MEASURES OF DEPENDENCE FOR ORNSTEIN–UHLENBECK PROCESSES WITH TEMPERED STABLE DISTRIBUTION

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In this paper we investigate the dependence structure for Ornstein–Uhlenbeck process with tempered stable distribution that is natural extension of the classical Ornstein–Uhlenbeck process with Gaussian and α-stable behavior. However, for the α-stable models the correlation is not defined, therefore in order to compare the structure of dependence for Ornstein–Uhlenbeck process with tempered stable and α-stable distribution, we need another measures of dependence defined for infinitely divisible processes such as Lévy correlation cascade or codifference. We show that for analyzed tempered stable process the rate of decay of the Lévy correlation cascade is different than in the stable case, while the codifference of the α-stable Ornstein–Uhlenbeck process has the same asymptotic behavior as in tempered stable case. As motivation of our study we calibrate the Ornstein–Uhlenbeck process with tempered stable distribution to real financial data.

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1. Introduction

In modern finance theorems continuous time models play a crucial role, because they allow handling unequally spaced data and even high frequency data, which are realistic for liquid financial markets. Probably the most famous example is the Ornstein–Uhlenbeck process that was originally introduced by Uhlenbeck and Ornstein [1] as a suitable model for the velocity process in Brownian diffusion. In other words, this process provides a stationary solution for the classical Klein–Kramers dynamics [2, 3]. It is an alternative to the classical Brownian motion in the case when some kind of mean reverting tendency is observed in the real data. The Ornstein–Uhlenbeck process has found many applications to the real financial data such as interest rates, currency exchange rates, and commodity prices. In finance it is best known in connection with the Vasiček interest rate model [4].

Many asset pricing models (such as classical Vasiček model) assume that the analyzed data have normal distribution. Unfortunately, the assumption of normality is unsatisfactory for many observed data. One approach is to replace the Brownian motion by a heavier tailed Lévy process. Many studies have shown that heavy-tailed distributions allow for modeling different kind of phenomena when the assumption of normality for the observations does not seem to be reasonable. Especially α-stable (stable) distributions have found many practical applications, for instance in finance [5], physics [6,7,8] and electrical engineering [9]. The Ornstein–Uhlenbeck process with α-stable distribution was analyzed in [10, 11] as a suitable model for description of different financial data.

However, the stable processes have infinite moments of the second or higher orders, therefore, there appear many problems especially in applications. In order to overcome this drawback, the processes with tempered stable distribution (and their modifications) have been introduced. There are many types of such processes, for example, classical tempered stable and modified tempered stable models, see [11,12,13]. The classical tempered stable models are known as Truncated Lévy Flight (see for instance [14,15,16,17]), KoBol [18] and CGMY processes [19, 20]. They found many applications especially in finance, see [21,22], biology [23], physics to description of anomalous diffusion and relaxation phenomena [24,25], turbulence [26] and in plasma physics [27], see also [28,29].

In this paper we consider the Ornstein–Uhlenbeck process with tempered stable distribution that is a natural extension of the classical (Gaussian) and α-stable Ornstein–Uhlenbeck process. However, for the stable models the correlation is not defined, therefore in order to compare the structure of dependence of the tempered and α-stable Ornstein–Uhlenbeck process, we examine another measures of dependence defined for infinitely divisible processes such as Lévy correlation cascade [30]. This is a useful tool for studying the ergodic and long-memory properties [31]. We study the asymptotic behavior of the mentioned measure for considered process and compare it to the α-stable case. As a main result, we show that the rate of decay of the Lévy correlation cascade in the considered case is different than in stable model. Moreover, for the tempered stable and α-stable Ornstein–Uhlenbeck process we compare alternative measure called codifference [32, 33, 34, 35], based on the Fourier transform of the distribution, that can be also useful for studying the long-range dependence [31]. We prove that for thease two analyzed processes this measure indicates the same asymptotic behavior.

The rest of the paper is organized as follows: in Section 2 we give the definition of Ornstein–Uhlenbeck process with tempered stable distribution. In order to present the motivation of the paper in Section 3, we calibrate the tempered stable Ornstein–Uhlenbeck process to the returns of closing price for Switzerland government bonds. Then, in Section 4, we review the measures of dependence for infinitely divisible processes: the codifference and the alternative measure called the Lévy correlation cascade. The measures of dependence for considered processes are studied in Section 5 and their asymptotic behavior is examined.

2. Ornstein–Uhlenbeck process with tempered stable distribution

The classical Ornstein–Uhlenbeck process is known as the mean-reverting process (*i.e.* over time, the process tends to drift towards its long-term mean) and it is given by the following stochastic differential equation

$$dY(t) = a(\mu - Y(t))dt + \sigma dB(t)\,, \tag{1}$$

where $\{B(t)\}_{t\geq 0}$ denotes the Brownian motion [36]. The parameter μ represents the long-term mean, a — the speed of mean-reversion, and σ — the volatility. If we extend the Brownian motion for the set $(-\infty, 0)$ and take the simplistic assumption $\mu = 0$ and $\sigma = 1$, then we can write the unique solution of equation (1)

$$Y(t) = \int_{-\infty}^{t} e^{-a(t-u)} dB^*(u)\,, \tag{2}$$

with $\{B^*(t)\}_{t\in R}$ — the Brownian motion extended to the set $(-\infty, 0)$, *i.e.*

$$B^*(t) = B(t)I_{(0,\infty)}(t) - M(-t)I_{(-\infty,0)}(t)\,, \qquad -\infty < t < \infty\,,$$

where $\{B(t),\ t \geq 0\}$ and $\{M(t),\ t \geq 0\}$ are two independent Brownian motions [37, 38].

An extension of the process (2) is an α-stable Ornstein–Uhlenbeck system defined as follows [10, 37]

$$Y(t) = \int_{-\infty}^{t} e^{-a(t-u)} dL_\alpha(u)\,, \tag{3}$$

where $\{L_\alpha(t)\}$ is a Lévy process with α-stable increments extended to the set $(-\infty, 0)$. Stable Ornstein–Uhlenbeck processes were analyzed for instance in [5, 10, 39] as a models describing real financial data.

In this paper we propose an extension of the mentioned Ornstein–Uhlenbeck processes and substitute the Lévy process with α-stable (or Gaussian) distribution extended to the set $(-\infty, 0)$ by the Lévy process with tempered stable increments. In this case the Ornstein–Uhlenbeck process can be represented by the following stochastic integral

$$Y(t) = \int_{-\infty}^{t} e^{-a(t-u)} dT(u) , \tag{4}$$

where $\{T(u)\}$ is a Lévy process with tempered stable increments extended to the set $(-\infty, 0)$.

An infinitely divisible distribution is called a tempered stable with parameters $1 < \alpha < 2$, $\lambda > 0$ and $C > 0$ if it has no Gaussian component and its Lévy measure is given by [40]

$$v(dx) = \frac{Ce^{-\lambda x}}{x^{1+\alpha}} 1_{x>0} dx . \tag{5}$$

In this case the Fourier transform ϕ_T of the tempered stable random variable T is given by the following formula [40]

$$\phi_T(u) = E \exp(iuT) = \exp\left(C\left((\lambda - iu)^\alpha - \lambda^\alpha + iu\alpha\lambda^{\alpha-1}\right)\right) . \tag{6}$$

When $\lambda = 0$, then the random variable T with the Fourier transform given in (6) has an α-stable distribution with the following values of the parameters

$$\alpha,\ \beta = 1 , \qquad \sigma = (-C\cos(\pi * \alpha/2))^{1/\alpha} , \qquad \mu = 0 .$$

By using the connection between the tempered stable and corresponding α-stable distribution it is easy to find the relation between the probability density functions (p.d.f.). Let $p_{\mathrm{T}}(x)$ and $p_{\mathrm{S}}(x)$ be p.d.f. of tempered stable with parameters α, λ, C and α-stable random variable with appropriate values of the parameters, respectively, then for $\alpha \neq 1$ we have

$$p_{\mathrm{T}}(x) = e^{-\lambda x + (\alpha-1)C\lambda^\alpha} p_{\mathrm{S}}\left(x - C\alpha\lambda^{\alpha-1}\right) .$$

In Fig. 1 we present such two p.d.f. functions with the the following parameters of the tempered stable distribution $\alpha = 1.2$, $\lambda = 1.3$ and $C = 1$.

The main properties as well as the procedures of simulation of the considered tempered stable distribution one can find for instance in [12] and [40].

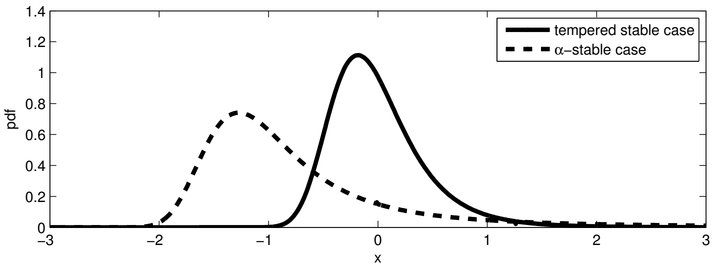

Fig. 1. The probability density functions for tempered stable and corresponding α-stable distributions. The parameters of tempered stable distribution are as follows: $\alpha = 1.2$, $\lambda = 1.3$ and $C = 1$.

3. Motivation

In order to present the motivation for using the Ornstein–Uhlenbeck process with tempered stable distribution, we analyze the real financial data that describe the closing price of Switzerland government bonds (CH644842) quoted daily in Swiss franc (CHF) between 02.01.1995 and 07.06.2010 (3288 observations). In Fig. 2 we present the analyzed data set. In order to re-

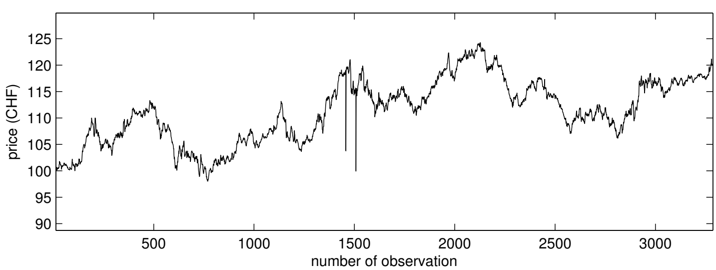

Fig. 2. The real data time series — the closing price of Switzerland government bonds quoted daily (in CHF) between 02.01.1995 and 07.06.2010.

move the deterministic trend before the further, analysis we differentiate the analyzed data. The five statistical tests based on the empirical distribution function, precisely described in [41], reject the hypothesis that the data can be described by Gaussian and strictly α-stable distribution. Therefore, as an alternative we propose to use the tempered stable distribution described in previous section. The autocorrelation function (ACF) and partial autocorrelation function (PACF) indicate the data can be described by autoregressive model of the order of 1 (AR(1)), that is a discrete version of the Ornstein–Uhlenbeck process. In this case (under the simplistic assumption $\mu = 0$ and $\sigma = 1$) the discretization of tempered stable Ornstein–Uhlenbeck model given in (4) has the following form

$$Y(t) = (1-a)Y(t-1) + \xi(t)\,, \qquad t = 1, 2, \ldots\,, \tag{7}$$

where $\{\xi(t)\}$ is a sequence of independent identically distributed (i.i.d.) random variables with tempered stable distribution with parameters α, λ and C.

To estimate the parameters a, α, λ, C we apply the following scheme:

- For given data by using the Yule Walker method [42] we estimate the parameter a in model (7).
- By using the obtained in point 1 estimator of a, namely $\hat{a}$, we calculate the sequence of innovations $\{\xi(t)\}$ (that represent the increments of the process $\{T(t)\}$) from the following equation

$$\xi(t) = Y(t) - (1-\hat{a})Y(t-1)\,, \qquad t = 1, 2, \ldots\,.$$

- Under the assumption the sequence $\{\xi(t)\}$ constitutes independent identically distributed random variables with tempered stable distribution we estimate the parameters α, λ and C by using the method of moments [43].

As a result we obtain

$$\hat{a} = 1.3037\,, \qquad \hat{\alpha} = 1.8731\,, \qquad \hat{\lambda} = 0.0358\,, \qquad \hat{C} = 0.1519\,.$$

In order to confirm that the estimated tempered stable distribution is a proper distribution of the residual series $\{\xi(t)\}$, in Fig. 3 we present the empirical p.d.f. of the residuals based on the kernel estimation method and theoretical p.d.f. calculated on the basis of the estimated parameters.

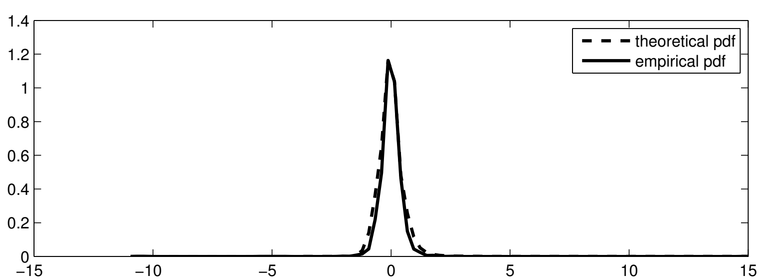

Fig. 3. The empirical p.d.f. of the residuals and theoretical p.d.f. calculated on the basis of the estimated parameters.

4. Measures of dependence for infinitely divisible processes

One of the important tool providing to construction of an appropriate mathematical model for the real-life data is correlation. However, for the large class of infinitely divisible processes, namely the strictly α-stable, the correlation is not defined. Therefore, Eliazar and Klafter in [30] introduced a new measure of dependence that is defined for infinitely divisible stochastic processes $\{Y(t),\ t \in R\}$ with the following integral representation

$$Y(t) = \int_X K(t,x)M(dx)\,,$$

where M is an independently scattered infinitely divisible random measure on some measurable space S with control measure m, see also [31].

The new measure was introduced as a concept of correlation cascades, which is a promising tool for exploiting the properties of the Poissonian part of $Y(t)$ and the dependence structure of this stochastic process. The Lévy correlation cascade is defined as follows [30, 31]

$$C_l(t_1, t_2, \ldots, t_n) = \int_X \Lambda\left(\frac{l}{\min\{K(t_1,x), \ldots, K(t_n,x)\}}\right) m(dx)\,, \tag{8}$$

where the tail function Λ is given by

$$\Lambda(l) = \int_{|x|>l} v(dx) \tag{9}$$

and v is a Lévy measure of the the process $\{Y(t)\}$.

Many significant properties and results connected with the Lévy correlation cascade for infinitely divisible processes are presented in [30] and [31]. We only mention here that the function $C_l(t_1, t_2, \ldots, t_n)$ tells us, how dependent the coordinates of the vector $(Y(t_1), Y(t_2), \ldots, Y(t_n))$ are. Therefore, $C_l(t_1, t_2, \ldots, t_n)$ can be considered as an appropriate measure of dependence for the Poissonian part of the infinitely divisible process [31]. In particular, the function $C_l(t_1, t_2)$ can serve as an analogue of the covariance, and the function

$$r_l(t_1, t_2) = \frac{C_l(t_1, t_2)}{\sqrt{C_l(t_1)C_l(t_2)}} \tag{10}$$

can play the role of the correlation function. The ergodic property, such as ergodicity, weak mixing and mixing, of a stationary infinitely divisible processes can be described in the language of the Lévy correlation cascade, therefore, this measure is a promising tool for studying the dependence structure for this large class of processes. Moreover, the Lévy correlation cascade is useful for detecting the long memory behavior, especially for the processes with infinite second moment for which the correlation is not defined. Namely a stationary process is said to have a long memory in terms of the Lévy correlation cascade if the following condition holds

$$\sum_{n=0}^{\infty} r_l(n,0) = \infty\,,$$

where $r_l(n,0)$ is given in (10), [31].

When the considered process is a moving-average with respect to the Lévy process $\{Z(t)\}$, *i.e.* it takes the following form

$$Y(t) = \int_{-\infty}^{t} f(t-u) Z(du)\,,$$

then the Lévy correlation cascade is defined as follows

$$C_l(0,t) = \int_{t}^{\infty} \Lambda\left(\frac{l}{f(y)}\right) dy\,.$$

The another measure, that is often considered as a tool of the dependence structure description, is the codifference (see for instance [32, 33, 34]). For the stationary infinitely divisible process $\{Y(t)\}$ codifference is defined as follows

$$\begin{aligned} &\mathrm{CD}(t,0) = \mathrm{CD}(Y(t), Y(0)) \\ &= \log E \exp\left(i(Y(t)-Y(0))\right) - \log E \exp\left(iY(t)\right) - \log E \exp\left(-iY(0)\right)\,. \end{aligned} \tag{11}$$

Codifference carries enough information to detect ergodic properties of the process $\{Y(t)\}$. It is also closely related to another measure — dynamical functional used in [44, 45] to investigate the chaotic behavior of the considered process. It is also used to examine the long-range dependence in the case when the correlation function is not defined [32, 46]. The stationary process is said to have a long memory in terms of the codifference function if the following condition holds

$$\sum_{n=0}^{\infty} \mathrm{CD}(n,0) = \infty\,,$$

where $\mathrm{CD}(n,0)$ is defined in (11). According to the results of the next section, the rates of decay of $C_l(t,0)$ and $\mathrm{CD}(t,0)$ computed for the same stationary process can be different. Therefore, it is hard to compare two definitions of long memory based on the Lévy correlation cascade and the codifference [31].

More properties of the codifference one can find in [32]. Let us mention here that there is a relationship between the asymptotic behavior of Lévy correlation cascade and codifference, namely for the stationary infinitely divisible process $\{Y(t)\}$ with the Lévy measure v_0 of $Y(0)$ without atoms in $2\pi Z$, the following two conditions are equivalent (see Theorem 2 in [31])

$$\begin{aligned} \lim_{t\to 0} C_l(t,0) &= 0 \quad \text{for every} \quad l>0\,, \\ \lim_{t\to 0} \mathrm{CD}(t,0) &= 0\,. \end{aligned}$$

In the next section, we examine the Ornstein–Uhlenbeck process with tempered stable distribution in the context of the mentioned measures of dependence. As a main result we show the asymptotic behavior of this process in the language of Lévy correlation cascade and compare it with the α-stable case. Moreover, we show that the codifference of the Ornstein–Uhlenbeck process with α-stable and tempered stable distribution has the same asymptotic properties. The obtained results indicate that the Ornstein–Uhlenbeck process with tempered stable increments does not demonstrate the long-range dependence in the sense of considered measures.

5. Structure of dependence of Ornstein–Uhlenbeck process with α- and tempered stable distribution

5.1. α-stable case

Let us consider the Ornstein–Uhlenbeck process given in (3) with α-stable distribution given in (3). In this case the Lévy measure of the α-stable Lévy process $\{L_\alpha(u)\}$ in equation (3) is given by (see [32])

$$v(dx) = \frac{1_{x>0}}{x^{1+\alpha}}dx\,.$$

Therefore, the Lévy correlation cascade and an analogue of the correlation given in (10) of the α-stable Ornstein–Uhlenbeck process $\{Y(t)\}$ have the following form [31]

$$C_l(0,t) = \frac{1}{a\alpha^2 l^\alpha}e^{-a\alpha t}\,, \qquad r_l(0,t) = e^{-a\alpha t}\,.$$

According to the fact that the Fourier transform of the α-stable random variable S with $\mu = 0$ and $\sigma = 1$ is given by

$$\phi_s(u) = Ee^{iuS} = \exp\left(-|u|^\alpha \left(1 - i\mathrm{sign}(u)\tan\frac{\pi\alpha}{2}\right)\right) .$$

The codifference CD$(t,0)$ of the Ornstein–Uhlenbeck process with α-stable Lévy motion can be expressed as [47]

$$\mathrm{CD}(t,0) = \frac{1 - \tan\frac{\pi\alpha}{2}}{a\alpha}\left(1 + e^{-a\alpha t} - \left|1 - e^{-at}\right|^\alpha\right) ,$$

that for large t gives

$$\mathrm{CD}(t,0) \sim \mathrm{const.}\, e^{-at} .$$

The exact description as well as the main properties and comparison between the Ornstein–Uhlenbeck process with α-stable Lévy motion and discrete autoregressive model in the language of the codifference function are presented in [47].

5.2. Tempered stable case

In the considered tempered stable case the tail function Λ given in (9) takes the following form

$$\Lambda(l) = C\lambda^\alpha \Gamma(-\alpha, \lambda l) ,$$

where $\Gamma(s,t)$ is an incomplete gamma function defined as follows

$$\Gamma(s,t) = \int_t^\infty x^{s-1}e^{-x}dx . \tag{12}$$

Using the form of the Λ function we obtain the following form of the Lévy correlation cascade for the tempered stable Ornstein–Uhlenbeck process defined in (4)

$$C_l(t,0) = \int_t^\infty C\lambda^\alpha \Gamma(-\alpha, \lambda l e^{au})du .$$

Let us consider the asymptotic behavior of such function for $t \to \infty$. Because the incomplete gamma function has the following property

$$\frac{\Gamma(s,x)}{x^{s-1}e^{-x}} \to 1 \quad \text{for} \quad x \to \infty$$

then for large t we obtain

$$C_l(t,0) \sim C\lambda^\alpha \int_t^\infty (\lambda l e^{au})^{-\alpha-1} \exp\{-\lambda l e^{au}\}\, du = \frac{C\lambda^\alpha}{a} \int_{\lambda l e^{at}}^\infty w^{-\alpha-2} e^{-w} dw$$
$$= \frac{C\lambda^\alpha}{a} \Gamma\left(-\alpha-1, l e^{at}\right) \sim \frac{C\lambda^\alpha}{a} \left(\lambda l e^{at}\right)^{-\alpha-2} \exp\left\{-\lambda l e^{at}\right\} \qquad (13)$$

that gives

$$r_l(t) = r_l(0,t) \sim e^{-at(\alpha+2)} \exp\left\{-\lambda l e^{at}\right\}\,.$$

This result indicates that the Ornstein–Uhlenbeck process with tempered stable distribution does not have long memory dependence in terms of Lévy correlation cascade. Because we have proved the financial time series examined in Section 3 can be modeled by using the considered process, therefore we can conclude the analyzed real data cannot be considered as process with the long memory behavior in the sense of Lévy correlation cascade.

For the tempered stable Ornstein–Uhlenbeck process the codifference defined in (11) is given by

$$\mathrm{CD}(t,0) = C \int_{-\infty}^0 \left(\lambda + i e^{as}\left(1-e^{-at}\right)\right)^\alpha$$
$$- \left(\lambda - i e^{-a(t-s)}\right)^\alpha ds - C \int_{-\infty}^0 (\lambda + i e^{as})^\alpha - \lambda^\alpha ds\,.$$

By using the following formula

$$(a+b)^\alpha = \sum_{k=0}^\infty \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} a^{\alpha-k} b^k$$

we can obtain

$$\mathrm{CD}(t,0) = C \int_{-\infty}^0 \sum_{k=1}^\infty i e^{ask} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)ak}$$
$$\times \left(\left(\lambda - i e^{-a(t-s)}\right)^{\alpha-k} - \lambda^{\alpha-k}\right) ds\,.$$

When $k > 0$, then the function

$$\left(\lambda - i e^{-a(t-s)}\right)^{\alpha-k} - \lambda^{\alpha-k}\,,$$

for large t, behaves like $i(\alpha - k)e^{-a(t-s)}$. Therefore, finally we obtain

$$\mathrm{CD}(t,0) \sim \text{const.}\, e^{-at}\,.$$

The result indicates the codifference of the Ornstein–Uhlenbeck process with tempered stable distribution have the same asymptotic behavior as in the α-stable case. This result also indicates that the real time series examined in Section 3 (and modeled by using the considered process) cannot be considered as a process with the long-range dependence in a sense of the codifference measure.

6. Conclusion

In this paper we have examined the structure of dependence of Ornstein–Uhlenbeck process with tempered stable distribution in the language of Lévy correlation cascade and codifference. As a main result, we have showed that rate of decay of the Lévy correlation cascade in the tempered stable case is different than in stable models while the codifference has the same asymptotic behavior. The obtained asymptotic results indicate that the Ornstein–Uhlenbeck process with tempered stable increments does not demonstrate long-range dependence in the sense of considered measures. As the motivation of our study we have presented the analysis of the real financial data that can be described by using the Ornstein–Uhlenbeck process with tempered stable distribution.

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