SUPERSYMMETRIC SOLUTION OF THE SCHRÖDINGER EQUATION FOR WOODS–SAXON POTENTIAL BY USING THE PEKERIS APPROXIMATION

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In this work, the three dimensional Woods–Saxon potential is studied within the context of Supersymmetry Quantum Mechanics. We have applied the SUSY method by using the Pekeris approximation to the centrifugal potential $l \neq 0$ states. By application of this method, it is possible to solve the Schrödinger equation for this potential. We obtain exactly bound state spectrum and wave function of Woods–Saxon potential for nonzero angular momentum. Hamiltonian hierarchy method and the shape invariance property are used in the calculations.

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1. Introduction

An analytical solution of the radial Schrödinger equation is of high importance in nonrelativistic quantum mechanics, because the wave function contains all necessary information for full description of a quantum system. There are only few potentials for which the radial Schrödinger equation can be solved explicitly for all n and l states. So far, many methods were developed, such as Supersymmetry (SUSY) [1, 2] and NU [3], to solve the radial Schrödinger equation exactly for $l \neq 0$ within these potentials. The idea of Supersymmetry in context of Quantum Mechanics was studied first by Nicolai and Witten [4, 5] and later by Cooper and Freedman [6]. Two decades ago Supersymmetric Quantum Mechanics, SUSYQM, was born to study SUSY breaking of higher dimension Quantum Field Theory. A developed structure of this method applied to unify the four fundamental interaction in nature, namely the electroweak, strong and gravitational interaction [7]. So far, SUSYQM has been extensively used to explore different aspects of nonrelativistic Quantum Mechanics systems [7]. This algebraic method was successful to study analytically solvable [7, 8], the partially solvable [9, 10], the isospectral, the periodic and exponential-type potential. Levai and Williams suggested a simple method for constructing potentials for which the Schrödinger equation can be solved exactly in terms of special functions [11] and showed relationship between the introduced formalism and Supersymmetric Quantum Mechanics [1]. In this work we try to solve Schrödinger equation for nuclear Woods–Saxon potential by using the Pekeris approximation to the centrifugal potential $l \neq 0$ states.

2. Supersymmetric Quantum Mechanics

In the $N = 2$ Supersymmetric Quantum Mechanics, it is necessary to define two nilpotent operators namely Q and Q_+, satisfying the algebra

$$\{Q, Q_+\} = H\,, \qquad Q^2 = Q_+^2 = 0\,, \tag{1}$$

where H is the supersymmetric Hamiltonian. These operators can be realized as

$$Q = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \qquad Q_+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}, \tag{2}$$

where A^+ and A^- are bosonic operators. The Hamiltonian, H, in terms of these operators is given by

$$H = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}. \tag{3}$$

Supersymmetric algebra allows us to write Hamiltonians as [12]

$$H_\pm = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_\pm(x)\,, \tag{4}$$

where the supersymmetric partner potentials $V_\pm(x)$ in terms of the superpotential $W(x)$ are given by

$$V_\pm(x) = W^2 \pm \frac{\hbar}{2m}\frac{dW}{dx}\,. \tag{5}$$

The superpotential has a definition

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\left(\frac{d\ln\psi_0^{(0)}(x)}{dx}\right), \tag{6}$$

where $\psi_0^{(0)}(x)$ denotes the ground state wave function that satisfies the relation

$$\psi_0^{(0)}(x) = N_0 \exp\left(-\frac{\sqrt{2m}}{\hbar}\int^{x} W\left(x'\right) dx'\right) . \tag{7}$$

The Hamiltonian $H_\pm$ can also be written in terms of the bosonic operators A^+ and A^-

$$H_\pm = A^\mp A^\pm \,, \tag{8}$$

where

$$A^\pm = \pm\frac{\hbar}{2m}\frac{d}{dx} + W(x)\,. \tag{9}$$

It is a remarkable result that the energy eigenvalues of H_- and H_+ are identical except for the ground state. In the case of unbroken supersymmetry, the ground state energy of the Hamiltonian H_- is zero $E_0^{(0)} = 0$ [12]. In the factorization of the Hamiltonian, Eqs. (4), (8) and (9) are used respectively. Hence we obtain

$$H_1(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x) = A_1^+ A_1^- + E_0^{(1)}\,. \tag{10}$$

Comparing each side of Eq. (10), term by term, we get the Riccati equation for the superpotential $W_1(x)$

$$W_1^2(x) - W_1'(x) = \frac{2m}{\hbar^2}\left(V_1(x) - E_0^{(1)}\right) . \tag{11}$$

Let us now construct the supersymmetric partner Hamiltonian H_2 as

$$H_2(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x) = A_2^+ A_2^- + E_0^{(2)} \tag{12}$$

and Riccati equation takes the form

$$W_2^2(x) - W_2'(x) = \frac{2m}{\hbar^2}\left(V_2(x) - E_0^{(2)}\right) . \tag{13}$$

Similarly, one can write, in general, the Riccati equation and Hamiltonians by iteration as

$$W_n^2(x) - W_n'(x) = \frac{2m}{\hbar^2}\left(V_n(x) - E_0^{(n)}\right) = A_n^\pm A_n^\mp + E_0^{(n)} \tag{14}$$

and

$$H_n(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_n(x) = A_n^+ A_n^- + E_0^{(n)}\,, \qquad n = 1, 2, 3, \ldots . \tag{15}$$

Because of the SUSY unbroken case, the partner Hamiltonians satisfy the following expressions [12]

$$E_0^{(n+1)} = E_1^{(n)}\,, \qquad n = 0, 1, 2\,; \qquad E_0^{(0)} = 0\,, \tag{16}$$

and also the wave function with the same eigenvalue can be written as [12]

$$\psi_n^{(1)}(x) = A_1^+ A_2^+ \ldots A_n^+ \psi_0^{(n+1)}(x)\,, \tag{17}$$

$$\psi_n^{(1)}(x) = \frac{A^- \psi_0^{(n+1)}(x)}{\sqrt{E_n^{(0)}(x)}}\,, \tag{18}$$

$$\psi_0^{(n+1)}(x) = \frac{A^+ \psi_1^{(n)}(x)}{\sqrt{E_n^{(0)}(x)}}\,. \tag{19}$$

3. Woods–Saxon potential

Woods and Saxon introduced a potential to study elastic scattering of 20 MeV protons by a heavy nuclei [15, 16]. The spherical Woods–Saxon potential that was used as a major part of nuclear shell model, has received a lot of attention in nuclear mean field model [14]. It can be used as central part of the interaction potential of neutron with heavy nucleus [18]. With the help of the axially-deformed Woods–Saxon potential, we may construct the structure of single-particle shell model [20]. The standard Woods–Saxon potential is given by [13]

$$V(r) = -\frac{V_0}{1 + \exp\left(\frac{r - R_0}{a}\right)}\,. \tag{20}$$

The parameter R_0 is interpreted as radius of a nucleus, the parameter a characterizes thickness of the superficial layer inside which the potential falls from value $V = 0$ outside of a nucleus up to value $V = -V_0$ inside a nucleus. At $a = 0$, one gets the simple potential well with jump of potential on the surface of a nucleus. The radial Schrödinger equation with Woods–Saxon potential is

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{L^2}{r^2}\right) R_{nl}(r) + \frac{2\mu}{\hbar^2}\left(E_{nl} + \frac{V_0}{1 + \exp(\frac{r - R_0}{a})}\right) R_{nl}(r) = 0\,, \tag{21}$$

where l is the angular momentum quantum number and μ is the reduced mass. Introducing a new function $U_{nl}(r) = rR_{nl}(r)$, Eq. (21) takes the form

$$\frac{d^2 U_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left(E_{nl} + \frac{V_0}{1 + \exp\left(\frac{r - R_0}{a}\right)} - \frac{\hbar^2 l(l+1)}{2\mu r^2}\right) U_{nl}(r) = 0\,. \tag{22}$$

Equation (22) has the same form as the equation for a particle in one dimension, except for two important differences. First, there is a repulsive effective potential proportional to the eigenvalue of $\hbar^2 l(l+1)$. Second, the radial function must satisfy the boundary condition $R(0) = 0$ and $R(\infty) = 0$. It is sometimes convenient to define in Eq. (22) the effective potential in the form

$$V_{\rm eff}(r) = V_r + \frac{\hbar^2 l(l+1)}{2\mu r^2} \,. \tag{23}$$

The effective potential together with the Woods–Saxon potential for $l \neq 0$ can be written as

$$V_{\rm eff}(r) = V_{\rm WS} + V_l(r) = -\frac{V_0}{1+\exp\left(\frac{r-R_0}{a}\right)} + \frac{\hbar^2 l(l+1)}{2\mu r^2} \,. \tag{24}$$

If to introduce in Eq. (24) the notations

$$r = R_0(x+1) \,, \tag{25}$$

$$\alpha = \frac{R_0}{a} \,, \tag{26}$$

then, the radial Schrödinger equation given by Eq. (22) takes the form

$$\frac{d^2 U_{nl}(x)}{dx^2} + \frac{2\mu}{\hbar^2}\left(E_{nl} - V_{\rm eff}\right) U_{nl}(x) = 0 \,. \tag{27}$$

It is known that the Schrödinger equation cannot be solved exactly for this potential at the value $l \neq 0$ using the standard methods as SUSY and NU. From Eq. (24) it is seen that the effective potential is a combination of the exponential and inverse square potentials, which cannot be solved analytically. Therefore, in order to solve this problem, we can take the most widely used and convenient for our purposes Pekeris approximation [16, 17]. This approximation is based on the expansion of the centrifugal barrier in a series of exponentials depending on the internuclear distance, taking into account terms up to second order, so that the effective l-dependent potential preserves the original form. It should be pointed out, however, that this approximation is valid only for low vibrational energy states. The centrifugal potential is expanded in the Taylor series around $x = 0 (r = R_0)$

$$V_l(x) = \frac{\hbar^2 l(l+1)}{2\mu r^2} = \frac{\hbar^2 l(l+1)}{2\mu R_0^2}\frac{1}{(x+1)^2} = \varOmega\left(1 - 2x + 3x^2 - 4x^3 + \ldots\right) , \tag{28}$$

where $\varOmega$ is equal to $\frac{\hbar^2 l(l+1)}{2\mu R_0^2}$. According to the Pekeris approximation, we shall replace potential $V_l(r)$ with expression

$$V_l(x) = \varOmega\left(d_0 + \frac{d_1}{1+\exp(\alpha x)} + \frac{d_2}{(1+\exp(\alpha x))^2}\right) . \tag{29}$$

In order to define the parameters d_0, d_1 and d_2, we also expand this potential in the Taylor series around the point $x = 0(r = R_0)$

$$\begin{aligned} V_l(x) \; = \; \Omega\Big[\Big(d_0 + \frac{d_1}{2} + \frac{d_2}{4} \Big) - \frac{\alpha}{4}(d_1 + d_2)x + \frac{\alpha^2}{16} d_2 x^2 \\ + \frac{\alpha^3}{48}(d_1 + d_2)x^3 - \frac{\alpha^4}{96} d_2 x^4 + \ldots \Big] . \end{aligned} \tag{30}$$

Comparing equal powers of x, Eqs. (28) and (30), we obtain the constants d_0, d_1 and d_2

$$d_0 = 1 - \frac{4}{\alpha} + \frac{12}{\alpha^2} , \qquad d_1 = \frac{8}{\alpha} - \frac{48}{\alpha^2} , \qquad d_2 = \frac{48}{\alpha^2} . \tag{31}$$

Now, the effective potential after Pekeris approximation becomes equal to

$$V_{\rm eff}(x) = \Omega d_0 - \frac{V_0 - \Omega d_1}{1 + \exp(\alpha x)} + \frac{\Omega d_2}{(1 + \exp(\alpha x))^2} . \tag{32}$$

Having inserted this new effective potential into Eq. (27), we obtain

$$\frac{d^2 U_{nl}(x)}{dx^2} + \frac{2\mu}{\hbar^2} \left(E_{nl} - \Omega d_0 + \frac{V_0 - \Omega d_1}{1 + \exp(\alpha x)} - \frac{\Omega d_2}{(1 + \exp(\alpha x))^2} \right) U_{nl}(x) = 0 . \tag{33}$$

If we introduce the notations

$$\frac{2\mu(E_{nl} - \Omega d_0)}{\hbar^2} = \varepsilon , \tag{34}$$

$$\frac{2\mu(V_0 - \Omega d_1)}{\hbar^2} = \beta , \tag{35}$$

$$\frac{2\mu \Omega d_2}{\hbar^2} = \gamma \tag{36}$$

we obtain

$$\frac{d^2 U_{nl}(x)}{dx^2} + \left(\varepsilon + \frac{\beta}{1 + \exp(\alpha x)} - \frac{\gamma}{(1 + \exp(\alpha x))^2} \right) U_{nl}(x) = 0 . \tag{37}$$

In Supersymmetric Quantum Mechanics the ground state eigenfunction $U_0(x)$ can be written as

$$U_0(x) = N_0 \exp\left(-\frac{\sqrt{2m}}{\hbar} \int^{x} W\left(x'\right) dx' \right) , \tag{38}$$

where N is a normalized constant. Through the superalgebra we make following Ansatz for the superpotential

$$W_1(x) = \frac{-\hbar}{\sqrt{2\mu}}\left(\frac{B}{1+\exp(\alpha x)} + A\right). \tag{39}$$

Satisfies the associated Riccati equation and substituting this expression into Eq. (11), we obtain the following identity

$$\begin{aligned} &A^2 + \frac{B^2}{(1+\exp(\alpha x))^2} + \frac{2AB}{1+\exp(\alpha x)} - \frac{\alpha B\exp(\alpha x)}{(1+\exp(\alpha x))^2} \\ &= \frac{2\mu}{\hbar^2}\left(V_1(x) - E_0^{(1)}\right) = -\varepsilon - \frac{\beta}{1+\exp(\alpha x)} + \frac{\gamma}{(1+\exp(\alpha x))^2}. \end{aligned} \tag{40}$$

With the comparison of the each side of Eq. (40), we obtain

$$A^2 = -\varepsilon\,, \qquad 2AB - \alpha B = -\beta\,, \qquad B^2 + \alpha B = \gamma\,. \tag{41}$$

Solving Eq. (41) yields

$$E_0 = -\frac{\hbar^2}{2\mu}\left(\frac{-\beta}{-\alpha \pm \sqrt{\alpha^2+4\gamma}} + \frac{\alpha}{2}\right)^2 + \Omega d_0\,, \tag{42}$$

$$A = \frac{-\beta}{-\alpha \pm \sqrt{\alpha^2+4\gamma}} + \frac{\alpha}{2}\,, \tag{43}$$

$$B = \frac{-\alpha \pm \sqrt{\alpha^2+4\gamma}}{2}\,. \tag{44}$$

In Eqs. (42), (43) and (44), we choose the positive-sign in calculations because this selection would, of course, be a right choice to ensure the well-behaved nature of the eigenfunction at the origin and the infinity. Using Eq. (38), the eigenfunction for ground state in terms of x will be obtained as

$$\begin{aligned} R_0(x) &= \frac{N}{(x+1)R_0}\exp\left(\left(\frac{-\beta}{-\alpha+\sqrt{\alpha^2+4\gamma}} + \frac{\alpha}{2}\right)x\right) \\ &\quad \times(\exp(-\alpha x)+1)^{\frac{-\alpha+\sqrt{\alpha^2+4\gamma}}{-2\alpha}}. \end{aligned} \tag{45}$$

We substituted Eq. (43) and Eq. (44) into Eq. (39) and then solved Eq. (5) for both $V_+(x)$ and $V_-(x)$. Consequently, these supersymmetric partner potentials are obtained as

$$V_+(x) = \frac{\hbar^2}{2\mu}\left(A^2 + \frac{B^2-\alpha B}{(1+\exp(\alpha x))^2} + \frac{2AB+\alpha B}{1+\exp(\alpha x)}\right), \tag{46}$$

$$V_-(x) = \frac{\hbar^2}{2\mu}\left(A^2 + \frac{B^2+\alpha B}{(1+\exp(\alpha x))^2} + \frac{2AB-\alpha B}{1+\exp(\alpha x)}\right). \tag{47}$$

The shape invariance concept that was introduced by Gendenshtein is [7]

$$V_+(a_0,x) = V_-(a_1,x) + R(a_1)\,, \tag{48}$$

where a_1 is a function of a_0 and $R(a_1)$ is independent of x. Hence, the energy spectrum becomes equal to

$$E_0^{(k)} = \sum_{i=0}^{k} R(a_i)\,, \tag{49}$$

$$E_{nl} = E_{nl}^{-} + E_0\,. \tag{50}$$

If we now consider a mapping of the form

$$B \longrightarrow B' = B - \alpha \tag{51}$$

in Eq. (47) and substitution $A = \frac{\gamma-\beta}{2B} - \frac{B}{2}$ into Eqs. (46) and (47), it is easily seen that apart from a constant, the partner potential are the same. In technical words, the chosen SUSY potential satisfies the shape invariance condition. On the other hand, we can obtain

$$B_1 = B_0 - \alpha\,, \qquad B_n = B_0 - n\alpha\,, \tag{52}$$

$$\begin{aligned} R(a_1) &= V_+(B,x) - V_-(B-\alpha,x) = \frac{-\hbar^2}{2\mu}\left[\left(\frac{\gamma-\beta}{2(B-\alpha)} - \frac{B-\alpha}{2}\right)^2\right. \\ &\quad \left. - \left(\frac{\gamma-\beta}{2B} - \frac{B}{2}\right)^2\right], \end{aligned} \tag{53}$$

$$\begin{aligned} R(a_i) &= \frac{-\hbar^2}{2\mu}\left[\left(\frac{\gamma-\beta}{2(B-i\alpha)} - \frac{B-i\alpha}{2}\right)^2 - \left(\frac{\gamma-\beta}{2\left(B-(i-1)\alpha\right)}\right.\right. \\ &\quad \left.\left. - \frac{B-(i-1)\alpha}{2}\right)^2\right], \end{aligned} \tag{54}$$

where the remainder $R(a_i)$ is independent of x. On repeatedly using the shape invariance condition Eq. (48), it is clear that

$$H^k(x) = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V_-(x,a_k) + \sum_{i=0}^{k} R(a_i)\,, \tag{55}$$

hence by using Eqs. (42), (49) and (50), the energy levels of the Woods–Saxon potential are found as

$$\begin{aligned} E_{nl} &= \frac{-\hbar^2}{2\mu}\left[\left(\frac{\gamma-\beta}{2(B-n\alpha)} - \frac{B-n\alpha}{2}\right)^2 - \left(\frac{\gamma-\beta}{2B} - \frac{B}{2}\right)^2\right. \\ &\quad \left. + \left(\frac{-\beta}{-\alpha+\sqrt{\alpha^2+4\gamma}} + \frac{\alpha}{2}\right)^2\right] + \varOmega d_0\,. \end{aligned} \tag{56}$$

On the other hand, we can obtain energy levels as follows

$$E_{nl} = \frac{-\hbar^2}{2\mu}\left[\left(\frac{\frac{l(l+1)(d_1+d_2)}{R_0^2}-\frac{2\mu V_0}{\hbar^2}}{-\alpha+\sqrt{\alpha^2+4\gamma}-2n\alpha}+\frac{\alpha-\sqrt{\alpha^2+4\gamma}}{4}+\frac{n\alpha}{2}\right)^2 - \frac{l(l+1)d_0}{R_0^2}-\left(\frac{\frac{l(l+1)(d_1+d_2)}{R_0^2}-\frac{2\mu V_0}{\hbar^2}}{-\alpha+\sqrt{\alpha^2+4\gamma}}+\frac{\alpha-\sqrt{\alpha^2+4\gamma}}{4}\right)^2 + \left(\frac{-\frac{2\mu(V_0-\Omega d_0)}{\hbar^2}}{-\alpha+\sqrt{\alpha^2+4\gamma}}+\frac{\alpha}{2}\right)^2\right]. \tag{57}$$

The other eigenfunctions can be obtained by repeatedly using Eq. (18) and Eq. (19).

4. Conclusions

In this paper, we have solved the radial part of Schrödinger equation in three dimensions by applying Hamiltonian Hierarchy Method within the frame work of SUSYQM. By using the Pekeris approximation to none centrifugal potential $l \neq 0$ states, we have applied the hierarchy of Hamiltonian method in the context of SUSYQM to get energy spectra of the Woods–Saxon potential. We have obtained the exact analytical eigenfunction and eigenvalue for the $l \neq 0$ case. In addition, we have also derived the solutions for $l = 0$ case, using an effective potential suggested by the $l \neq 0$ case. Finally, we would like to point out that although the SUSYQM scheme works quite well for Woods–Saxon potential, extensive applications to other effective Woods–Saxon-like potentials are needed to test the credibility of the method. We can conclude that our results are interesting not only for pure theoretical physicist but also for experimental physicist, because the results are exact and more general and useful to study nuclear scattering.

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