

CLUSTER STATISTICS OF BTW AUTOMATA

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The cluster statistics of BTW automata in the SOC states are obtained by extensive computer simulation. Various moments of the clusters are calculated and few results are compared with earlier available numerical estimates and exact results. Reasonably good agreement is observed. An extended statistical analysis has been made.

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1. Introduction

Bak, Tang and Wiesenfeld (BTW) [1] have introduced (1987) the concept of Self Organised Criticality to describe the appearance of long range spatio-temporal correlations observed in extended, dissipative dynamical systems in nature. This phenomena of SOC is characterised by spontaneous evolution into a steady state which shows long range temporal and spatial correlation. The simple lattice automata model of sandpile which shows this SOC behaviour was studied by BTW [1]. Substantial developments have been made on the study of BTW model. Several properties of this critical state, *e.g.*, entropy, height correlation, height probabilities *etc.* have been calculated analytically. But the critical exponents have not been calculated analytically. Extensive numerical efforts have been performed to estimate various exponents. The studies regarding this model have been reviewed by Dhar [2].

Here, in this paper, the statistics of the clusters, formed by the sites having particular value of the automaton, have been studied, by computer simulation. We have calculated the various statistical quantities *e.g.*, average size of clusters, total number of clusters, maximum size of the clusters *etc.* in the SOC state and found the distribution function of the clusters. Related to the statistics of the clusters of the automation values, Manna has estimated

the fraction of sites, having automation values, 0,1,2 and 3, present in the SOC state [3,4]. Majumder and Dhar calculated exactly the fraction of sites having minimum value of automation in the SOC state [5].

The paper is organised as follows: in Sec. 2 the model and the simulation technique have been described, Sec. 3 contains the numerical results obtained here, and the paper ends with a concluding section (Sec. 4).

2. The model and simulation

BTW lattice automata model [1] evolves to a stationary state in a self-organised (having no tunable parameter) way. This state has no scale of length and time, hence called critical. Altogether the state is called self-organised critical state. The description of the lattice automata model is as follows: at each site of this lattice, a variable (automaton) $z(i, j)$ is associated which can take positive integer values. Starting from the initial condition (at every site $z(i, j) = 0$), the value of $z(i, j)$ is increased (so-called addition of one “sand” particle) at randomly chosen site (i, j) of the lattice in steps of unity as

$$z(i, j) = z(i, j) + 1.$$

When the value of z at any site reaches a maximum z_m , its value decreases by four units (*i.e.*, it topples) and each of the four nearest neighbours gets one unit of z (maintaining local conservation) as follows

$$z(i, j) = z(i, j) - 4, \tag{1}$$

$$z(i \pm 1, j \pm 1) = z(i \pm 1, j \pm 1) + 1 \tag{2}$$

for $z(i, j) \geq z_m$. Each boundary site is attached with an additional site which acts as a sink (*i.e.*, open boundary condition is used). In this simulation, the value of $z_m = 4$.

It has been observed that, as the time goes on the average value (\bar{z}) of $z(i, j)$, over the space, increases and ultimately reaches a steady value (\bar{z}_c) characterising the SOC state. At any time (*i.e.*, a particular value of \bar{z}), after the addition of one such “sand” particle, when the system becomes stable (*i.e.*, all the sites have $z(i, j) < 4$), the lattice contains the sites having the automaton values 0, 1, 2 and 3 only. These sites (having a particular value of $z(i, j)$) form clusters connected via the nearest neighbours. These clusters have some size distribution described by the function $\rho_k(s_k)$, where $\rho_k(s_k)$ denotes the number of clusters (formed via nearest neighbour connection of sites having their automaton value k) of size s_k . From this distribution $\rho_k(s_k)$, the various statistical moments of such clusters have been calculated here and studied as a function of \bar{z} . The various statistical quantities (moments), for a particular class of cluster, are defined as follows:

- Total fraction of sites ($F_k(\bar{z})$) having value k at a particular value of \bar{z}

$$F_k(\bar{z}) = \frac{1}{L^2} \left(\sum s_k \rho_k(s_k) \right).$$

- Average size ($S_k(\bar{z})$) of clusters formed by the sites of automation value k . If $\rho_k(s_k)$ is the number of clusters (formed by the sites having their automation value z) of size s_k

$$S_k(\bar{z}) = \frac{\sum s_k \rho_k(s_k)}{\sum \rho_k(s_k)}.$$

- Total number of clusters ($N_k(\bar{z})$) formed by the sites having their automaton value $z(i, j) = k$

$$N_k(\bar{z}) = \sum \rho_k(s_k).$$

- Largest size ($s_k^{\max}(\bar{z})$) of cluster formed by the sites of their automation value k

$$L_k(\bar{z}) = \text{Max}[s_k].$$

3. Results

We have calculated the cluster size distribution, $\rho_k(s_k)$, in the SOC state for different automaton values ($K = 0, 1, 2$ and 3). We did it by using the program [6] to calculate the cluster size distribution used in standard percolation problem [7]. From the distribution all the statistical quantities (or moments) *i.e.*, F_k , S_k , N_k and L_k are calculated in the SOC state. All these data have been obtained for $L = 400$, by averaging over 20 different random samples. The values of F_k have been estimated earlier [3,4,5] and our values agree well with the previous estimates (see Table I).

Figure 1 shows the semilog plot of the normalised cluster size distribution, ($n_s = \rho_k(s_k)/L^2$, is the number of clusters of size s per lattice site), for the automation value, $k = 3$, and for $L = 400$. The distribution function fits with the curve

$$n_s = 0.000004 s^{-1.5} \exp(-0.045 s).$$

To compare the cluster size distribution with the known result of percolation, we have plotted here the normalised cluster size distribution, $n_s = \rho_k(s_k)/L^2$, instead of $\rho_k(s_k)$. We see that this distribution function (for $k = 3$) has the same functional form as that of the percolation [7] (below p_c) apart from the exponent θ of the pre-exponential factor, which may

TABLE I

k	F_k^{SOC}	S_k^{SOC}	N_k^{SOC}	L_k^{SOC}
0	0.0741 0.0730 [4] 0.0736 [5] 0.0736 [3] 0.07363 [8]	1.000	11856.2	1.00
1	0.1746 0.1740 [4] 0.1740 [5] 0.1739 [8]	1.429	19552.4	10.85
2	0.3069 0.307 [4] 0.3062 [5] 0.3063 [8]	2.376	20670.2	37.90
3	0.444 0.446 [4] 0.4462 [5] 0.4461 [8]	4.326	16430.5	109.84

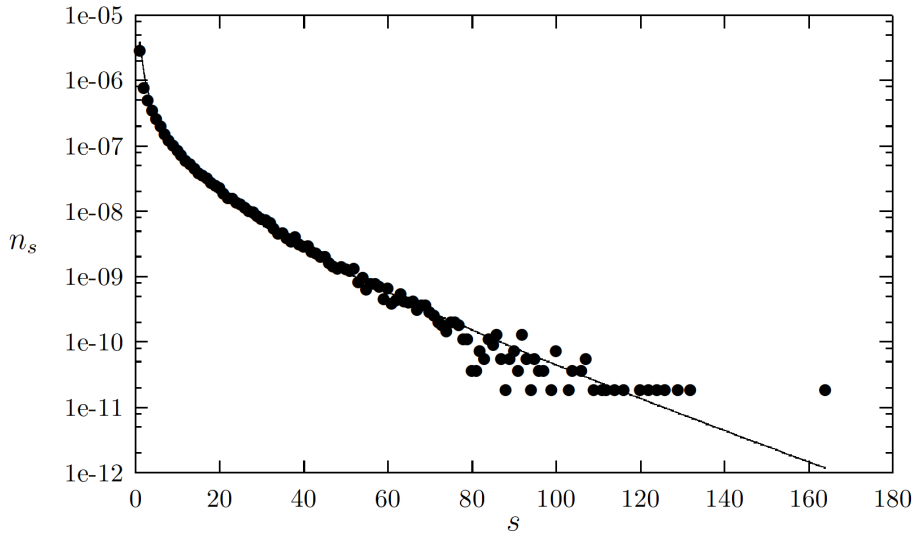


Fig. 1. The normalised distribution (n_s) of clusters of size s for the automaton value $k = 3$. Solid line represents the best fit.

be due to the fact that the sites are not randomly filled and they are correlated [9]. Another interesting point to note here is that the clusters (for $k = 0, 1, 2, 3$) formed by automaton values are not system spanning (see the first column of Table I), although it can give a system spanning (scale invariant) avalanche in SOC state. It is to be mentioned that the other two distribution functions of the clusters, for automaton value $k = 2$ and 1, have also been observed. These two distribution functions are also of the same nature as that of $k = 3$. Since the maximum size of the cluster decreases as the automaton value decreases, the fluctuation in the distribution function for large clusters increases and hence these results have not shown explicitly.

The basic features of this simulational studies may be described briefly as follows: in the self organised critical state the BTW automata will assume the values 0, 1, 2 and 3. These automata will form the clusters. The statistics of such clusters are studied here. The distribution of the sizes of clusters, average size, maximum size *etc.* are studied by numerical simulation. An algebraic form of the cluster size distribution is proposed. Interestingly, it was observed that the clusters are not system spanning in the SOC state.

4. Concluding remarks

The statistical aspects of BTW cellular automata is investigated by computer simulation. In the SOC state the automaton can take the values 0, 1, 2 and 3. The cluster formed by the fixed value of automaton (say 3) is identified and measured its size. The clusters are defined via nearest neighbour connections. The statistical distribution of these clusters are calculated and from this distribution various moments are calculated in SOC state. Few results (for example the fraction of sites having a fixed value) are compared with the earlier numerical estimates (Ref. [3–4]) and exact calculations (Ref. [5]). The values of S_k^{SOC} and L_k^{SOC} for $k = 0$ listed in Table I are exactly known from Dhar's [2] burning algorithm. This study is a generalisation of earlier study (which estimates only fraction of sites having a fixed value) extending the calculation of entire statistics of BTW automata. It would be interesting to study the dynamical evolution (towards SOC state) of the cluster, its formation nucleation, coalescence *etc.*

REFERENCES

- [1] P. Bak, C. Tang, K. Wiesenfeld, *Phys. Rev. Lett.* **59**, 381 (1987); P. Bak, C. Tang, K. Wiesenfeld, *Phys. Rev.* **A38**, 364 (1988).
- [2] D. Dhar, [arXiv:cond-mat/9902137](https://arxiv.org/abs/cond-mat/9902137).
- [3] S.N. Majumder, D. Dhar, *J. Phys. A: Math. Gen.* **24**, L357 (1991).

- [4] S.S. Manna, *J. Stat. Phys.* **59**, 509 (1990).
- [5] P. Grassberger, S.S. Manna, *J. Phys. (Paris)* **51**, 1077 (1990).
- [6] K. Binder, D.W. Heermann, *Monte Carlo Simulation in Statistical Physics*, Springer, 1992, pp. 135–138.
- [7] D. Stauffer, *Introduction to Percolation Theory*, Taylor and Francis, 1985.
- [8] V.B. Priezzhev, *J. Stat. Phys.* **74**, 955 (1994).
- [9] E.V. Ivashkevich, *J. Phys. A: Math. Gen.* **27**, 3643 (1994).