TWO-MODE SUPERPOSITION COHERENT STATES: ENTANGLEMENT AND NONCLASSICALITY

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(Received September 22, 2011; revised version received October 10, 2011)

In this paper, we have analyzed the entanglement and nonclassicality of two-mode superposition coherent states based on two coherent states shifted in phase by $\pi/2$. Here, the relative phase of the superposition will be taken equal to the phase shift between the two coherent states *i.e.* $\phi = \pi/2$. Entanglement-sensitivity is investigated and it was found that out of four, only one state is maximally entangled. Moreover, it is also revealed that the considered states have stronger nonclassical features than those of even–odd entangled coherent states.

DOI:10.5506/APhysPolB.42.2077
PACS numbers: 03.67.Ac, 03.65.Ud, 42.50.–p

1. Introduction

Nonclassical features of two-mode superposition coherent states have much importance due to their potential applications in quantum information processing [1,2,3]. There are various manifestation to study the nonclassical features of two-mode superposition coherent states such as oscillatory and sub-Poissonian photon statistics [4,5], violation of Cauchy–Schwartz inequality [6], and the degree of quadrature squeezing [7]. In fact, the entangled coherent states are more robust against photon absorption than the standard bi-photon polarization entangled states [8]. The nonclassical features and entanglement of entangled coherent states have already been discussed in [9, 10] and a variety of schemes have been anticipated to generate such states [11,12].

A new family of Schrödinger cat state can be defined as [13]

$$|\psi\rangle = \frac{N}{\sqrt{2}}\left(|\alpha\rangle + e^{i\phi}\,|i\alpha\rangle\right) , \qquad (1)$$

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where $N^2 = [1 + e^{-|\alpha|^2}\cos(\phi - |\alpha|^2)]^{-1}$. Nonclassical features of the state given in Eq. (1) have already been discussed in [13]. Recently, nonclassical features of two-mode superposition coherent states based on Eq. (1) have been discussed in [14]. The authors considered the relative phase equal to the average photon number and found that the states have strong nonclassical features. Following their work, we make a different assumption that the relative phase will be equal to the phase difference between coherent states *i.e.* $\phi = \pi/2$.

Hence, by fixing the relative phase equal to the phase shift, the quasi-Bell states can be written as

$$\begin{aligned} |\phi_1\rangle_{12} &= \frac{ig_1}{\sqrt{2}}\left(|\alpha\rangle_1 |i\alpha\rangle_2 + |i\alpha\rangle_1 |\alpha\rangle_2\right) , \\ |\phi_2\rangle_{12} &= \frac{ig_2}{\sqrt{2}}\left(|\alpha\rangle_1 |i\alpha\rangle_2 - |i\alpha\rangle_1 |\alpha\rangle_2\right) , \\ |\phi_3\rangle_{12} &= \frac{g_3}{\sqrt{2}}\left(|\alpha\rangle_1 |\alpha\rangle_2 - |i\alpha\rangle_1 |i\alpha\rangle_2\right) , \\ |\phi_4\rangle_{12} &= \frac{g_4}{\sqrt{2}}\left(|\alpha\rangle_1 |\alpha\rangle_2 + |i\alpha\rangle_1 |i\alpha\rangle_2\right) , \end{aligned} \tag{2}$$

where $g^2_{1,2} = \frac{1}{1 \pm e^{-2|\alpha|^2}}$ and $g^2_{3,4} = \frac{1}{1 \mp e^{-2|\alpha|^2}\cos(2|\alpha|^2)}$.

The quasi-Bell states for even–odd entangled coherent states have already been defined in [8], such that

$$\begin{aligned} |\psi_1\rangle_{12} &= h_1\left(|\alpha\rangle_1 |-\alpha\rangle_2 + |-\alpha\rangle_1 |\alpha\rangle_2\right) , \\ |\psi_2\rangle_{12} &= h_2\left(|\alpha\rangle_1 |-\alpha\rangle_2 - |-\alpha\rangle_1 |\alpha\rangle_2\right) , \\ |\psi_3\rangle_{12} &= h_3\left(|\alpha\rangle_1 |\alpha\rangle_2 + |-\alpha\rangle_1 |-\alpha\rangle_2\right) , \\ |\psi_4\rangle_{12} &= h_4\left(|\alpha\rangle_1 |\alpha\rangle_2 - |-\alpha\rangle_1 |-\alpha\rangle_2\right) , \end{aligned} \tag{3}$$

where $h^2_{1,3} = \frac{1}{2+2e^{-4|\alpha|^2}}$ and $h^2_{2,4} = \frac{1}{2-2e^{-4|\alpha|^2}}$.

This paper is organized in the following way: in Sec. 2, we will measure the amount of entanglement. In Sec. 3, we analyze the oscillatory and sub-Poissonian photon statistics. In Sec. 4, two-mode quadrature squeezing have been measured. In Sec. 5, we discuss the violation of Cauchy–Schwartz inequality and finally in Sec. 6, we conclude the results.

2. Amount of entanglement

Entanglement is the most strangest and enthralling feature of quantum theory. It is the essential resource for the realization of quantum information processing [16, 17].

The reduced density operator is so useful for the analysis of composite quantum systems because the partial trace operation is a unique operation which gives rise to the correct description of observable quantities for subsystems of a composite system [18]. The reduces density operator can be defined as

$$\rho_1^{(i)} = \mathrm{Tr}_2 \left|\phi_i\right\rangle_{12} \left\langle\phi_i\right| . \tag{4}$$

Hence, in order to obtain the amount of entanglement, the reduces density operator for considered quasi-Bell states can be calculated as

$$\begin{aligned}
\rho_1^{(1)} &= \frac{g_1^2}{2}\left[|\alpha\rangle\langle\alpha| + |i\alpha\rangle\langle i\alpha| + e^{-(1+i)|\alpha|^2}|\alpha\rangle\langle i\alpha| + e^{-(1-i)|\alpha|^2}|i\alpha\rangle\langle\alpha|\right] ,\\
\rho_1^{(2)} &= \frac{g_2^2}{2}\left[|\alpha\rangle\langle\alpha| + |i\alpha\rangle\langle i\alpha| - e^{-(1+i)|\alpha|^2}|\alpha\rangle\langle i\alpha| - e^{-(1-i)|\alpha|^2}|i\alpha\rangle\langle\alpha|\right] ,\\
\rho_1^{(3)} &= \frac{g_3^2}{2}\left[|\alpha\rangle\langle\alpha| + |i\alpha\rangle\langle i\alpha| - e^{-(1-i)|\alpha|^2}|\alpha\rangle\langle i\alpha| - e^{-(1+i)|\alpha|^2}|i\alpha\rangle\langle\alpha|\right] ,\\
\rho_1^{(4)} &= \frac{g_4^2}{2}\left[|\alpha\rangle\langle\alpha| + |i\alpha\rangle\langle i\alpha| + e^{-(1-i)|\alpha|^2}|\alpha\rangle\langle i\alpha| + e^{-(1+i)|\alpha|^2}|i\alpha\rangle\langle\alpha|\right] .
\end{aligned} \tag{5}$$

The Gram matrix $G_{lm} = |_{12}\langle\phi_l||\phi_m\rangle_{12}|$ for the considered states will take the form, such that

$$\begin{pmatrix} 1 & 0 & K & L \\ 0 & 1 & 0 & 0 \\ K & 0 & 1 & M \\ L & 0 & M & 1 \end{pmatrix} ,$$

where

$$\begin{aligned}
K &= 2g_1 g_3 e^{-|\alpha|^2}\sin\left(|\alpha|^2\right) , \qquad L = 2ig_1 g_4 e^{-|\alpha|^2}\cos\left(|\alpha|^2\right) ,\\
M &= 2ig_3 g_4 e^{-2|\alpha|^2}\sin\left(2|\alpha|^2\right) ,
\end{aligned} \tag{6}$$

whereas, the eigenvalues of the reduced density operators can be defined as

$$\lambda_{1,2}^{(1)} = \frac{\left(1 \pm e^{-|\alpha|^2}\right)^2}{2(1+e^{-2|\alpha|^2})} ; \qquad \lambda_{1,2}^{(2)} = \tfrac{1}{2} , \tag{7}$$

$$\lambda_{1,2}^{(3)} = \tfrac{1}{2}\left(1 \pm \sqrt{1-4A_-}\right) ; \qquad \lambda_{1,2}^{(4)} = \tfrac{1}{2}\left(1 \pm \sqrt{1-4A_+}\right) , \tag{8}$$

where

$$A_\pm = \frac{\left(1 - e^{-2|\alpha|^2}\right)^2}{4\left[1 \pm e^{-2|\alpha|^2}\cos(2|\alpha|^2)\right]^2} . \tag{9}$$

To measure the amount of entanglement, we substitute the eigenvalues into the following relation

$$E^{(i)} = -\sum_{m=1,2} \lambda_m^{(i)} \log_2 \lambda_m^i \,. \tag{10}$$

After substitution, one can clearly see that only the state $|\phi_2\rangle_{12}$ is maximally entangled by having 1 ebit of entanglement. Hence, this state can be used for useful quantum information processing.

3. Oscillatory and sub-Poissonian photon statistics

In this section, we will measure the photon-count probability for the considered two-mode superposition coherent states, such that

$$P^{|\phi_1\rangle,|\phi_2\rangle} = |\langle n_1, n_2|\phi_{1,2}\rangle|^2 = \frac{g_{1,2}^2}{2}\left[\frac{e^{-|\alpha|^2}|\alpha|^{2n_1}}{n_1!} \times \frac{e^{-|\alpha|^2}|\alpha|^{2n_2}}{n_2!}\right], \tag{11}$$

$$P^{|\phi_3\rangle,|\phi_4\rangle} = |\langle n_1, n_2|\phi_{3,4}\rangle|^2 = g_{3,4}^2 \frac{e^{-|\alpha|^2}|\alpha|^{2(n_1+n_2)}}{n_1!n_2!} \times \left[1 \mp \cos\left(\frac{\pi}{2}(n_1+n_2)\right)\right]. \tag{12}$$

It is observed that the states $|\phi_3\rangle_{12}$ and $|\phi_4\rangle_{12}$ exhibit oscillatory photon statistics in both modes 1 and 2. Here, in Eq. (12) it is evident that the cosine term is responsible for the oscillations in photon numbers which is a clear signature of nonclassicality.

Now, oscillatory photon statistics for the even–odd entangled coherent states are

$$P^{|\psi_1\rangle,|\psi_2\rangle} = |\langle n_1, n_2|\psi_{1,2}\rangle|^2 = h_{1,2}^2 \frac{e^{-2|\alpha|^2}|\alpha|^{2(n_1+n_2)}}{n_1!n_2!} \left|(-1)^{n_2} \pm (-1)^{n_1}\right|^2, \tag{13}$$

$$P^{|\psi_3\rangle,|\psi_4\rangle} = |\langle n_1, n_2|\psi_{3,4}\rangle|^2 = h_{3,4}^2 \frac{e^{-2|\alpha|^2}|\alpha|^{2(n_1+n_2)}}{n_1!n_2!} \left|1 \pm (-1)^{n_1+n_2}\right|^2. \tag{14}$$

However, in the case of even–odd entangled coherent states, only the states $|\psi_3\rangle$ and $|\psi_4\rangle$ exhibit oscillatory photon statistics which is, however, less manifest as compared to the states $|\phi_3\rangle$ and $|\phi_4\rangle$.

The sub-Poissonian photon statistics in each mode is defined by $Q_i < 1$, where

$$Q_i = \frac{\left\langle a_i^{\dagger 2} a_i^2 \right\rangle - \left\langle a_i^\dagger a_i \right\rangle^2}{\left\langle a_i^\dagger a_i \right\rangle} \qquad (i = 1, 2)\,. \tag{15}$$

Hence, the Mandal Q parameters for all states $\left\{|\phi_j\rangle_{12}\right\}$ are given by

$$Q_{1(2)}^{|\phi_1\rangle,|\phi_2\rangle} = -\frac{|\alpha|^2 e^{-4|\alpha|^2}}{1 \pm e^{-2|\alpha|^2}}\,, \tag{16}$$

$$Q_{1(2)}^{|\phi_3\rangle,|\phi_4\rangle} = -\frac{g_{3,4}^2|\alpha|^2 e^{-2|\alpha|^2}\left[e^{-2|\alpha|^2} \pm 2\sin\left(2|\alpha|^2\right)\right]}{\left(1 + e^{-2|\alpha|^2}\sin\left(2|\alpha|^2\right)\right)}\,. \tag{17}$$

By plotting Q parameter, it is observed that the considered states do exhibit sub-Poissonian photon statistics in both modes, as shown in Fig. 1.

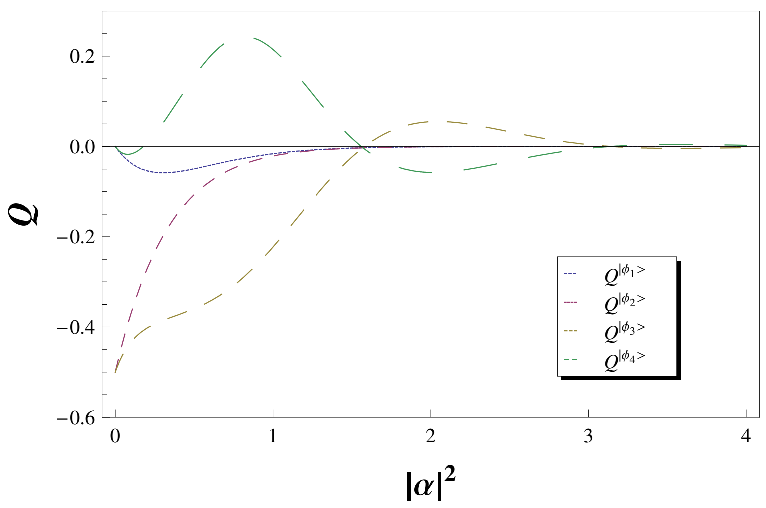

Fig. 1. Q parameter *versus* $|\alpha|^2$ for the considered states.

The Q parameter for the even–odd entangled coherent states is given as

$$Q_{A(B)}^{|\psi_1\rangle,|\psi_2\rangle} = \pm\frac{4|\alpha|^2 e^{-4|\alpha|^2}}{1 - e^{-8|\alpha|^2}}\,, \tag{18}$$

$$Q_{A(B)}^{|\psi_3\rangle,|\psi_4\rangle} = \pm\frac{4|\alpha|^2 e^{-4|\alpha|^2}}{1 - e^{-8|\alpha|^2}}\,, \tag{19}$$

whereas, in the case of even–odd entangled coherent states only two states exhibit sub-Poissonian photon statistics, as shown in Fig. 2.

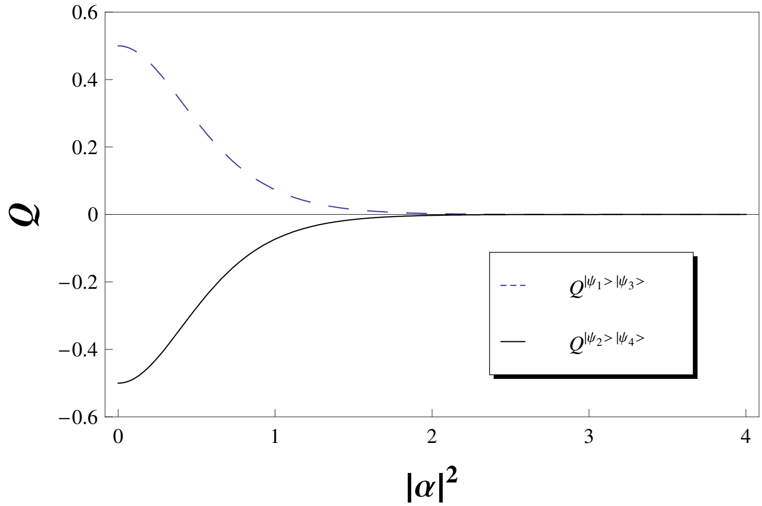

Fig. 2. Q parameter *versus* $|\alpha|^2$ for the even–odd entangled coherent states.

4. Two-mode quadrature squeezing

Squeezing is the key ingredient to measure the nonclassical light. Moreover, squeezed light has wide applications in quantum information processing, specially in quantum communication. In order to attain the squeezing effect for the two-mode superposition coherent states, we have quadrature operators such that

$$U_1 = \frac{1}{2\sqrt{2}}\left(a_1 + a_1^\dagger + a_2 + a_2^\dagger\right) , \tag{20a}$$

$$U_2 = \frac{1}{2\sqrt{2i}}\left(a_1 - a_2^\dagger + a_2 - a_2^\dagger\right) . \tag{20b}$$

Condition for existing two-mode squeezing is

$$(\Delta U_j)^2 < \tfrac{1}{4} , \qquad j = 1, 2 \tag{21}$$

and degree of squeezing exists in the limit

$$D_j = 4\left(\Delta U_j\right)^2 - 1 < 0 , \qquad j = 1, 2 . \tag{22}$$

The degree of squeezing for the $|\phi_3\rangle$ and $|\phi_4\rangle$ can be written as

$$D_{1,2}^{|\phi_3\rangle} = g_3^2 4|\alpha|^2 - 2[x_1 \pm x_2]^2 \left[1 + g_3^2 e^{-4|\alpha|^2} \sin^2\left(2|\alpha|^2\right)\right] , \tag{23}$$

$$D_{1,2}^{|\phi_4\rangle} = g_4^2 4|\alpha|^2 - 2[x_1 \pm x_2]^2 \left[1 + g_4^2 e^{-4|\alpha|^2} \sin^2\left(2|\alpha|^2\right)\right] . \tag{24}$$

The degree of squeezing for the even–odd entangled coherent states is found to be

$$D_{1,2}^{|\psi_3\rangle} = \pm 4\left(x_1^2 - x_2^2\right) + h_3^2 4|\alpha|^2 \left(1 - e^{-4|\alpha|^2}\right), \tag{25a}$$

$$D_{1,2}^{|\psi_4\rangle} = \pm 4\left(x_1^2 - x_2^2\right) + h_4^2 4|\alpha|^2 \left(1 + e^{-4|\alpha|^2}\right), \tag{25b}$$

where $x_1 = (\alpha + \alpha^*)/2$, $x_2 = (\alpha - \alpha^*)/2i$. Squeezing effects can be analyzed as a function of x_1 and x_2 and the squeezing region can only be analyzed in the negative regions of the phase space. Squeezing effects for the state $|\phi_4\rangle$ are shown in Fig. 3 from which it is evident that squeezing appears along the line $x_1 = -x_2$. Also, the states $|\phi_1\rangle$ and $|\phi_2\rangle$ have zero degree of squeezing and the state $|\phi_3\rangle$ exhibits very less amount of squeezing as compared to the state $|\phi_4\rangle$. However, in the case of even–odd entangled coherent states, the two-mode quadrature squeezing for states $|\psi_1\rangle$ and $|\psi_2\rangle$ is zero. Also, the state $|\psi_4\rangle$ do not exhibit two-mode quadrature squeezing. Hence, only state $|\psi_3\rangle$ of even–odd entangled coherent states exhibits two-mode quadrature squeezing as shown in Fig. 4.

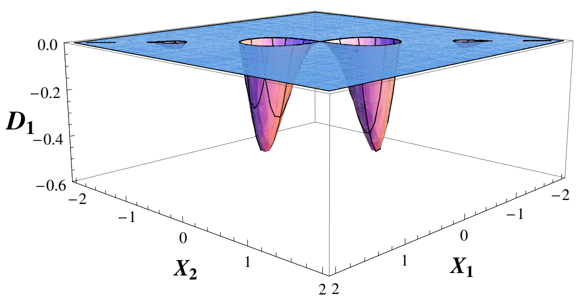

Fig. 3. The degree of squeezing in both quadratures X_1 and X_2 for the state $|\phi_4\rangle$.

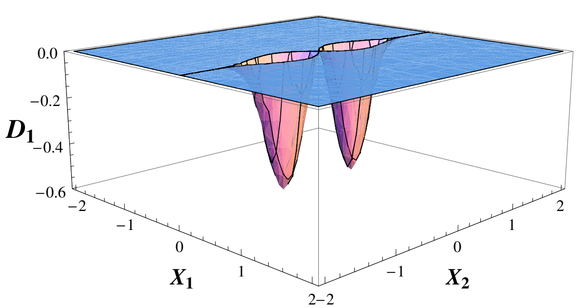

Fig. 4. The degree of squeezing in both quadratures X_1 and X_2 for the even–odd entangled coherent state $|\psi_3\rangle$.

Furthermore, by using analytical approach, it is also analyzed that sub-Poissonian photon statistics and the squeezing effect simultaneously exists in the case of our considered states. Whereas, in the case of even and odd coherent states both squeezing and sub-Poissonian photon statistics do not appear simultaneously.

5. Cauchy–Schwarz inequality

We know that, second-order coherence function can be defined as

$$\gamma_{ij}^{(2)} = \frac{\left\langle a_i^\dagger a_j^\dagger a_j a_i \right\rangle}{\left\langle a_i^\dagger a_i \right\rangle \left\langle a_j^\dagger a_j \right\rangle}\,, \qquad j = 1, 2\,. \tag{26}$$

By using the above mentioned function, we will define the Cauchy–Schwarz inequality according to which "the square of the inner product of two vectors does not exceed the product of the squares of their norms"

$$\left(\gamma_{12}^{(2)}\right)^2 \leq \gamma_{11}^{(2)} \gamma_{22}^{(2)}\,. \tag{27}$$

Otherwise we can consider the inequality as

$$V = \left\langle a_A^\dagger a_B^\dagger a_B a_A \right\rangle^2 - \left\langle a_A^\dagger a_A^\dagger a_A a_A \right\rangle \left\langle a_B^\dagger a_B^\dagger a_B a_B \right\rangle \leq 0\,. \tag{28}$$

The Cauchy–Schwartz inequality for the considered states $\{|\phi_i\rangle_{12}\}$ is given below

$$V^{|\phi_1\rangle} = |\alpha|^8 \left[1 - \left(\frac{1 - e^{-2|\alpha|^2}}{1 + e^{-2|\alpha|^2}}\right)^2\right] > 0\,, \tag{29a}$$

$$V^{|\phi_2\rangle} = |\alpha|^8 \left[1 - \left(\frac{1 + e^{-2|\alpha|^2}}{1 - e^{-2|\alpha|^2}}\right)^2\right] < 0\,, \tag{29b}$$

$$V^{|\phi_3\rangle} = V^{|\phi_4\rangle} = 0\,. \tag{29c}$$

From this, it is clear that state $|\phi_1\rangle$ violate the Cauchy–Schwartz inequality. This violation of the Cauchy–Schwarz inequality clearly demonstrates the non-classical character of the correlations between photons in a coherent state. The fact is that, the second-order coherence function takes on classically forbidden values which may be interpreted as a quantum-mechanical violation of the Cauchy–Schwarz inequality [19, 20, 21].

However, in the case of even–odd entangled coherent states, not even a single state violate the Cauchy–Schwartz inequality

$$V^{|\psi_1\rangle} = V^{|\psi_2\rangle} = V^{|\psi_3\rangle} = V^{|\psi_4\rangle} = 0\,. \tag{30}$$

6. Summary and conclusions

In this paper, we have studied the entanglement and nonclassicality of entangled coherent states based on two coherent states $\pi/2$ out of phase for which the relative phase is taken equal to the phase difference between the states *i.e.* $\phi = \pi/2$ by keeping in view their role in quantum information processing, especially in quantum teleportation. The quantitative analysis has shown that nonclassicality exists in states under consideration. It is found that the state $|\phi_2\rangle$ has perfect entanglement by having 1 ebit of entanglement. It has also shown that all the states of considered entangled coherent states exhibit sub-Poissonian photon statistics, whereas, in the case of even–odd entangled coherent states only two states exhibit sub-Poissonian photon statistics. Furthermore, both squeezing and sub-Poissonian photon statistics exhibit simultaneously for the states under consideration which is in contrast to the even and odd coherent states. Hence, the strong nonclassicality of considered states provides a clear evidence that these states may prove to be used as a physical resource in quantum information processing.

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