CORRELATED LÉVY NOISE IN LINEAR DYNAMICAL SYSTEMS

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(Received September 29, 2010; revised version received November 10, 2010)

Linear dynamical systems, driven by a non-white noise which has the Lévy distribution, are analysed. Noise is modelled by a specific stochastic process which is defined by the Langevin equation with a linear force and the Lévy distributed symmetric white noise. Correlation properties of the process are discussed. The Fokker–Planck equation driven by that noise is solved. Distributions have the Lévy shape and their width, for a given time, is smaller than for processes in the white noise limit. Applicability of the adiabatic approximation in the case of the linear force is discussed.

DOI:10.5506/APhysPolB.42.3
PACS numbers: 02.50.Ey, 05.40.Ca, 05.40.Fb

1. Introduction

Stochastic dynamical equation (the Langevin equation) describes motion of a particle which is subjected to both deterministic and stochastic force. The latter one can be understood either as a result of elimination of internal degrees of freedom or as some external physical process. The external noise possesses its own time scale and relaxation properties. If relaxation time of processes in the environment is relatively short, the white noise may be a good approximation: the noise variables change rapidly, compared to the particle variables. Otherwise the Langevin description must involve the correlated ('coloured') noise. This problem was widely discussed for the Gaussially distributed noise. Well-known physical examples involve a phenomenon of narrowing of magnetic resonance lines due to the thermal fluctuations [1] and the fluctuations of dye laser light [2]. The problem of correlated noise also emerges when one eliminates some variables in a multi-dimensional dynamical system; then the effective low-dimensional description involves correlations even if the original many-dimensional system

is Markovian [3]. The Langevin equation with the correlated Gaussian noise, both additive and multiplicative, corresponds to a non-Markovian process and it resolves itself to an integro-differential Fokker–Planck equation which can be solved exactly for simple potentials; otherwise approximate methods may be applied [3, 4].

Recently, the Lévy processes — which constitute a general class of the stable processes with the Gaussian process as a special case — attract a considerable interest. They are characterised by long tails, which make the variance divergent, and can be observed in many systems from various fields: biology [5], hydrology [6], sociology [7] and finance [8]. As a result, long jumps may appear and the standard central limit theorem is no longer valid. Realistic problems are usually characterised by high complexity and they exhibit collective phenomena; they involve long-range correlations, nonlocal interactions and a complicated, nonhomogeneous (in particular fractal or multifractal) structure of the medium. For example, diffusion in the porous media, which display the fractal structure, can be described by a stochastic equation driven by the Lévy process [9].

It is natural to expect that processes which are driven by a noise with long jumps are correlated. As an example can serve an experimental study on spontaneous electrical activity of neuronal networks with different sizes [10]. It was found that all networks exhibited scale-invariant Lévy distributions. The authors conclude that different-size networks self-organise to adjust their activities over many time scales. The power spectrum, calculated from the experimental time series, indicates correlations: it obeys a power-law decay at low frequencies for all network sizes.

The non-Markovian master equation governs probability distributions in the framework of the decoupled continuous time random walk theory [11]. If jumps are Lévy distributed and the waiting time distribution is algebraic, the Fokker–Planck equation is fractional both in time and position. The integral operators introduce a competition between subdiffusion and accelerated diffusion; the latter one results from the infinite variance. Integral Fokker–Planck equations were solved for both fast and slowly decaying memory kernels [12]. They can be generalised to the fractional orders and to the case of a variable diffusion coefficient [13].

In this paper, we consider a linear dynamical system which is defined by the Langevin equation with the Lévy distributed non-white noise. That problem was solved by Hänggi and Jung [3] (and references therein) for an arbitrary autocorrelation function $\mathcal{C}(t)$ in the case of the Gaussian noise. However, a method which directly deals with the autocorrelation function cannot be applied for $\alpha < 2$ since then $\mathcal{C}(t)$ does not exist; we will discuss that difficulty in Section 2. Therefore, we introduce a specific model of the correlated noise; we require that the model process should have the

Lévy distribution and be correlated (in a sense which will be explained in Section 2). Moreover, it should be as simple as possible. We define that process in Section 2 by an additional Langevin equation which corresponds to the Ornstein–Uhlenbeck process with the white symmetric Lévy noise. We discuss its correlation properties. The Langevin equation, driven by that process, is analysed in Section 3 for simple forms of the potential: the free Lévy motion, the constant force and the linear force. The problem resolves itself to solving a set of two Langevin equations. Results are summarised in Section 4.

2. Ornstein–Uhlenbeck process with Lévy noise

Motion of a particle, which is subjected to the linear force and the Lévy noise, is described by the following linear Langevin equation

$$\dot{\xi}(t) = -\gamma\xi(t) + \dot{L}(t)\,, \tag{1}$$

where the uncorrelated and symmetric noise $L(t)$ is the α-stable Lévy process and $\gamma = \text{const.} > 0$. Eq. (1), with the initial condition $\xi(0) = 0$, can be formally solved [14]

$$\xi(t) = \int_0^t K(t-\tau)L(d\tau)\,, \tag{2}$$

where $K(t) = \exp(-\gamma t)$. Eq. (2) transforms an uncorrelated input noise into a correlated output process. The well-known theory of the Brownian motion corresponds to the case of $\alpha = 2$. Generalisation to the non-Gaussian stable cases, which are defined by Eq. (1), constitutes the Ornstein–Uhlenbeck–Lévy process (OULP). If $\alpha = 2$, trajectories are continuous and Eq. (1) corresponds to the standard Fokker–Planck equation. Otherwise jumps — in a sense of violation of the Lindeberg condition — emerge [4, 15] and their presence requires introducing integral operators. The Fokker–Planck equation, which is suited for problems with jumps, contains the fractional operator

$$\frac{\partial}{\partial t}\,p(\xi,t) = \gamma\frac{\partial}{\partial\xi}\,[\xi p(\xi,t)] + D\frac{\partial^\alpha}{\partial|\xi|^\alpha}p(\xi,t)\,, \tag{3}$$

where $0 < \alpha \leq 2$ denotes the stability index of the Lévy distribution and $D \geq 0$ is a constant noise intensity. The Lévy distribution itself is given by the following Fourier transform

$$P(L) = \frac{1}{\pi}\int_0^\infty \exp\left(-Dk^\alpha\right)\cos(kL)dk\,. \tag{4}$$

Distribution $p(\xi,t)$ can be evaluated either directly from Eq. (2) [16] or by solving Eq. (3) [17]. The characteristic function reads

$$\widetilde{p}(k,t) \equiv \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} p(\xi,t)\, e^{-ik\xi} d\xi = \exp\left[-\frac{D}{\alpha\gamma}|k|^{\alpha}\left(1-e^{-\gamma\alpha t}\right)\right]. \tag{5}$$

Expression (5) corresponds to the Lévy stable and symmetric process and the width converges with time to a constant, producing a stationary distribution. The second moment is divergent, unless $\alpha = 2$, and also the mean is divergent if $\alpha < 1$.

The Langevin equation driven by the white non-Gaussian noise was studied by several authors, both for linear and nonlinear systems [17,18,19,20]. It was generalised to the asymmetric Lévy noise [16] and to the multiplicative noise [21,22]. OULP was also discussed in Ref. [23] where several fractional generalisations were presented.

Dynamical relation (1) introduces a dependence among process values ξ at different times: the process $\xi(t)$ possesses memory. For the Gaussian case, the autocorrelation function serves as a measure of the memory loss. It is defined [4] as the average along a stochastic trajectory

$$G(\tau) = \lim_{T\to\infty}\frac{1}{T}\int\limits_{0}^{T}\xi(t)\xi(t+\tau)\,dt\,. \tag{6}$$

$G(\tau)$ can be evaluated as the inverse Fourier transform from the spectral function

$$S(\omega) = \lim_{T\to\infty}\frac{1}{2\pi T}\left|\widetilde{\xi}(\omega)\right|^2, \tag{7}$$

where $\widetilde{\xi}(\omega)$ stands for the Fourier transform from $\xi(t)$, by means of the Wiener–Khinchin theorem

$$G(\tau) = \mathcal{F}^{-1}[S(\omega)]\,. \tag{8}$$

For the ordinary Ornstein–Uhlenbeck process, which is given by Eq. (1) with $\alpha = 2$, the stationary autocovariance function $G(\tau)$ follows directly from Eq. (2). It assumes the exponential form [4]

$$G(\tau) = \frac{D}{\gamma}e^{-\gamma|\tau|} \tag{9}$$

which corresponds to the Lorentzian shape of $S(\omega)$. The correlation time $\tau_{\rm c} = 1/\gamma$ measures the decay rate of $G(\tau)$.

Applying the above formalism to the case of $\alpha < 2$ is problematic since the variance $\sigma^2 = G(0)$ becomes infinite. To overcome that difficulty, some modifications of the standard covariance definition were introduced. One can define [24, 25] the 'codifference' $\tau_{X,Y} = \sigma_X^\alpha + \sigma_Y^\alpha - \sigma_{X-Y}^\alpha$, where X, Y are stable and symmetric processes. For independent X and Y, $\tau_{X,Y} = 0$; codifference resolves itself to the standard covariance if $\alpha = 2$. On the other hand, one can utilise the Poissonian structure of the Lévy process to introduce an infinite cascade of Poissonian correlation functions which correspond to the autocorrelation function [26, 14]. That function depends exponentially on time for OULP, Eq. (1). Standard correlation formalism of the general Lévy case may be applied if Lévy measure in the Lévy–Khinchine formula [15] possesses a cut-off [27]; all moments are then finite. Solutions of the Langevin equation, which is driven by noise with such a truncated distribution, are identical with those for the stable noise up to arbitrarily large distances [28].

The usual definition of the autocorrelation function, Eq. (6), may still be applicable to the general stable Lévy case, despite divergent variance. The characteristic function of the increment $\xi(t_2) - \xi(t_1)$ can be formally derived [17]; that function contains all information about two-point correlations. Special methods of spectral analysis were developed to handle experimental time series which involve long jumps, *e.g.* calculating the count-based periodogram [29]. That method allows one to calculate the autocorrelation function and power spectrum for long signals, also containing nonstationary trends [10]. We will demonstrate, by means of numerical simulation of stochastic trajectories, that speed of memory loss for the process (1) can be determined by means of the ordinary spectral analysis. Let us calculate the power spectrum, Eq. (7), from a trajectory which follows from Eq. (1) and has a given length T; the Fourier transform is simultaneously evaluated. The relative normalisation of $S(\omega)$, $S_0 = S(0)\gamma^2$, is finite in any calculation since T is always finite. However, it depends on T and then cannot be determined, as expected. The analysis shows that the quantity $S(\omega)/S_0$ is well determined in the limit $T \to \infty$, it obeys the Lorentz function

$$\lim_{T\to\infty} \frac{S(\omega)}{S_0} = \frac{1}{\gamma^2 + \omega^2}\,. \tag{10}$$

The renormalised $S(\omega)$ is presented in Fig. 1 for $T = 10^4$ and some values of α and γ. All curves follow the Lorentzian shape. The value of S_0, which emerges from that calculation, may be large, it ranges from 1 ($\alpha = 2$) to 10^3 ($\alpha = 1.2$).

On the other hand, the covariance can be evaluated by averaging over the ensemble if one introduces a cut-off in the distribution (4). We define the ensemble-averaged autocorrelation function

$$\mathcal{C}(\tau) = \frac{\langle \xi(0)\xi(\tau)\rangle}{\langle \xi(0)^2\rangle} \tag{11}$$

on the assumption that $P(L) = 0$ for $L > L_c$. Figure 1 presents that quantity; it demonstrates that also $\mathcal{C}(\tau)$ obeys the exponential dependence (9). Equivalence of the expression (6) with the ensemble averaged covariance is not obvious for $\alpha < 1$ since then the system may exhibit the weakly non-ergodic behaviour [30].

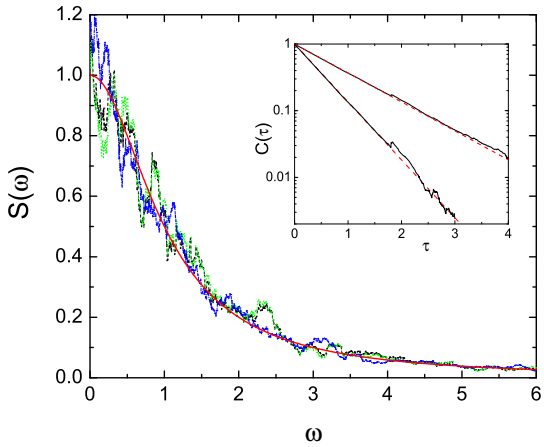


Fig. 1. Renormalised spectral function for OULP, Eq. (1), calculated from evolution of a trajectory up to $t = 10^4$, for the following cases: $\alpha = 1.2$ (dashed line), $\alpha = 1.5$ ((green) dots) and $\alpha = 2$ ((blue) dash-dotted line). Red solid line denotes the Lorentz function (10). Upper and lower curves correspond to $\gamma = 1$ and 2, respectively. Inset: $\mathcal{C}(\tau)$, calculated from an ensemble of 10^6 trajectories with $L_c = 10^4$, for $\gamma = 1$ and 2 (solid lines). Dashed lines (red) represent the function $e^{-\gamma\tau}$.

3. Langevin equation with coloured noise

In this section, we study the stochastic dynamics of a particle which is subjected to the Lévy correlated noise and the linear deterministic force. The noise $\xi(t)$ is represented by OULP, Eq. (1). Then we have to solve a set of two Langevin equations

$$\begin{aligned} \dot{x}(t) &= f_0 - \lambda x(t) + \gamma\xi(t)\,, \\ \dot{\xi}(t) &= -\gamma\xi(t) + \dot{L}(t)\,, \end{aligned} \tag{12}$$

where $\gamma \geq 0$, $\lambda \geq 0$ and f_0 are constants. In the presence of jumps, the system remains far from the thermal equilibrium and the detailed balance is violated. Then $\xi(t)$ can be regarded as an external noise which has its own

time scale, determined by the parameter γ. In general, processes which obey Langevin equation with the correlated noise are non-Markovian since the process values are evaluated from mutually dependent noise increments [3]. For large γ (short correlation time), ξ is a fast, rapidly relaxing variable and the process can be approximated by a corresponding white-noise problem, by using the methods of adiabatic elimination of fast variables [3, 4].

3.1. The case without deterministic force and with a constant force

The force-free motion, with the white Lévy noise, is a generalisation of the Wiener process; it describes simple diffusion if $\alpha = 2$. Generalisation to the coloured noise is defined by Eq. (12) with $f_0 = \lambda = 0$. We assume the initial conditions $x(0) = \xi(0) = 0$. Our aim is to find the probability distribution of the variable x. One can solve Eq. (12) and utilise the fact that $x(t)$ — as a superposition of the Lévy distributions — is still a process with independent increments, though multiplied by some function of time; then convolution of densities can be performed. That method was applied in Ref. [18] to the second order Langevin equation for the case $\alpha = 1$. We apply van Kampen's method of compound master equations [20] which consists in solving the joint fractional Fokker–Planck equation for the two-dimensional system, (x, ξ), and integrating over the internal noise ξ. That method is relatively simple in the case without potential and formally applicable also to nonlinear systems with a multiplicative noise. In the linear case, the existence, uniqueness and positiveness of the solution is ensured [21].

The Langevin equations (12) correspond to the fractional Fokker–Planck equation for a joint probability distribution $p(x, \xi, t)$ [31, 21]

$$\frac{\partial}{\partial t} p\left(x, \xi, t\right) = \left[-\gamma \frac{\partial}{\partial x} \xi + \gamma \frac{\partial}{\partial \xi} \xi + D \frac{\partial^{\alpha}}{\partial |\xi|^{\alpha}}\right] p\left(x, \xi, t\right) . \tag{13}$$

Knowing the solution of Eq. (13), the probability distribution of the variable x can be obtained by integration over all possible realisations of the noise ξ

$$p(x, t) = \int_{-\infty}^{\infty} p\left(x, \xi, t\right) d\xi \, . \tag{14}$$

Fourier transformation of Eq. (13), in respect to both x and ξ, produces the equation for the characteristic function $\widetilde{p}(k, \kappa, t)$

$$\frac{\partial}{\partial t} \widetilde{p} - \gamma \left(k - \kappa\right) \frac{\partial}{\partial \kappa} \widetilde{p} = -D |\kappa|^{\alpha} \widetilde{p} \, , \tag{15}$$

which can be solved exactly by the method of characteristics; details are presented in Appendix. The Fourier transform of the solution, Eq. (14),

follows from Eq. (A.6)

$$\widetilde{p}\,(k,t) = \widetilde{p}\,(k,0,t) = e^{-D\sigma(t)|k|^{\alpha}}\,, \tag{16}$$

where

$$\sigma(t) = \frac{1}{\gamma}\int\limits_0^g \frac{\kappa^{\alpha}}{1-\kappa}d\kappa \tag{17}$$

and $g = 1 - e^{-\gamma t}$. Eq. (16) predicts the Lévy shape with the stability index α. The width parameter $\sigma(t)$ can be estimated in the limit $\gamma t \gg 1$, when the main contribution to the integral comes from the vicinity of the upper integration limit, since then the denominator is close to zero

$$\sigma(t) \approx \frac{1}{\gamma}\left(1-e^{-\gamma t}\right)^{\alpha}\int\limits_0^g \frac{d\kappa}{1-\kappa} = t\left(1-e^{-\gamma t}\right)^{\alpha}\,. \tag{18}$$

In the limit $\gamma t \to \infty$, σ rises linearly with time and $p(x,t)$ coincides with the solution of the uncorrelated problem. Convergence to that solution depends on α: it is faster for smaller α.

The integral (17) can be exactly evaluated if α is a rational number. In particular, for $\alpha = 3/2$ it yields

$$\sigma(t) = \frac{2}{\gamma}\left[-\left(1-e^{-\gamma t}\right)^{1/2} - \left(1-e^{-\gamma t}\right)^{3/2} + \operatorname{arctanh}\sqrt{1-e^{-\gamma t}}\right]\,. \tag{19}$$

In the limit $\gamma t \gg 1$, the expression (19) predicts a time shift, in respect to the white noise case, since it can be approximated by $\sigma \approx t - (8/3 - 2\ln 2)/\gamma$.

Numerical values of the probability distribution $p(x,t)$, which result from inversion of the characteristic function (16), can be obtained from the series expansion [32],

$$p(x,t) = \frac{1}{\pi\sigma^{1/\alpha}\alpha}\sum_{n=0}^{\infty}\frac{\Gamma[1+(2n+1)/\alpha]}{(2n+1)!!}(-1)^n\left(\frac{x}{\sigma^{1/\alpha}}\right)^{2n}\,, \tag{20}$$

if $|x|$ is not too large. Figure 2 presents those distributions for the case $\alpha = 1.5$ at $t = 1$, $\sigma(t)$ was calculated from Eq. (19). Figure shows that the memory affects the rate of spreading of the distribution: $p(x,t)$ is broadest for the white noise case, $\gamma = \infty$, and it contracts to the delta function in the limit $\gamma \to 0$. Results are compared with the Monte Carlo simulations of individual trajectories, according to the stochastic equations (12). For that purpose, a simple Euler algorithm was applied. The white noise value at ith integration step, L_i, was represented by the term $\tau^{1/\alpha}L_i$, where τ was

the step size [33]. Probability distributions were obtained by averaging over an statistical ensemble of the individual trajectories. Since the analytical result does not contain any approximation, agreement with the simulations is exact.

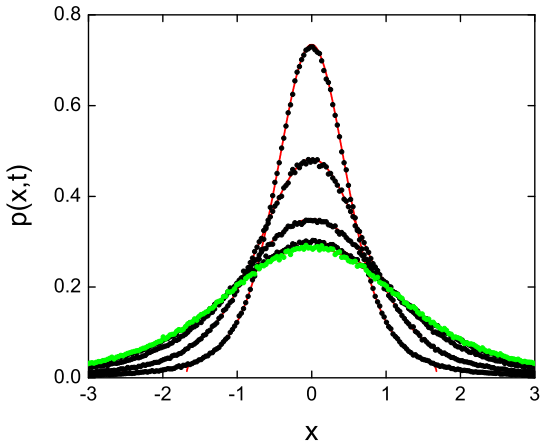


Fig. 2. Probability distributions at $t = 1$ for the force-free case calculated by the Monte Carlo simulations (points) for $\gamma = 1, 2, 5, 20$ (from top to bottom); the most diffused case corresponds to the white noise limit ($\gamma = \infty$). Analytical results, calculated from Eq. (20) with σ from Eq. (19), are presented as solid lines. The stability index $\alpha = 1.5$. Numerical simulations were performed with the time step $\tau = 0.005$ and averaged over 10^7 events.

Problem of the linear potential, $-f_0 x$, where $f_0 = \text{const.}$, can be reduced to the force-free case which was discussed above. The first equation in Eq. (12) takes the form $\dot{x}(t) = f_0 + \gamma\xi(t)$. From the corresponding fractional Fokker–Planck equation,

$$\frac{\partial}{\partial t}p(x,\xi,t) = \left[-\frac{\partial}{\partial x}(f_0 + \gamma\xi) + \gamma\frac{\partial}{\partial \xi}\xi + D\frac{\partial^\alpha}{\partial|\xi|^\alpha}\right] p(x,\xi,t)\,, \tag{21}$$

we derive equation for the characteristic function

$$\frac{\partial}{\partial t}\widetilde{p} - \gamma(k-\kappa)\frac{\partial}{\partial \kappa}\widetilde{p} = -\left(if_0 k + D|\kappa|^\alpha\right)\widetilde{p}\,. \tag{22}$$

Its solution, $\widetilde{p}(k,\kappa,t) = e^{-if_0 kt}\widetilde{p}_0$, where $\widetilde{p}_0$ is given by Eq. (A.6), follows from the general theory [21]. It can be also obtained by separation of real and imaginary parts of $\widetilde{p}(k,\kappa,t)$ and by solving the resulting set of two equations. Integration over the variable ξ produces the final result

$$\widetilde{p}(k,t) = e^{-if_0 kt}\widetilde{p}_0\,, \tag{23}$$

where $\widetilde{p}_0(k,t)$ follows from Eq. (16). The distribution $p(x,t)$ has the same shape, for any time, as that for the case $f_0 = 0$ but it is shifted by $f_0 t$. That means that the average rises linearly with time, $\langle x \rangle = f_0 t$ (if $\alpha > 1$), and the distribution widens with time according to the function $\sigma(t)$, Eq. (17). In the limit $\gamma \to 0$, $p_0(x,t) = \delta(x)$ which corresponds to a deterministic motion with velocity f_0. Probability distributions which follow from the Monte Carlo simulations (not presented) agree with the solution (23).

In the limit $\gamma t \to \infty$, Eq. (23) coincides with the solution of fractional Fokker–Planck equation with the constant force for the white noise case [17]. The problem of transport in an effective constant force field emerges in the framework of the continuous time random walk theory when one considers a biased walk [34]. It resolves itself to the fractional Fokker–Planck equation with a drift term.

3.2. Linear force

The system is defined by Eq. (12) with $f_0 = 0$, where $\lambda > 0$ measures intensity of the deterministic force. The aim of this section is a comparison of exact probability distributions, obtained by numerical simulation of two-dimensional stochastic trajectories from Eq. (12), with predictions of the adiabatic approximation.

Figure 3 presents examples of stochastic trajectories for two cases: the Lévy distribution with $\alpha = 1.5$ and for the normal distribution. In the former case, large jumps, typical for the Lévy processes, are visible along the horizontal direction which represents OULP (Eq. (1)). The process $x(t)$, in turn, is stronger localised for both values of α. The plot shrinks in the

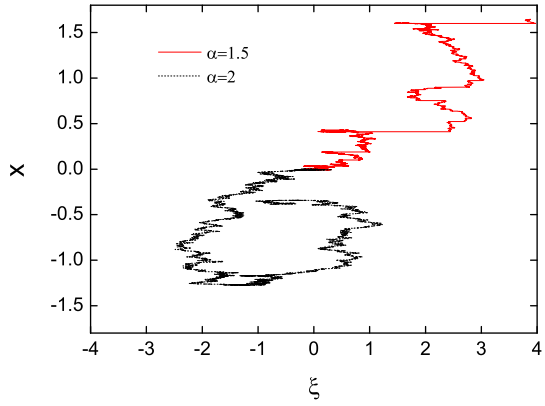


Fig. 3. Exemplary stochastic trajectories in the space (ξ, x), calculated from Eq. (12) with time step $\tau = 5 \times 10^{-4}$ up to $t = 3$, for $\lambda = 1$ and $\gamma = 1$. The trajectory for the case $\alpha = 1.5$ is positioned in upper-right quarter of the figure.

horizontal direction with increasing γ (not shown) which reflects the fact that ξ becomes the fast variable: it relaxes rapidly to $\xi = 0$. Averaging over a large number of trajectories produces the probability distribution $p(x,t)$. Figure 4 demonstrates that it converges with time to the stationary distribution, as in the white noise case. Comparison of the distribution for consecutive times shows that the time which is needed to reach the steady state equals 5 for the case presented in the figure; for larger times the distribution remains unchanged. The shape of $p(x,t)$ coincides with the Lévy distribution for any γ and its stability index α corresponds to that of the driving noise $L(t)$. The apparent width rises with γ and, for large γ, the white-noise limit is reached.

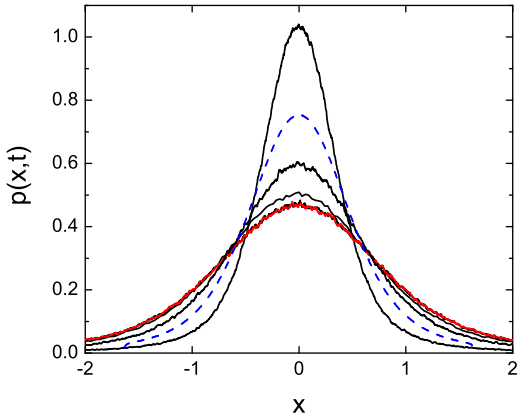


Fig. 4. Time evolution of the probability distribution for the system with linear force, Eq. (12), calculated for the following times: 1, 2, 3, 5 (black solid lines from top to bottom). The case $t = 10$, which corresponds to the stationary solution, is marked by (red) solid line. The stationary solution which is predicted by the adiabatic approximation, Eq. (25), is shown as (blue) dashed line. The other parameters: $\alpha = 1.5$, $\lambda = 1$ and $\gamma = 1$.

To estimate the dependence $\sigma(\gamma)$ the characteristic function $\exp(-\sigma(t)|k|^\alpha)$ was evaluated. Results are presented in Fig. 5. The distribution very slowly converges with γ to the white-noise value whereas it shrinks to the delta function for $\gamma \to 0$.

The adiabatic approximation in the case of the normally distributed noise was discussed in Ref. [35]; we apply a similar procedure. Combination of equations (12) yields a single second order stochastic equation

$$\ddot{x}(t) = -(\lambda + \gamma)\dot{x}(t) - \lambda\gamma x(t) + \gamma\dot{L}(t)\,. \tag{24}$$

One can demonstrate, by introducing a new time variable $t' = \sqrt{\gamma}t$, that the term $\ddot{x}$ is small both for $\gamma \to 0$ and ∞. Therefore, Eq. (24) can be

approximated by the following equation

$$\dot{x}(t) = -\lambda c_\gamma x(t) + c_\gamma \dot{L}(t)\,, \tag{25}$$

where $c_\gamma = 1/(1+\lambda/\gamma)$. The corresponding fractional Fokker–Planck equation is analogous to Eq. (3) and it can be easily solved. Fourier transform of the solution is $\widetilde{p}_{\rm a}(k,t) = \exp(-\sigma_{\rm a}(t)|k|^\alpha)$, where the apparent width

$$\sigma_{\rm a}(t) = \frac{c_\gamma^\alpha D}{\alpha\lambda}\left(1 - e^{-\alpha\lambda t}\right)\,. \tag{26}$$

The adiabatic solution, $p_{\rm a}(x,t)$, converges with time to the steady state and it coincides with the uncorrelated process in the limit $\gamma \to \infty$; Eq. (26) implies that $\sigma_{\rm a}$ rises with γ. Eq. (25) is exact both for $\gamma \to 0$ — when the delta function is the solution — and in the limit $\gamma \to \infty$ (the Smoluchowski limit). For intermediate values of γ, one can expect that Eq. (25) is a good approximation on time scales $t > 1/(\lambda+\gamma)$ and at distances $\gg D^{-1/2}/(\gamma^{1/2}+\lambda\gamma^{-1/2})$ [3].

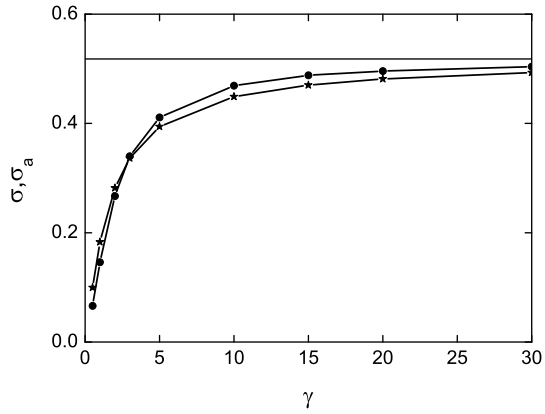


Fig. 5. Width parameter σ, evaluated from the characteristic function for $t = 1$, as a function of memory parameter γ (points). Results of the adiabatic approximation, Eq. (26), are marked by stars. The parameters are: $\alpha = 1.5$ and $\lambda = 1$. Horizontal line marks the white noise limit.

The width parameter $\sigma(t)$ for the exact solution is compared with $\sigma_{\rm a}$, predicted by Eq. (26), in Fig. 5. Some differences are visible but qualitative agreement of the functions $\sigma(\gamma)$ for both cases is good in the entire range of presented γ values. In general, however, discrepancies may be more pronounced. For example, the adiabatic approximation underestimates the width of the steady-state distribution for $\gamma = 1$, which is shown in Fig. 4, by a factor of two (0.24 *versus* 0.48).

4. Summary and conclusions

We have studied the linear dynamical systems which are driven by the additive, non-white Lévy noise. That noise is modelled by a concrete, simple stochastic process, OULP. Then the system is defined in terms of two Langevin equations. OULP reveals the memory effects, as for the ordinary Ornstein–Uhlenbeck process, but their quantitative description is more difficult because of the divergent variance. We have presented a numerical example which demonstrates that the renormalised autocorrelation function $G(t)$ can be useful as a measure of the memory loss; it falls exponentially with time for any stability index α. The same result was obtained for the ensemble-averaged autocorrelation function on the assumption that the Lévy distribution is truncated.

In the absence of any deterministic force, the non-Markovian problem resolves itself to the Wiener–Lévy process (correlated Lévy motion). The resulting probability distribution has the Lévy shape, with parameter α, and it converges with time to that for the uncorrelated case. Correlation time $\tau_{\mathrm{c}} = 1/\gamma$ determines the distribution width: the larger τ_{c}, the narrower the distribution. The case of the constant force f_0 is similar; shape and width of the distribution is the same but the time-dependent shift $f_0 t$ emerges.

Solution for the case of the linear force converges with time to the steady state, as for the white-noise problem, and its shape is Lévy with parameter α. Inclusion of the finite correlation time narrows the distribution, analogously to the case without a force. The above observations agree with the adiabatic approximation approach. That method deals with a corresponding, effective white-noise process and resolves itself to the Langevin equation of the first order. It is supposed to be accurate if γ is sufficiently large or if $\gamma \to 0$. For intermediate values of γ, overall predictions of the adiabatic approximation in respect to the distribution shape and its dependence on γ are still correct, nevertheless some quantitative discrepancies have been found.

Appendix

In Appendix, we solve the fractional Fokker–Planck equation, Eq. (15), by means of the method of characteristics.

First, we put the equation into the form

$$|\kappa|^{-\alpha}\frac{\partial}{\partial t}\widetilde{p}(k,\kappa,t) - \gamma(k-\kappa)|\kappa|^{-\alpha}\frac{\partial}{\partial \kappa}\widetilde{p}(k,\kappa,t) = -D\widetilde{p}(k,\kappa,t)\,. \tag{A.1}$$

Eq. (A.1) is the linear partial differential equation of the first order with only two variables, t and κ, since k can be regarded as a constant parameter. The equation can be handled by the method of characteristics [36]. The method consists in reducing the problem to solution of a system of ordinary

differential equations (characteristic equations). Those equations determine variables t, ξ and z, as functions of parameters s and r, on a characteristic curve. They are of the form

$$\begin{aligned} \frac{dt(r,s)}{ds} &= |\kappa|^{-\alpha}\,, \\ \frac{d\kappa(r,s)}{ds} &= -\gamma(k-\kappa)|\kappa|^{-\alpha}\,, \\ \frac{dz(r,s)}{ds} &= -Dz \end{aligned} \tag{A.2}$$

with the initial conditions

$$\begin{aligned} t(r,0) &= 0\,, \\ \kappa(r,0) &= r\,, \\ z(r,0) &= 1\,; \end{aligned} \tag{A.3}$$

the third condition reflects the requirement that $p(x,\xi,0)$ is to be the delta function in the variable ξ. We must solve the system (A.2) and then eliminate the parameters $r(t,\kappa)$ and $s(t,\kappa)$. The final solution of Eq. (A.1) is given by $\widetilde{p}(k,\kappa,t) = z(r,s)$. Combination of the first and second equation gives the relation between t and κ on the characteristic curve: $t = \ln[(\kappa-k)/(r-k)]/\gamma$, where the initial conditions (A.3) were taken into account. The above relation determines the parameter r

$$r(t,\kappa) = k - (k-\kappa)e^{-\gamma t}\,. \tag{A.4}$$

Integration of the third equation (A.2) is straightforward, $z(r,s) = e^{-Ds}$, and s, as a function of the variables κ and t, follows from the second equation

$$s(t,\kappa) = \frac{1}{\gamma}\int_r^\kappa \frac{|\kappa'|^\alpha}{\kappa'-k}d\kappa'\,. \tag{A.5}$$

The final solution reads

$$\widetilde{p}(k,\kappa,t) = e^{-Ds}\,, \tag{A.6}$$

where s is given by Eq. (A.5). The solution (A.6) can be verified by a direct inserting into Eq. (A.1) and applying the Leibniz rule for differentiation of the integral.

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