon U^{(1)} v\left(\vec{q},\dot{\vec{q}},t\right) \tag{16}$$

with

$$U^{(1)} = U + \sum_{i=1}^{n}\left(\dot{\eta}_i\left(\vec{q},t\right) - \dot{\xi}\left(\vec{q},t\right)\dot{q}_i\right)\frac{\partial}{\partial\dot{q}_i}. \tag{17}$$

To write the Noether's theorem we consider, among the general set of point transformations defined by (11), only those that leave the action $\int L\,dt$ invariant. In other words, we demand that

$$L\left(\vec{q}_i,\dot{\vec{q}}_i,t\right) \overset{!}{=} L'\left(\vec{q}_i^{\,\prime},\dot{\vec{q}}_i^{\,\prime},t'\right). \tag{18}$$

In order to satisfy the condition in (18), we allow the Lagrangian to change its functional form ($L \to L'$). The functional relation between L' and L may be expressed by introducing a gauge function $f(\vec{q},t)$ [10] such that

$$L'\left(\vec{q}_i^{\,\prime},\dot{\vec{q}}_i^{\,\prime},t'\right) = L\left(\vec{q}_i^{\,\prime},\dot{\vec{q}}_i^{\,\prime},t'\right) - \epsilon\,\frac{df\left(\vec{q},t\right)}{dt}. \tag{19}$$

From (18) and (19) we have

$$L\left(\vec{q}_i^{\,\prime},\dot{\vec{q}}_i^{\,\prime},t'\right)dt' = L\left(\vec{q}_i,\dot{\vec{q}}_i,t\right)dt + \epsilon\,\frac{df\left(\vec{q},t\right)}{dt}dt. \tag{20}$$

On the other hand, using L for v in (16) we have

$$L\left(\vec{q}_i^{\,\prime}, \dot{\vec{q}}_i^{\,\prime}, t'\right) = L\left(\vec{q}_i, \dot{\vec{q}}_i, t\right) + \epsilon\, U^{(1)} L\left(\vec{q}_i, \dot{\vec{q}}_i, t\right) . \tag{21}$$

From (20) and (21) it is easy to see that

$$\frac{df\left(\vec{q}, t\right)}{dt} = \dot{\xi} L + \xi \frac{\partial L}{\partial t} + \sum_{i=1}^{n} \left(\eta_i \frac{\partial L}{\partial q_i} + \left(\dot{\eta}_i - \dot{\xi} \dot{q}_i \right) \frac{\partial L}{\partial \dot{q}_i} \right) . \tag{22}$$

In writing (22) we have made use of the results in (15) and (17). We, therefore, infer that the action is invariant under those point transformations whose constituents ξ and η_i satisfy (22). The terms of (22) can be rearranged to write

$$\frac{dI}{dt} + \sum_{i=1}^{n} \left(\xi \dot{q}_i - \eta_i \right) \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) = 0 \tag{23}$$

with

$$I = \sum_{i=1}^{n} \left(\xi \dot{q}_i - \eta_i \right) \frac{\partial L}{\partial \dot{q}_i} - \xi L + f\left(\vec{q}, t\right) . \tag{24}$$

Along the trajectory of the system, the Euler–Lagrange equations hold good such that the second term in (23) is zero. Thus I given in (24) is a conserved quantity or a constant of the motion. The invariant given by (24) and the differential equations for the gauge function in (22) are commonly stated as the Noether's theorem.

3. Conserved quantities using Noether's theorem

Here we shall calculate the conserved quantities corresponding to variational symmetries implied by actions written in terms of L^{B} and L^{S}. For the time-independent Lagrangian L^{B} we shall work directly with the Lie symmetries of the Bateman dual system. On the other hand, we shall use the results of Subsection 2.2. to calculate the conserved densities that follow from the invariance of the action associated with L^{S}.

3.1. Time-independent Lagrangian L^{B}

The symmetry Lie algebra of the systems (1) and (3) is spanned by [11]

$$U_1 = e^{\gamma t} x \frac{\partial}{\partial y} , \tag{25a}$$

$$
\begin{aligned}
U_2 &= e^{-\gamma t} y \frac{\partial}{\partial x}\,, && \text{(25b)}\\
U_3 &= e^{\frac{\gamma}{2}t} x \cos\bar\omega t \frac{\partial}{\partial t} - \frac{1}{2} e^{\frac{\gamma}{2}t} x^2 \left(\gamma \cos\bar\omega t + 2\bar\omega \sin\bar\omega t\right) \frac{\partial}{\partial x} \\
&\quad + \frac{1}{2} e^{\frac{\gamma}{2}t} xy \left(\gamma \cos\bar\omega t + 2\bar\omega \sin\bar\omega t\right) \frac{\partial}{\partial y}\,, && \text{(25c)}\\
U_4 &= e^{\frac{\gamma}{2}t} x \sin\bar\omega t \frac{\partial}{\partial t} - \frac{1}{2} e^{\frac{\gamma}{2}t} x^2 \left(\gamma \sin\bar\omega t - 2\bar\omega \cos\bar\omega t\right) \frac{\partial}{\partial x} \\
&\quad + \frac{1}{2} e^{\frac{\gamma}{2}t} xy \left(\gamma \sin\bar\omega t + 2\bar\omega \cos\bar\omega t\right) \frac{\partial}{\partial y}\,, && \text{(25d)}\\
U_5 &= e^{\frac{\gamma}{2}t} \cos\bar\omega t \frac{\partial}{\partial y}\,, && \text{(25e)}\\
U_6 &= e^{\frac{\gamma}{2}t} \sin\bar\omega t \frac{\partial}{\partial y}\,, && \text{(25f)}\\
U_7 &= \frac{1}{2\omega^2} e^{-\frac{\gamma}{2}t} y \left(\gamma \cos\bar\omega t + 2\bar\omega \sin\bar\omega t\right) \frac{\partial}{\partial t} - \frac{1}{2\omega^2} e^{-\frac{\gamma}{2}t} xy \\
&\quad \times \left\{ \left(\gamma^2 - 2\omega^2\right) \cos\bar\omega t + 2\gamma\bar\omega \sin\bar\omega t \right\} \frac{\partial}{\partial x} + e^{-\frac{\gamma}{2}t} y^2 \cos\bar\omega t \frac{\partial}{\partial y}\,, && \text{(25g)}\\
U_8 &= \frac{1}{2\omega^2} e^{-\frac{\gamma}{2}t} y \left(\gamma \sin\bar\omega t - 2\bar\omega \cos\bar\omega t\right) \frac{\partial}{\partial t} - \frac{1}{2\omega^2} e^{-\frac{\gamma}{2}t} xy \\
&\quad \times \left\{ \left(\gamma^2 - 2\omega^2\right) \sin\bar\omega t - 2\gamma\bar\omega \cos\bar\omega t \right\} \frac{\partial}{\partial x} + e^{-\frac{\gamma}{2}t} y^2 \cos\bar\omega t \frac{\partial}{\partial y}\,, && \text{(25h)}\\
U_9 &= e^{-\frac{\gamma}{2}t} \cos\bar\omega t \frac{\partial}{\partial x}\,, && \text{(25i)}\\
U_{10} &= e^{-\frac{\gamma}{2}t} \sin\bar\omega t \frac{\partial}{\partial x}\,, && \text{(25j)}\\
U_{11} &= -\frac{1}{4\bar\omega^2} \left(\gamma \frac{\partial}{\partial t} - 2\omega^2 x \frac{\partial}{\partial x} + 2\omega^2 y \frac{\partial}{\partial y} \right)\,, && \text{(25k)}\\
U_{12} &= \frac{1}{4\bar\omega^2} \left(-2 \frac{\partial}{\partial t} + \gamma x \frac{\partial}{\partial x} - \gamma y \frac{\partial}{\partial y} \right)\,, && \text{(25l)}\\
U_{13} &= \frac{1}{2\omega^2} \left(\gamma \cos 2\bar\omega t + 2\bar\omega \sin 2\bar\omega t\right) \frac{\partial}{\partial t} - \frac{x}{2\omega^2} \left\{ \left(\gamma^2 - 2\omega^2\right) \cos 2\bar\omega t \right. \\
&\quad \left. + 2\gamma\bar\omega \sin 2\bar\omega t \right\} \frac{\partial}{\partial x} + y \cos 2\bar\omega t \frac{\partial}{\partial y} && \text{(25m)}
\end{aligned}
$$

and

$$
\begin{aligned}
U_{14} &= \frac{1}{2\omega^2} \left(-\gamma \sin 2\bar\omega t + 2\bar\omega \cos 2\bar\omega t\right) \frac{\partial}{\partial t} + \frac{x}{2\omega^2} \left\{ \left(\gamma^2 - 2\omega^2\right) \sin 2\bar\omega t \right. \\
&\quad \left. - 2\gamma\bar\omega \sin 2\bar\omega t \right\} \frac{\partial}{\partial x} - y \sin 2\bar\omega t \frac{\partial}{\partial y}\,. && \text{(25n)}
\end{aligned}
$$

Here $\bar{\omega} = (\omega^2 - \frac{\gamma^2}{4})^{\frac{1}{2}}$. By evaluating the expression $(\dot{\xi} L + U^{(1)} L)$ with the time-independent Lagrangian in (2) we have found that U_1–U_4, U_7, and U_8 do satisfy neither the invariance test in (7) nor the divergence condition in (9). Thus we are left with only eight vector fields, namely, U_5, U_6 and U_9–U_{14} of which U_5, U_6, U_9, U_{10}, U_{13} and U_{14} stand for divergence symmetries while U_{11} and U_{12} are pure variational symmetries. In the following we present results for conserved quantities associated with these symmetries. We have

$$I^0_{U_5} = e^{\frac{\gamma}{2} t}\left(\dot{x}\cos\bar{\omega}t + \frac{\gamma}{2} x\cos\bar{\omega}t + \bar{\omega}x\sin\bar{\omega}t\right), \tag{26a}$$

$$I^0_{U_6} = e^{\frac{\gamma}{2} t}\left(\dot{x}\sin\bar{\omega}t + \frac{\gamma}{2} x\sin\bar{\omega}t - \bar{\omega}x\cos\bar{\omega}t\right), \tag{26b}$$

$$I^0_{U_9} = e^{-\frac{\gamma}{2} t}\left(\dot{y}\cos\bar{\omega}t - \frac{\gamma}{2} y\cos\bar{\omega}t + \bar{\omega}y\sin\bar{\omega}t\right), \tag{26c}$$

$$I^0_{U_{10}} = e^{-\frac{\gamma}{2} t}\left(\dot{y}\sin\bar{\omega}t - \frac{\gamma}{2} y\sin\bar{\omega}t - \bar{\omega}y\cos\bar{\omega}t\right), \tag{26d}$$

$$I^0_{U_{11}} = -\frac{1}{4\bar{\omega}^2}\left[\gamma\dot{x}\dot{y} + 2\omega^2\left(x\dot{y} - y\dot{x}\right) - \gamma\omega^2 xy\right], \tag{26e}$$

$$I^0_{U_{12}} = \frac{1}{4\bar{\omega}^2}\left[2\dot{x}\dot{y} - \frac{1}{2}\gamma^2\left(x\dot{y} - y\dot{x}\right) + 2\omega^2 xy\right], \tag{26f}$$

$$\begin{aligned} I^0_{U_{13}} = & -\frac{\bar{\omega}}{\omega^2}\left(\dot{x}\dot{y} + \gamma x\dot{y} - \omega^2 xy\right)\sin 2\bar{\omega}t \\ & -\left[\frac{\gamma}{2\omega^2}\left(\dot{x}\dot{y} + \gamma x\dot{y} - \omega^2 xy\right) + \dot{x}y + x\dot{y}\right]\cos 2\bar{\omega}t \end{aligned} \tag{26g}$$

and

$$\begin{aligned} I^0_{U_{14}} = & -\frac{\bar{\omega}}{\omega^2}\left(\dot{x}\dot{y} + \gamma x\dot{y} - \omega^2 xy\right)\cos 2\bar{\omega}t \\ & +\left[\frac{\gamma}{2\omega^2}\left(\dot{x}\dot{y} + \gamma x\dot{y} - \omega^2 xy\right) - \dot{x}y - x\dot{y}\right]\sin 2\bar{\omega}t\,. \end{aligned} \tag{26h}$$

Here we have used the superscript 0 to denote conserved quantities related to the time-independent Lagrangian. Understandably, the subscripts U_i refer to the corresponding symmetries. It is of interest to note that the symmetry vector U_{11} becomes a squeezing operator for $\gamma = 0$ and corresponding conserved quantity in this limit is the angular momentum. For $\gamma = 0$, U_{12} gives the time translation operator, the corresponding conserved quantity being the total energy of the two uncoupled harmonic oscillators. In view of this, $I^0_{U_{12}}$ in (26f) may be called the Jacobi integral of the system represented by (1) and (3).

We shall now study the group properties of U_i. Since the vector fields U_i are the generators of symmetries (variational) we shall, for clarity of presentation, use $G_1 = U_5$, $G_2 = U_6$, $G_3 = U_9$, $G_4 = U_{10}$, $G_5 = U_{11}$, $G_6 = U_{12}$, $G_7 = U_{13}$, and $G_8 = U_{14}$. To satisfy the requirement of Lie algebra the generators G_i will be expected to obey the closure relation $[G_i, G_j] = C_{ij}^k G_k$ which defines the algebra. Here C_{ij}^ks are the structure constants and $[G_i, G_j]$ stands for the commutation relations of the symmetry generators. In the following table we display the results for the closure relations.

TABLE I

Commutation relation for the generators in (25). Each element G_{ij} in the table being represented by $G_{ij} = [G_i\,,\, G_j]$.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
G_1	0	0	0	0	$-\frac{1}{2}G_1 - cG_2$	$-\frac{1}{2\bar{\omega}}G_2$	$aG_1 - bG_2$	$-aG_2 - bG_1$
G_2	0	0	0	0	$-\frac{1}{2}G_2 + cG_1$	$\frac{1}{2\bar{\omega}}G_2$	$-aG_2 - bG_1$	$-aG_1 + bG_2$
G_3	0	0	0	0	$\frac{1}{2}G_3 - cG_4$	$\frac{1}{2\bar{\omega}}G_4$	$aG_3 - bG_4$	$-aG_4 - bG_3$
G_4	0	0	0	0	$\frac{1}{2}G_4 + cG_3$	$\frac{1}{2\bar{\omega}}G_3$	$-aG_4 - bG_3$	$-aG_3 + bG_4$
G_5	$\frac{1}{2}G_1 + cG_2$	$\frac{1}{2}G_2 - cG_1$	$-\frac{1}{2}G_3 + cG_4$	$-\frac{1}{2}G_4 - cG_3$	0	0	$-2cG_8$	$2cG_7$
G_6	$\frac{1}{2\bar{\omega}}G_2$	$-\frac{1}{2\bar{\omega}}G_1$	$-\frac{1}{2\bar{\omega}}G_4$	$-\frac{1}{2\bar{\omega}}G_3$	0	0	$-\frac{1}{\bar{\omega}}G_8$	$\frac{1}{\bar{\omega}}G_7$
G_7	$-aG_1 + bG_2$	$aG_2 + bG_1$	$-aG_3 + bG_4$	$aG_4 + bG_3$	$2cG_8$	$\frac{1}{\bar{\omega}}G_8$	0	$4\bar{\omega}aG_8$
G_8	$aG_2 + bG_1$	$aG_1 - bG_2$	$aG_4 + bG_3$	$aG_3 - bG_4$	$-2cG_7$	$-\frac{1}{\bar{\omega}}G_7$	$-4\bar{\omega}aG_8$	0

In Table I, $a = \frac{\bar{\omega}^2}{\omega^2}$, $b = \frac{\gamma\bar{\omega}}{\omega^2}$ and $c = \frac{\gamma}{4\bar{\omega}}$. From this table it is clear that the algebra is closed. To classify the algebra we construct the metric tensor (Killing form) $g_{ij} = C_{ik}^m C_{jm}^k$ and find that the determinant of g_{ij} is non-vanishing. Thus according to Cartan's criterion, Lie algebra is semi-simple [12].

Interestingly, using the conserved quantities $I_{U_5}^0$, $I_{U_6}^0$, $I_{U_9}^0$ and $I_{U_{10}}^0$ associated with linearly independent vector fields we can get the solutions of the damped oscillators

$$x = \frac{e^{-\frac{\gamma}{2}t}}{\bar{\omega}}\left(I_{U_5}^0 \sin\bar{\omega}t - I_{U_6}^0 \cos\bar{\omega}t\right) \tag{27a}$$

and

$$y = \frac{e^{\frac{\gamma}{2}t}}{\bar{\omega}}\left(I_{U_9}^0 \sin\bar{\omega}t - I_{U_{10}}^0 \cos\bar{\omega}t\right)\ . \tag{27b}$$

Since x and y represents the general solution of the damped harmonic oscillators in (1) and (3), the system is completely specified by the four-parameter Abelian symmetry group generated by G_1, G_2, G_3 and G_4.

3.2. Time-dependent alternative Lagrangian L^{S}

We shall make use of (15) and (24) to derive the symmetries and conservation laws that follow from the invariance of the action written in terms of the time-dependent Lagrangian in (4). In this case $\vec{q} \equiv (x, y)$ so that

$$\begin{aligned} I \;=\; & f(x,y,t) - \xi\left[e^{\gamma t}\left(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2\right)\right. \\ & + e^{-\gamma t}\left(\tfrac{1}{2}\dot{y}^2 - \tfrac{1}{2}\omega^2 y^2\right) + \left(\xi\dot{x} - \eta_1\right)\dot{x}e^{\gamma t} \\ & \left. + \left(\xi\dot{y} - \eta_2\right)\dot{y}e^{-\gamma t}\right]. \end{aligned} \tag{28}$$

In (28)

$$\xi = \xi(x,y,t) \qquad \text{and} \qquad \eta_i = \eta_i(x,y,t)\,. \tag{29}$$

From $\frac{dI}{dt} = 0$, we get

$$\begin{aligned} & f_t + \dot{x}f_x + \dot{y}f_y - \tfrac{1}{2}\dot{x}^2 e^{\gamma t}\xi_t + \tfrac{1}{2}\omega^2 x^2 e^{\gamma t}\xi_t \\ & -\tfrac{1}{2}\dot{y}^2 e^{-\gamma t}\xi_t + \tfrac{1}{2}\omega^2 y^2 e^{-\gamma t}\xi_t - \tfrac{1}{2}\dot{x}^3 e^{\gamma t}\xi_x \\ & +\tfrac{1}{2}\omega^2 x^2 \dot{x} e^{\gamma t}\xi_x - \tfrac{1}{2}\dot{x}\dot{y}^2 e^{-\gamma t}\xi_x + \tfrac{1}{2}\omega^2 \dot{x} y^2 e^{-\gamma t}\xi_x \\ & -\tfrac{1}{2}\dot{x}^2\dot{y} e^{\gamma t}\xi_y + \tfrac{1}{2}\omega^2 x^2 \dot{y} e^{\gamma t}\xi_y - \tfrac{1}{2}\dot{y}^3 e^{-\gamma t}\xi_y \\ & +\tfrac{1}{2}\omega^2 y^2 \dot{y} e^{-\gamma t}\xi_y - \frac{\gamma}{2}\dot{x}^2 e^{\gamma t}\xi + \frac{\gamma}{2}\omega^2 x^2 e^{\gamma t}\xi \\ & +\frac{\gamma}{2}\dot{y}^2 e^{-\gamma t}\xi - \frac{\gamma}{2}\omega^2 y^2 e^{-\gamma t}\xi + \gamma\dot{x}^2 e^{\gamma t}\xi + \omega^2 x\dot{x} e^{\gamma t}\xi \\ & +\omega^2 x\dot{x} e^{\gamma t}\xi - \gamma\dot{y}^2 e^{\gamma t}\xi + \omega^2 y\dot{y} e^{-\gamma t}\xi + \omega^2 y\dot{y} e^{-\gamma t}\xi \\ & +\dot{x}^2 e^{\gamma t}\xi_t + \dot{x}^3 e^{\gamma t}\xi_x + \dot{x}^2\dot{y} e^{\gamma t}\xi_y - \gamma\dot{x}^2 e^{\gamma t}\xi \\ & -\omega^2 x\dot{x} e^{\gamma t}\xi - \dot{x} e^{\gamma t}\eta_{1t} - \dot{x}^2 e^{\gamma t}\eta_{1x} - \dot{x}\dot{y} e^{\gamma t}\eta_{1y} \\ & -\gamma\dot{x}^2 e^{\gamma t}\xi - \omega^2 x\dot{x} e^{\gamma t}\xi + \gamma\dot{x} e^{\gamma t}\eta_1 + \omega^2 x e^{\gamma t}\eta_1 \\ & +\gamma\dot{x}^2 e^{\gamma t}\xi - \gamma\dot{x} e^{\gamma t}\eta_1 + \dot{y}^2 e^{-\gamma t}\xi_t + \dot{x}\dot{y}^2 e^{-\gamma t}\xi_x \\ & +\dot{y}^3 e^{-\gamma t}\xi_y + \gamma\dot{y}^2 e^{-\gamma t}\xi + \omega^2 y\dot{y} e^{-\gamma t}\xi - \dot{y} e^{-\gamma t}\eta_{2t} \\ & -\dot{x}\dot{y} e^{-\gamma t}\eta_{2x} - \dot{y}^2 e^{-\gamma t}\eta_{2y} + \gamma\dot{y}^2 e^{-\gamma t}\xi - \omega^2 y\dot{y} e^{-\gamma t}\xi \\ & -\gamma\dot{y} e^{-\gamma t}\eta_2 + \omega^2 y e^{-\gamma t}\eta_2 - \gamma\dot{y}^2 e^{-\gamma t}\xi \\ & +\gamma\dot{y} e^{-\gamma t}\eta_2 \;=\; 0\,. \end{aligned} \tag{30}$$

In writing (30), we have used the equations of the Bateman dual system. Equation (30) can be globally satisfied for any particular choice of the velocities provided the sum of velocity-independent terms, the coefficients of linear, quadratic and cubic terms in $\dot{x}$ and $\dot{y}$ vanish separately. Following this viewpoint we write

$$\begin{aligned}
\dot{x}^0\,,\dot{y}^0 \;&:\; f_t+\frac{\omega^2}{2}\left(x^2e^{\gamma t}+y^2e^{-\gamma t}\right)\xi_t+\frac{\gamma\omega^2}{2}\\
&\quad\times\left(x^2e^{\gamma t}-y^2e^{-\gamma t}\right)\xi+\omega^2\eta_1xe^{\gamma t}+\omega^2\eta_2xe^{-\gamma t}=0\,, && \text{(31a)}\\
\dot{x} \;&:\; f_x+\tfrac{1}{2}\left(\omega^2x^2e^{\gamma t}+\omega^2y^2e^{-\gamma t}\right)\xi_x-e^{\gamma t}\eta_{1t}=0\,, && \text{(31b)}\\
\dot{y} \;&:\; f_y+\tfrac{1}{2}\left(\omega^2x^2e^{\gamma t}+\omega^2y^2e^{-\gamma t}\right)\xi_y-e^{-\gamma t}\eta_{2t}=0\,, && \text{(31c)}\\
\dot{x}^2 \;&:\; \tfrac{1}{2}e^{\gamma t}\xi_t-\frac{\gamma}{2}e^{\gamma t}\xi-e^{\gamma t}\eta_{1x}=0\,, && \text{(31d)}\\
\dot{y}^2 \;&:\; \tfrac{1}{2}e^{-\gamma t}\xi_t-\frac{\gamma}{2}e^{-\gamma t}\xi-e^{-\gamma t}\eta_{2y}=0\,, && \text{(31e)}\\
\dot{x}\dot{y} \;&:\; e^{\gamma t}\eta_1y+e^{-\gamma t}\eta_2x=0\,, && \text{(31f)}\\
\dot{x}\dot{y}^2 \;&:\; \tfrac{1}{2}e^{-\gamma t}\xi_x=0\,, && \text{(31g)}\\
\dot{x}^2\dot{y} \;&:\; \tfrac{1}{2}e^{\gamma t}\xi_y=0\,, && \text{(31h)}\\
\dot{x}^3 \;&:\; \tfrac{1}{2}e^{\gamma t}\xi_x=0 && \text{(31i)}
\end{aligned}$$

and

$$\dot{y}^3 \;:\; \tfrac{1}{2}e^{-\gamma t}\xi_y=0\,. \tag{31j}$$

Equation in (31a) signifies that we have equated the sum of $\dot{x}$ or $\dot{y}$-independent terms to zero while (31b)–(31j) have been obtained by equating the sum of the coefficients of $\dot{x}$, $\dot{y}$, $\dot{x}^2$ *etc.* to zero. From (31g)–(31j) we see that ξ is not a function of x and y. Thus *e.g.*,

$$\xi(x,\,y,\,t)\equiv\xi(t)=\beta(t)\,. \tag{32}$$

Sum of (31d), (31e) and (31f) can be written in a compact form

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\left[\tfrac{1}{2}e^{(-1)^i\gamma t}\left(\dot{\beta}+(-1)^i\gamma\beta\right)\delta_{ij}-e^{(-1)^j\gamma t}\tfrac{\partial\eta_i}{\partial x_j}\right]=0$$
$$\text{with}\qquad x_1=x\,,x_2=y\,. \tag{33}$$

Equation (33) will be globally satisfied provided $e^{(-1)^j\gamma t}\frac{\partial\eta_i}{\partial x_j}$ cancels the other term in the squared bracket up to a constant element a_{ij} of an antisymmetric matrix (a_{ij}) . Thus we write

$$\frac{\partial\eta_i}{\partial x_j}=\tfrac{1}{2}e^{\left((-1)^i-(-1)^j\right)\gamma t}\left[\dot{\beta}+(-1)^i\gamma\beta\right]\delta_{ij}+a_{ij}e^{(-1)^j\gamma t}\,. \tag{34}$$

On integration (34) reads

$$\eta_i=\tfrac{1}{2}\left(\dot{\beta}+(-1)^i\gamma\beta\right)x_i+\sum_{i=1}^{2}a_{ij}e^{-(-1)^j\gamma t}x_j+\psi_i(t)\,, \tag{35}$$

where $\psi_i(t)$ is a constant of integration. In view of (32), we can write (31a), (31b) and (31c) as

$$\begin{aligned}&\frac{\partial f}{\partial t}+\frac{\omega^2}{2}\left(x^2e^{\gamma t}+y^2e^{-\gamma t}\right)\dot{\beta}+\frac{\gamma\omega^2}{2}\left(x^2e^{\gamma t}-y^2e^{-\gamma t}\right)\beta\\&+\omega^2\eta_1xe^{\gamma t}+\omega^2\eta_2ye^{-\gamma t}=0\,,\end{aligned}\tag{36}$$

$$e^{-\gamma t}\frac{\partial f}{\partial x}-\frac{\partial\eta_1}{\partial t}=0\tag{37}$$

and

$$e^{\gamma t}\frac{\partial f}{\partial y}-\frac{\partial\eta_2}{\partial t}=0\,.\tag{38}$$

For η_i in (35), we see that

$$\begin{aligned}f\;=\;&\tfrac14\left(\ddot{\beta}-\gamma\dot{\beta}\right)x^2e^{\gamma t}+\tfrac14\left(\ddot{\beta}+\gamma\dot{\beta}\right)y^2e^{-\gamma t}-\gamma a_{12}xy\\&+\dot{\psi}_1e^{\gamma t}x+\dot{\psi}_2e^{-\gamma t}y\end{aligned}\tag{39}$$

represents a general solution of (37) and (38). Using the expressions for η_i and f from (35) and (39) in (24) we obtain the invariant I in the form

$$I=I_\beta+I_{\psi_1}+I_{\psi_2}+I_a\,,\tag{40}$$

where

$$\begin{aligned}I_\beta\;=\;&\tfrac14\left(x^2e^{\gamma t}+y^2e^{-\gamma t}\right)\ddot{\beta}-\frac{\gamma}{4}\left(x^2e^{\gamma t}-y^2e^{-\gamma t}\right)\dot{\beta}-\tfrac12\left(x\dot{x}e^{\gamma t}+y\dot{y}e^{-\gamma t}\right)\dot{\beta}\\&+\tfrac12\left\{e^{\gamma t}\left(\dot{x}^2+\omega^2x^2\right)+e^{-\gamma t}\left(\dot{y}^2+\omega^2\right)\right\}\beta+\gamma\left(x\dot{x}e^{\gamma t}-y\dot{y}e^{-\gamma t}\right)\beta\,,\end{aligned}\tag{41a}$$

$$I_{\psi_i}\;=\;\dot{\psi}_ie^{\gamma t}x_i-\psi_ie^{\gamma t}x_i\,,\qquad i=1,2\tag{41b}$$

and

$$I_a\;=\;-a_{12}\dot{x}y-a_{21}x\dot{y}-a_{12}\gamma xy\,.\tag{41c}$$

In writing (40) we also used (4) and (32). Each of the Is in (40) is expected to form a separate constant. This can be seen as follows.

Substituting the values of η_i and f in (36) we get

$$J_\beta+J_{\psi_1}+J_{\psi_2}+J_a=0\,,\tag{42}$$

where

$$J_\beta\;=\;\tfrac14\left(x^2e^{\gamma t}+y^2e^{-\gamma t}\right)\left(\dddot{\beta}+\left(4\omega^2-\gamma^2\right)\dot{\beta}\right)\,,\tag{43a}$$

$$J_{\psi_i}\;=\;\left(\ddot{\psi}_i-(-1)^i\gamma\dot{\psi}_i+\omega^2\psi_i\right)x_ie^{-(-1)^i\gamma t}\,,\qquad i=1,\,2\tag{43b}$$

and

$$J_a\;=\;\omega^2\left(a_{12}xy+a_{21}xy\right)\,.\tag{43c}$$

It is easy to verify that

$$\int J_\beta dt \;=\; I_\beta \tag{44a}$$

and

$$\int J_{\psi_i} dt \;=\; I_{\psi_i}\,, \qquad i = 1, 2\,. \tag{44b}$$

Equations (44a) and (44b) verify our conjecture. The matrix (a_{ij}) is antisymmetric. Therefore, $a_{11} = a_{22} = 0$ and $a_{12} = -a_{21}$. Thus for the two dimensional case under consideration (a_{ij}) cannot contain more than one independent element. In view of this (43c) becomes identically equal to zero and (41c) gives

$$I_a = x\dot{y} - \dot{x}y - \gamma xy \qquad \text{for} \qquad a_{12} = 1\,. \tag{45}$$

The generator of the infinitesimal transformations leading to the conserved quantity in (45) as obtained from (15) reads

$$U_a = e^{-\gamma t} y \frac{\partial}{\partial x} - e^{\gamma t} x \frac{\partial}{\partial y}\,. \tag{46}$$

Clearly, in the case of no damping ($\gamma = 0$), U_a becomes the well known rotation operator of two uncoupled harmonic oscillators with the angular momentum $x\dot{y} - \dot{x}y$ as the conserved quantity.

A similar treatment also applies for I_β and I_{ψ_i}. We first need to calculate the special values of $\beta(t)$ and $\psi_i(t)$ from

$$J_\beta \;=\; 0 \tag{47a}$$

and

$$J_{\psi_i} \;=\; 0\,. \tag{47b}$$

Equations (47a) and (47b) give

$$\beta \;=\; 1 \qquad \text{and} \qquad \beta^{\pm} = e^{\pm 2i\bar{\omega}t}\,, \qquad \bar{\omega} = \sqrt{\omega^2 - \frac{\gamma^2}{4}}\,, \tag{48a}$$

and

$$\psi_i^{\pm} \;=\; e^{(-1)^i \frac{\gamma}{2} \pm i\bar{\omega}t}\,, \qquad i = 1, 2\,. \tag{48b}$$

Equation (48) clearly shows that we are interested in the underdamped oscillator. For $\beta = 1$ the conserved quantity

$$I_{\beta=1} = \tfrac{1}{2}\Big\{ e^{\gamma t}\left(\dot{x}^2 + \omega^2 x^2\right) + e^{-\gamma t}\left(\dot{y}^2 + \omega^2 y^2\right) \Big\} + \gamma\left(x\dot{x}e^{\gamma t} - y\dot{y}e^{-\gamma t}\right) \tag{49}$$

has the associated generator

$$U_{\beta=1} = \frac{\partial}{\partial t} - \frac{\gamma}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) . \tag{50}$$

The generator U_β in (50) consists of two parts. The first one is the usual time translation operator. The second part arises due to damping and tends to reduce the temporal operator by the action of a squeezing operator $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. For $\gamma = 0$, $I_{\beta=1}$ represents the total energy of two uncoupled harmonic oscillators. For finite values of γ, however, $I_{\beta=1}$ stands for the energy function of the system plus γ times a conserved quantity arising from squeezing. For brevity, we shall call this a Jacobi's integral [13]. Results similar to (45), (46) and (49), (50) for $\beta^{\pm}$, $\psi_i^{\pm}$ are given bellow.

For $\beta^+ = e^{+2i\bar{\omega}t}$, the invariant I_β gives rise to two real invariants

$$\begin{aligned} I_{\beta^1} &= \mathrm{Re}\, I_{\beta^+=e^{+2i\bar{\omega}t}} \\ &= \tfrac{1}{2}\Big\{e^{\gamma t}\left(\dot{x}^2 - \omega^2 x^2\right) + e^{-\gamma t}\left(\dot{y}^2 - \omega^2 y^2\right)\Big\}\cos 2\bar{\omega}t \\ &\quad + \tfrac{1}{2}\left(\gamma^2 x^2 e^{\gamma t} - \gamma^2 y^2 e^{-\gamma t} + \gamma x\dot{x}e^{\gamma t} - \gamma y\dot{y}e^{-\gamma t}\right)\cos 2\bar{\omega}t \\ &\quad + \bar{\omega}\left(\frac{\gamma}{2}x^2 e^{\gamma t} - \frac{\gamma}{2}y^2 e^{-\gamma t} + x\dot{x}e^{\gamma t} - y\dot{y}e^{-\gamma t}\right)\sin 2\bar{\omega}t \end{aligned} \tag{51}$$

and

$$\begin{aligned} I_{\beta^2} &= \mathrm{Im}\, I_{\beta^+=e^{+2i\bar{\omega}t}} \\ &= \tfrac{1}{2}\Big\{e^{\gamma t}\left(\dot{x}^2 - \omega^2 x^2\right) + e^{-\gamma t}\left(\dot{y}^2 - \omega^2 y^2\right)\Big\}\sin 2\bar{\omega}t \\ &\quad + \tfrac{1}{2}\left(\gamma^2 x^2 e^{\gamma t} - \gamma^2 y^2 e^{-\gamma t} + \gamma x\dot{x}e^{\gamma t} - \gamma y\dot{y}e^{-\gamma t}\right)\sin 2\bar{\omega}t \\ &\quad - \bar{\omega}\left(\frac{\gamma}{2}x^2 e^{\gamma t} - \frac{\gamma}{2}y^2 e^{-\gamma t} + x\dot{x}e^{\gamma t} - y\dot{y}e^{-\gamma t}\right)\cos 2\bar{\omega}t\,. \end{aligned} \tag{52}$$

The generators of I_{β^1} and I_{β^2} as found from (15) are given by

$$\begin{aligned} U_{\beta^1} &= \mathrm{Re}\, U_{\beta^+=e^{+2i\bar{\omega}t}} \\ &= \cos 2\bar{\omega}t\left\{\frac{\partial}{\partial t} - \frac{\gamma}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\right\} - \bar{\omega}\,\sin 2\bar{\omega}t\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \end{aligned} \tag{53}$$

and

$$\begin{aligned} U_{\beta^2} &= \mathrm{Im}\, U_{\beta^+=e^{+2i\bar{\omega}t}} \\ &= \sin 2\bar{\omega}t\left\{\frac{\partial}{\partial t} - \frac{\gamma}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\right\} + \bar{\omega}\,\cos 2\bar{\omega}t\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) . \end{aligned} \tag{54}$$

For $\beta^- =^{-2i\bar{\omega}t}$, the results similar to those in (51), (52) and (53), (54) read

$$\begin{aligned} I_{\beta^3} &= \operatorname{Re} I_{\beta^-=e^{-2i\bar{\omega}t}} = I_{\beta^1}\,, & (55)\\ I_{\beta^4} &= \operatorname{Im} I_{\beta^-=e^{-2i\bar{\omega}t}} = -I_{\beta^2} & (56)\end{aligned}$$

and

$$\begin{aligned} U_{\beta^3} &= \operatorname{Re} U_{\beta^-=e^{-2i\bar{\omega}t}} = U_{\beta^1}\,, & (57)\\ U_{\beta^4} &= \operatorname{Im} U_{\beta^-=e^{-2i\bar{\omega}t}} = -U_{\beta^2}\,. & (58)\end{aligned}$$

From (15), (41b) and (48b) we get the following invariants and generators.

$$\begin{aligned}
I_{\Psi_1^1} &= \operatorname{Re} I_{\psi_1^+} = e^{\left(-\frac{\gamma}{2}+i\bar{\omega}\right)t} \\
&= -\left(\frac{\gamma}{2}xe^{\frac{\gamma}{2}t} + \dot{x}e^{\frac{\gamma}{2}t}\right)\,\cos\bar{\omega}t - \bar{\omega}xe^{\frac{\gamma}{2}t}\sin\bar{\omega}t\,, & (59)\\
I_{\Psi_1^2} &= \operatorname{Im} I_{\psi_1^+} = e^{\left(-\frac{\gamma}{2}+i\bar{\omega}\right)t} \\
&= -\left(\frac{\gamma}{2}xe^{\frac{\gamma}{2}t} + \dot{x}e^{\frac{\gamma}{2}t}\right)\,\sin\bar{\omega}t + \bar{\omega}xe^{\frac{\gamma}{2}t}\cos\bar{\omega}t\,, & (60)\\
U_{\Psi_1^1} &= \operatorname{Re} U_{\psi_1^+=e^{(-\frac{\gamma}{2}+i\bar{\omega})t}} = e^{-\frac{\gamma}{2}t}\cos\bar{\omega}t\,\frac{\partial}{\partial x}\,, & (61)\\
U_{\Psi_1^2} &= \operatorname{Im} U_{\psi_1^+=e^{(-\frac{\gamma}{2}+i\bar{\omega})t}} = e^{-\frac{\gamma}{2}t}\sin\bar{\omega}t\,\frac{\partial}{\partial x}\,, & (62)\\
I_{\Psi_1^3} &= \operatorname{Re} I_{\psi_1^-=e^{(-\frac{\gamma}{2}-i\bar{\omega})t}} = I_{\Psi_1^1}\,, & (63)\\
I_{\Psi_1^4} &= \operatorname{Im} I_{\psi_1^-=e^{(-\frac{\gamma}{2}-i\bar{\omega})t}} = -I_{\Psi_1^2}\,, & (64)\\
U_{\Psi_1^3} &= \operatorname{Re} U_{\psi_1^-=e^{(-\frac{\gamma}{2}-i\bar{\omega})t}} = U_{\Psi_1^1}\,, & (65)\\
U_{\Psi_1^4} &= \operatorname{Im} U_{\psi_1^-=e^{(-\frac{\gamma}{2}-i\bar{\omega})t}} = -U_{\Psi_1^2}\,, & (66)\\
I_{\Psi_2^1} &= \operatorname{Re} I_{\psi_2^+=e^{(\frac{\gamma}{2}+i\bar{\omega})t}} \\
&= \left(\frac{\gamma}{2}ye^{-\frac{\gamma}{2}t} - \dot{y}e^{-\frac{\gamma}{2}t}\right)\,\cos\bar{\omega}t - \bar{\omega}ye^{-\frac{\gamma}{2}t}\sin\bar{\omega}t\,, & (67)\\
I_{\Psi_2^2} &= \operatorname{Im} I_{\psi_2^+=e^{(\frac{\gamma}{2}+i\bar{\omega})t}} \\
&= \left(\frac{\gamma}{2}ye^{-\frac{\gamma}{2}t} - \dot{y}e^{-\frac{\gamma}{2}t}\right)\,\sin\bar{\omega}t + \bar{\omega}ye^{-\frac{\gamma}{2}t}\,\cos\bar{\omega}t\,, & (68)\\
U_{\Psi_2^1} &= \operatorname{Re} U_{\psi_2^+=e^{(\frac{\gamma}{2}+i\bar{\omega})}} = e^{\frac{\gamma}{2}t}\cos\bar{\omega}t\,\frac{\partial}{\partial y}\,, & (69)\\
U_{\Psi_2^2} &= \operatorname{Im} U_{\psi_2^+=e^{(\frac{\gamma}{2}+i\bar{\omega})}} = e^{\frac{\gamma}{2}t}\sin\bar{\omega}t\,\frac{\partial}{\partial y}\,, & (70)\\
I_{\Psi_2^3} &= \operatorname{Re} I_{\psi_2^-=e^{(\frac{\gamma}{2}-i\bar{\omega})t}} = I_{\Psi_2^1}\,, & (71)
\end{aligned}$$

$$I_{\Psi_2^4} \;=\; \operatorname{Im} I_{\psi_2^- = e^{(\frac{\gamma}{2} - i\bar{\omega})t}} = -I_{\Psi_2^2}\,, \tag{72}$$

$$U_{\Psi_2^3} \;=\; \operatorname{Re} U_{\psi_2^- = e^{(\frac{\gamma}{2} - i\bar{\omega})t}} = U_{\Psi_2^1} \tag{73}$$

and

$$U_{\Psi_2^4} \;=\; \operatorname{Im} U_{\psi_2^- = e^{(\frac{\gamma}{2} - i\bar{\omega})t}} = -U_{\Psi_2^2}\,. \tag{74}$$

In the above, the odd and even superscripts on β and ψ_i refer to real and imaginary part of the invariants and the generators as the case may be. Looking closely into equations in (49)–(74) we find that there are only eight vector fields given by $G_1 = U_{\Psi_2^1}$, $G_2 = U_{\Psi_2^2}$, $G_3 = U_{\Psi_1^1}$, $G_4 = U_{\Psi_1^2}$, $G_5 = U_a$, $G_6 = U_{\beta=1}$, $G_7 = U_{\beta^2}$, and $G_8 = U_{\beta^1}$, as the variational symmetries of the alternative Lagrangian L^{S}.

We have already pointed out that G_6 represents the generator of the symmetry transformation that conserves the Jacobi's integral of the system. We further note that for $\gamma = 0$, G_5 is a generator of the rotation in the $(x\,,\,y)$ plane. The system is rotationally invariant and the corresponding conserved quantity is the angular momentum. For $x = y$ and $\gamma = 0$, equations in (1) and (3) reduce to the equation for a single oscillator. In this case, the generators G_1 and G_2 coalesce with G_3 and G_4, respectively. The generator G_5 vanishes altogether. This leaves us with only five linearly independent group generators of the one dimensional harmonic oscillator [14]. The algebra of our eight parameter Lie group is given in Table II.

TABLE II

Commutation relation for the generators in (75). Each element G_{ij} in the table being represented by $G_{ij} = [G_i\,,\,G_j]$.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
G_1	0	0	0	0	G_3	$\bar{\omega}G_2$	$\bar{\omega}G_1$	$-\bar{\omega}G_2$
G_2	0	0	0	0	G_4	$-\bar{\omega}G_1$	$-\bar{\omega}G_2$	$-\bar{\omega}G_1$
G_3	0	0	0	0	$-G_1$	$\bar{\omega}G_4$	$\bar{\omega}G_3$	$-\bar{\omega}G_4$
G_4	0	0	0	0	$-G_2$	$-\bar{\omega}G_3$	$-\bar{\omega}G_4$	$-\bar{\omega}G_3$
G_5	$-G_3$	$-G_4$	G_1	G_2	0	0	0	0
G_6	$-\bar{\omega}G_2$	$\bar{\omega}G_1$	$-\bar{\omega}G_4$	$\bar{\omega}G_3$	0	0	$2\bar{\omega}G_5$	$-2\bar{\omega}G_7$
G_7	$-\bar{\omega}G_1$	$\bar{\omega}G_2$	$-\bar{\omega}G_3$	$\bar{\omega}G_4$	0	$2\bar{\omega}G_5$	0	$-2\bar{\omega}G_6$
G_8	$\bar{\omega}G_2$	$\bar{\omega}G_1$	$\bar{\omega}G_4$	$\bar{\omega}G_3$	0	$2\bar{\omega}G_7$	$2\bar{\omega}G_6$	0

To each of the one parameter subgroups in Table II there corresponds a constant of the motion as represented above by I_ψ, I_β *etc.* As in the case of time-independent Lagrangian the solutions of the damped harmonic oscillators can be obtained as

$$x = \frac{e^{-\frac{\gamma}{2}t}}{\bar{\omega}}\left(I_{\Psi_1^2}\cos\bar{\omega}\mathrm{t} - I_{\Psi_1^1}\sin\bar{\omega}\mathrm{t}\right) \tag{75a}$$

and

$$y = \frac{e^{\frac{\gamma}{2}t}}{\bar{\omega}}\left(I_{\Psi_2^2}\cos\bar{\omega}\mathrm{t} - I_{\Psi_2^1}\sin\bar{\omega}\mathrm{t}\right). \tag{75b}$$

4. Summary and concluding remarks

Our objective in this work was to demonstrate how the association between symmetries and conservation laws differs in the presence of alternative Lagrangian representations for the damped harmonic oscillator. To clearly visualise the ambiguities we display, in Tables III and IV, results for symmetry vectors and conserved quantities corresponding to the invariance of the action functionals for L^{B} and L^{S}, respectively.

TABLE III

Symmetries and conserved quantities for L^{B}.

Symmetry	Conserved quantity
$G_1 = e^{\frac{\gamma}{2}t}\cos\bar{\omega}t\frac{\partial}{\partial y}$	$I^0_{U_5} = e^{\frac{\gamma}{2}t}\left(\dot{x}\cos\bar{\omega}t + \frac{\gamma}{2}x\cos\bar{\omega}t + \bar{\omega}x\sin\bar{\omega}t\right)$
$G_2 = e^{\frac{\gamma}{2}t}\sin\bar{\omega}t\frac{\partial}{\partial y}$	$I^0_{U_6} = e^{\frac{\gamma}{2}t}\left(\dot{x}\sin\bar{\omega}t + \frac{\gamma}{2}x\sin\bar{\omega}t - \bar{\omega}x\cos\bar{\omega}t\right)$
$G_3 = e^{-\frac{\gamma}{2}t}\cos\bar{\omega}t\frac{\partial}{\partial x}$	$I^0_{U_9} = e^{-\frac{\gamma}{2}t}\left(\dot{y}\cos\bar{\omega}t - \frac{\gamma}{2}y\cos\bar{\omega}t + \bar{\omega}y\sin\bar{\omega}t\right)$
$G_4 = e^{-\frac{\gamma}{2}t}\sin\bar{\omega}t\frac{\partial}{\partial x}$	$I^0_{U_{10}} = e^{-\frac{\gamma}{2}t}\left(\dot{y}\sin\bar{\omega}t - \frac{\gamma}{2}y\sin\bar{\omega}t - \bar{\omega}y\cos\bar{\omega}t\right)$
$G_5 = -\frac{1}{4\bar{\omega}^2}\left(\gamma\frac{\partial}{\partial t} - 2\omega^2 x\frac{\partial}{\partial x} + 2\omega^2 y\frac{\partial}{\partial y}\right)$	$I^0_{U_{11}} = -\frac{1}{4\bar{\omega}^2}\left[\gamma\dot{x}\dot{y} + 2\omega^2\left(x\dot{y} - y\dot{x}\right) - \gamma\omega^2 xy\right]$
$G_6 = \frac{1}{4\bar{\omega}^2}\left(-2\frac{\partial}{\partial t} + \gamma x\frac{\partial}{\partial x} - \gamma y\frac{\partial}{\partial y}\right)$	$I^0_{U_{12}} = \frac{1}{4\bar{\omega}^2}\left[2\dot{x}\dot{y} - \frac{1}{2}\gamma^2\left(x\dot{y} - y\dot{x}\right) + 2\omega^2 xy\right]$
$G_7 = \frac{1}{2\omega^2}(\gamma\cos 2\bar{\omega}t + 2\bar{\omega}\sin 2\bar{\omega}t)\frac{\partial}{\partial t} -$ $\frac{x}{2\omega^2}\left\{(\gamma^2 - 2\omega^2)\cos 2\bar{\omega}t + 2\gamma\bar{\omega}\sin 2\bar{\omega}t\right\}\frac{\partial}{\partial x}$ $+y\cos 2\bar{\omega}t\frac{\partial}{\partial y}$	$I^0_{U_{13}} = -\frac{\bar{\omega}}{\omega^2}\left(\dot{x}\dot{y} + \gamma x\dot{y} - \omega^2 xy\right)\sin 2\bar{\omega}t -$ $\left[\frac{\gamma}{2\omega^2}\left(\dot{x}\dot{y} + \gamma x\dot{y} - \omega^2 xy\right) - \dot{x}y - x\dot{y}\right]\sin 2\bar{\omega}t$
$G_8 = \frac{1}{2\omega^2}(-\gamma\sin 2\bar{\omega}t + 2\bar{\omega}\cos 2\bar{\omega}t)\frac{\partial}{\partial t} +$ $\frac{x}{2\omega^2}\left\{(\gamma^2 - 2\omega^2)\sin 2\bar{\omega}t - 2\gamma\bar{\omega}\sin 2\bar{\omega}t\right\}\frac{\partial}{\partial x}$ $-y\sin 2\bar{\omega}t\frac{\partial}{\partial y}$	$I^0_{U_{14}} = -\frac{\bar{\omega}}{\omega^2}\left(\dot{x}\dot{y} + \gamma x\dot{y} - \omega^2 xy\right)\cos 2\bar{\omega}t +$ $\left[\frac{\gamma}{2\omega^2}\left(\dot{x}\dot{y} + \gamma x\dot{y} - \omega^2 xy\right) - \dot{x}y - x\dot{y}\right]\sin 2\bar{\omega}t$

TABLE IV

Symmetries and conserved quantities for L^{S}.

Symmetry	Conserved quantity
$G_1 = e^{\frac{\gamma}{2}t}\cos\bar{\omega}t\frac{\partial}{\partial y}$	$I_{\Psi_2^1} = -e^{-\frac{\gamma}{2}t}\left(\dot{y}\cos\bar{\omega}t - \frac{\gamma}{2}y\cos\bar{\omega}t + \bar{\omega}y\sin\bar{\omega}t\right)$
$G_2 = e^{\frac{\gamma}{2}t}\sin\bar{\omega}t\frac{\partial}{\partial y}$	$I_{\Psi_2^2} = -e^{-\frac{\gamma}{2}t}\left(\dot{y}\sin\bar{\omega}t - \frac{\gamma}{2}y\sin\bar{\omega}t - \bar{\omega}y\cos\bar{\omega}t\right)$
$G_3 = e^{-\frac{\gamma}{2}t}\cos\bar{\omega}t\frac{\partial}{\partial x}$	$I_{\Psi_1^1} = -e^{\frac{\gamma}{2}t}\left(\dot{x}\cos\bar{\omega}t + \frac{\gamma}{2}x\cos\bar{\omega}t + \bar{\omega}x\sin\bar{\omega}t\right)$
$G_4 = e^{-\frac{\gamma}{2}t}\sin\bar{\omega}t\frac{\partial}{\partial x}$	$I_{\Psi_1^2} = -e^{\frac{\gamma}{2}t}\left(\dot{x}\sin\bar{\omega}t + \frac{\gamma}{2}x\sin\bar{\omega}t - \bar{\omega}x\cos\bar{\omega}t\right)$
$G_5 = e^{-\gamma t}y\frac{\partial}{\partial x} - e^{\gamma t}x\frac{\partial}{\partial y}$	$I_a = x\dot{y} - \dot{x}y - \gamma xy$
$G_6 = \frac{\partial}{\partial t} - \frac{\gamma}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$	$I_{\beta=1} = \frac{1}{2}\left\{e^{\gamma t}\left(\dot{x}^2+\omega^2x^2\right) + e^{-\gamma t}\left(\dot{y}^2+\omega^2y^2\right)\right\}$ $+\gamma\left(x\dot{x}e^{\gamma t} - y\dot{y}e^{-\gamma t}\right)$
$G_7 = \sin 2\bar{\omega}t\left\{\frac{\partial}{\partial t} - \frac{\gamma}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\right\}$ $+\bar{\omega}\,\cos 2\bar{\omega}t\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$	$I_{\beta^2} = \frac{1}{2}\left\{e^{\gamma t}\left(\dot{x}^2-\omega^2x^2\right) + e^{-\gamma t}\left(\dot{y}^2-\omega^2y^2\right)\right\}\sin 2\bar{\omega}t$ $+\frac{1}{2}\left(\gamma^2x^2e^{\gamma t} - \gamma^2y^2e^{-\gamma t} + \gamma x\dot{x}e^{\gamma t} - \gamma y\dot{y}e^{-\gamma t}\right)\sin 2\bar{\omega}t$ $-\bar{\omega}\left(\frac{\gamma}{2}x^2e^{\gamma t} - \frac{\gamma}{2}y^2e^{-\gamma t} + x\dot{x}e^{\gamma t} - y\dot{y}e^{-\gamma t}\right)\cos 2\bar{\omega}t$
$G_8 = \cos 2\bar{\omega}t\left\{\frac{\partial}{\partial t} - \frac{\gamma}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\right\}$ $-\bar{\omega}\,\sin 2\bar{\omega}t\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$	$I_{\beta^1} = \frac{1}{2}\left\{e^{\gamma t}\left(\dot{x}^2-\omega^2x^2\right) + e^{-\gamma t}\left(\dot{y}^2-\omega^2y^2\right)\right\}\cos 2\bar{\omega}t$ $+\frac{1}{2}\left(\gamma^2x^2e^{\gamma t} - \gamma^2y^2e^{-\gamma t} + \gamma x\dot{x}e^{\gamma t} - \gamma y\dot{y}e^{-\gamma t}\right)\cos 2\bar{\omega}t$ $+\bar{\omega}\left(\frac{\gamma}{2}x^2e^{\gamma t} - \frac{\gamma}{2}y^2e^{-\gamma t} + x\dot{x}e^{\gamma t} - y\dot{y}e^{-\gamma t}\right)\sin 2\bar{\omega}t$

From these tables we see that symmetries G_1–G_4 are the same for the case of both L^{B} and L^{S} and they form an Abelian subgroup. The corresponding conserved quantities are linearly independent such that the solutions of the Bateman dual system could be constructed using these conserved quantities only. It is interesting to note that $I^0_{U_5}|^{L^{\mathrm{B}}}_{G_1} = -I_{\psi^1_1}|^{L^{\mathrm{S}}}_{G_3}$ which means that the conserved quantity $I^0_{U_5}$ corresponding to the symmetry vector G_1 for the Lagrangian L^{B} is related to the conserved quantity $I_{\psi^1_1}$ corresponding to the symmetry vector G_3 for L^{S}. Similarly, we have $I^0_{U_6}|^{L^{\mathrm{B}}}_{G_2} = -I_{\psi^2_1}|^{L^{\mathrm{S}}}_{G_4}$, $I^0_{U_9}|^{L^{\mathrm{B}}}_{G_3} = -I_{\psi^1_2}|^{L^{\mathrm{S}}}_{G_1}$ and $I^0_{U_{10}}|^{L^{\mathrm{B}}}_{G_4} = -I_{\psi^2_2}|^{L^{\mathrm{S}}}_{G_2}$. Thus we have ambiguities in the association of symmetries and conservation laws. Except G_5, other G values are the same in the limit of no damping ($\gamma = 0$). In this limit both $I^0_{U_{12}}|^{L^{\mathrm{B}}}_{G_6}$ and $I_{\beta=1}|^{L^{\mathrm{S}}}_{G_6}$ give the Hamiltonian/energy of the uncoupled harmonic oscillators with time translation as the symmetry vector. The conserved quantities $I^0_{U_{13}}|^{L^{\mathrm{B}}}_{G_7}$, $I^0_{U_{14}}|^{L^{\mathrm{B}}}_{G_8}$, $I_{\beta^2}|^{L^{\mathrm{S}}}_{G_7}$ and $I_{\beta^1}|^{L^{\mathrm{S}}}_{G_8}$ do not admit simple physical realization. The vector field G_5 for L^{B} is a squeezing operator for $\gamma = 0$ and the corresponding conserved quantity is the angular momentum. A squeezing-like operator is, however, not at all a symmetry vector of the action implied by L^{S}. In contrast, G_5 for L^{S} gives a rotation operator for $\gamma = 0$ and angular momentum is the corresponding conserved quantity. Once again this shows that the alternative Lagrangian representations bring in a point of contrast for the association of symmetries and conservation laws. Further, the Lie algebra (g) of the symmetry vectors that leaves the action corresponding to L^{B} invariant is semi-simple. On the other hand, g is only a simple Lie algebra for the action characterised by L^{S}.

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