FLUCTUATIONS OF COSMIC PARAMETERS DUE TO INHOMOGENEITIES*

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We present the fluctuations of the cosmic parameters on different scales. For large scales we use standard perturbation theory, for small scales the relativistic Zel'dovich approximation. We find that 1% curvature fluctuations reach out to scales of $600\,\mathrm{h}^{-1}$ Mpc and that backreaction contributes up to 15% to the cosmic energy budget on scales of 50 Mpc.

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1. Introduction

Over the last decade, ever increasing galaxy surveys as the 2dF and SDSS survey, have shown that the Universe is inhomogeneous up to scales of at least $100\,\mathrm{h}^{-1}$ Mpc. This raised the question, whether these inhomogeneities influenced the expansion history of the Universe as a whole, or only the local evolution. This is the cosmic backreaction problem recalled in Sect. 2.

In [1] we derived a condition on the initial backreaction under which it leads to important effects today. Analysing N-body data we found, that it is only satisfied on small scales. As today's simulations are Newtonian, this was expected, because in a Newtonian set-up backreaction vanishes on the scale of homogeneity. Thus, also a Newtonian perturbative study [2] found that only the small scale evolution is modified by the effective sources.

In general relativity (GR) this is not necessarily true. This motivated the extension of the analysis of [2] to GR in [3]. There we found that also in GR the expected deviation of a domain from the background behaviour is small. The fluctuations between different domains in the Universe, however, may

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be quite large, which confirms results of [4, 5]. They also showed, that even if backreaction itself contributes significantly only below the homogeneity scale, curvature still has sizeable fluctuations above that scale.

This shows that one has to take averaging effects into account in the interpretation of the observations. As we make all our observations in a special region of the Universe, *i.e.* around the earth, we have to know how typical this region is. Only CMB observations allow to make statements about the (observable) Universe as a whole. All the others are more or less local and so subject to inevitable uncertainties by the Universe's inhomogeneity. Due to their potential importance to the interpretation of observations, we quantified them in [6]. Here we summarize these results on large scale fluctuations together with results of [3] on the evolution of typical inhomogeneous regions on small scales.

2. Cosmic parameters as local averages

To quantify the fluctuations of cosmic parameters, we first have to define them for limited domains. For inhomogeneous domains the approach of [7,8] allows this. It defines averages of scalar quantities by the Riemannian average over a domain $\mathcal{D}$ on a spatial hypersurface $\langle f\rangle_{\mathcal{D}} := \int_{\mathcal{D}} f\left(t,\boldsymbol{x}\right) d\mu_g / V_{\mathcal{D}}$ with $d\mu_g := \left({}^{(3)}g\left(t,\boldsymbol{x}\right)\right)^{1/2} d^3x$. It can then be shown that the domain's volume scale factor $a_{\mathcal{D}}$ evolves through Friedmann like equations

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -4\pi G\left\langle\varrho\right\rangle_{\mathcal{D}} + \Lambda + \mathcal{Q}_{\mathcal{D}}\,, \tag{1}$$

$$3H_{\mathcal{D}}^2 = 8\pi G\left\langle\varrho\right\rangle_{\mathcal{D}} - \tfrac{1}{2}\left\langle\mathcal{R}\right\rangle_{\mathcal{D}} + \Lambda - \tfrac{1}{2}\mathcal{Q}_{\mathcal{D}}\,, \tag{2}$$

that contain the new, effective kinematical backreaction term $\mathcal{Q}_{\mathcal{D}}$

$$\mathcal{Q}_{\mathcal{D}} := \tfrac{2}{3}\left(\left\langle\theta^2\right\rangle_{\mathcal{D}} - \left\langle\theta\right\rangle_{\mathcal{D}}^2\right) - 2\left\langle\sigma^2\right\rangle_{\mathcal{D}}\,, \qquad \partial_t\left(a_{\mathcal{D}}^6\mathcal{Q}_{\mathcal{D}}\right) = -a_{\mathcal{D}}^4\partial_t\left(a_{\mathcal{D}}^2\langle\mathcal{R}\rangle_{\mathcal{D}}\right)\,. \tag{3}$$

If the expansion fluctuations $\left\langle\theta^2\right\rangle_{\mathcal{D}} - \left\langle\theta\right\rangle_{\mathcal{D}}^2$ on $\mathcal{D}$ outweigh the shear fluctuations $\left\langle\sigma^2\right\rangle_{\mathcal{D}}$, by (1) $\mathcal{Q}_{\mathcal{D}}$ drives acceleration. It also modifies the curvature evolution of $\mathcal{D}$ by its coupling to the average three Ricci scalar in Eq. (3).

Therefore, the local cosmic parameters on $\mathcal{D}$, evolve differently from the global ones. The former are defined in analogy to the latter by

$$\Omega_m^{\mathcal{D}} := \frac{8\pi G}{3H_{\mathcal{D}}^2}\langle\varrho\rangle_{\mathcal{D}}\,, \quad \Omega_\Lambda^{\mathcal{D}} := \frac{\Lambda}{3H_{\mathcal{D}}^2}\,, \quad \Omega_{\mathcal{R}}^{\mathcal{D}} := -\frac{\langle\mathcal{R}\rangle_{\mathcal{D}}}{6H_{\mathcal{D}}^2}\,, \quad \Omega_{\mathcal{Q}}^{\mathcal{D}} := -\frac{\mathcal{Q}_{\mathcal{D}}}{6H_{\mathcal{D}}^2}\,. \tag{4}$$

It is the typical variations of these parameters between different places in the Universe, that we are interested in. Formally, one may characterize these fluctuations by the variance of an ensemble average $\overline{O_{\mathcal{D}}}$, *i.e.* $\sigma^2(O_{\mathcal{D}}) := \overline{O^2_{\mathcal{D}}} - \overline{O_{\mathcal{D}}}^2$, where $\sigma(O_{\mathcal{D}})$ measures the variation in the average, over a domain $\mathcal{D}$ of a specific size and shape, of an observable O, compared to its value averaged over the same kind of domain at a different location.

3. Calculation of the intrinsic fluctuations of cosmic parameters

To quantify the fluctuations of the cosmic parameters (4) we use different techniques adapted to the scale under consideration. On large scales, standard perturbation theory is sufficient. On small scales we use the relativistic generalization of the Zel'dovich approximation. As we found that on large scales $\mathcal{Q}_{\mathcal{D}}$ is negligible (see Fig. 1, l.h.s.), we will only include it in the plots of the small scale results.

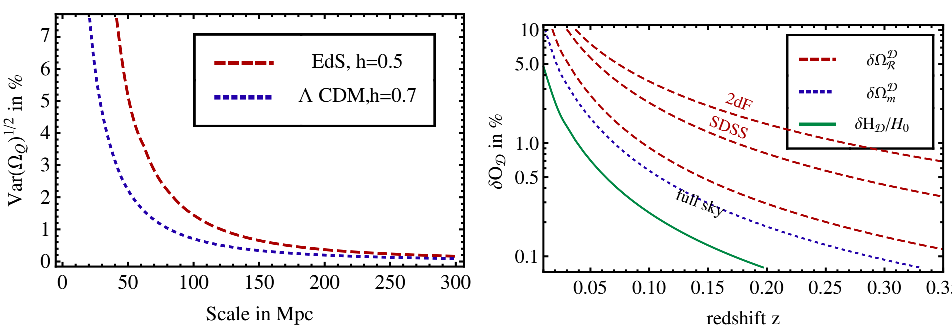

Fig. 1. Left: Fluctuations in the kinematical backreaction parameter between different domains $\mathcal{D}$. Above the homogeneity scale of the order of 100 Mpc, $\mathcal{Q}_{\mathcal{D}}$ gives typically only sub percent corrections to the evolution of the volume scale factor $a_{\mathcal{D}}$. Right: The top three lines show the r.m.s. fluctuation of the curvature parameter, $\delta\Omega^{\mathcal{D}}_{\mathcal{R}}$, for geometries resembling the 2dFGRS, the SDSS and a full sky survey. The x-axis measures the radial extension of the domain $\mathcal{D}$.

3.1. Large scale calculation

For evaluating the large scale fluctuations we use standard cosmological perturbation theory in comoving synchronous gauge. Starting from the perturbed metric one can derive the dependence of the local quantities on the background and the perturbation. One then takes the $\mathcal{D}$-average of Sect. 2 and calculates the ensemble variance of the resulting domain dependent cosmic parameter $\Omega^{\mathcal{D}}_X$. The fluctuations obtained may be summarized by

$$
\begin{aligned}
\delta H_{\mathcal{D}} &= \tfrac{1}{3}\overline{H_{\mathcal{D}}}\left(a_{\mathcal{D}}\right)\, f\left(a_{\mathcal{D}}\right)\, \sigma_{\mathcal{D}} = 0.17 H_0 \sigma_{\mathcal{D}}\,,\\
\delta \Omega_m^{\mathcal{D}} &= \overline{\Omega_m^{\mathcal{D}}}\left(a_{\mathcal{D}}\right)\left(1+\tfrac{2}{3}\, f\left(a_{\mathcal{D}}\right)\right)\, \sigma_{\mathcal{D}} = 0.40\sigma_{\mathcal{D}}\,,\\
\delta \Omega_{\mathcal{R}}^{\mathcal{D}} &= \overline{\Omega_m^{\mathcal{D}}}\left(a_{\mathcal{D}}\right)\left(1+\tfrac{2}{3}\frac{f\left(a_{\mathcal{D}}\right)}{\overline{\Omega_m^{\mathcal{D}}}\left(a_{\mathcal{D}}\right)}\right)\, \sigma_{\mathcal{D}} = 0.64\sigma_{\mathcal{D}}\,,\\
\delta \Omega_{\Lambda}^{\mathcal{D}} &= \tfrac{2}{3}\overline{\Omega_{\Lambda}^{\mathcal{D}}}\left(a_{\mathcal{D}}\right)\, f\left(a_{\mathcal{D}}\right)\, \sigma_{\mathcal{D}} = 0.24\sigma_{\mathcal{D}}\,,\\
\delta \Omega_{\mathcal{Q}}^{\mathcal{D}} &= \mathcal{O}\left(\sigma_{\mathcal{D}}^2\right)\,,
\end{aligned}
\tag{5}
$$

where the last column indicates the values for a ΛCDM Universe, today. As by the Einstein equations, the inhomogeneities in the matter distribution induced the ones in the average Hubble rate $H_{\mathcal{D}}$ and the average curvature $\langle\mathcal{R}\rangle_{\mathcal{D}}$, all fluctuations depend on the underlying matter fluctuations on $\mathcal{D}$

$$
\sigma_{\mathcal{D}}^2 := \int_{\mathbb{R}^3} d^3k\; P_{\boldsymbol{i}}(k)\widetilde{W}_{\mathcal{D}}\left(\boldsymbol{k}\right)\widetilde{W}_{\mathcal{D}}\left(-\boldsymbol{k}\right)\,, \tag{6}
$$

$\sigma_{\mathcal{D}}$ is characterized by their power spectrum $P_{\boldsymbol{i}}\left(k\right)$ and the window function of the domain $\mathcal{D}$, $W_{\mathcal{D}}\left(\boldsymbol{r}\right)$.

The evolution of the fluctuations depends on the cosmology under consideration. The modified growth rate

$$
f\left(a_{\mathcal{D}}\right) := \tfrac{3}{2}\overline{\Omega_m^{\mathcal{D}}}\left(a_{\mathcal{D}}\right)\left(\tfrac{5}{3D_0}\tfrac{a_{\mathcal{D}}}{a_{\mathcal{D}_0}} - 1\right) \approx \begin{cases} 0.5 & \Lambda\text{CDM}\\ 1.0 & \text{EdS}\end{cases} \tag{7}
$$

is bigger for a pure Einstein–de-Sitter model (the approximate values are calculated for "today"). In the general case, the growth function D depends on the matter content of the background one chooses. Also the evolution of the average domain $\mathcal{D}$, $\overline{\Omega_m^{\mathcal{D}}} = \left(1 + c\left(a_{\mathcal{D}}/a_{\mathcal{D}_0}\right)^3\right)^{-1}$, depends on the background via the ratio $c = \Omega_\Lambda/\Omega_m$.

3.2. Small scale calculation

For domains smaller than the homogeneity scale, Fig. 1 shows that backreaction has to be taken into account. Therefore, we need more refined methods than the simple first order calculation of the previous section. One such method is the general relativistic Zel'dovich approximation discussed in [9].

Decomposing the metric into coframes $g_{ij} = G_{ab}\eta^a{}_i\eta^b{}_j$ allows to perturb it analogously to the Newtonian Zel'dovich approximation (see [2])

$$
{}^{\mathrm{RZA}}\eta^a{}_i(t,\boldsymbol{X}^k) := a(t)\left(\delta^a{}_i + \xi(t)\dot{\mathscr{P}}^a{}_i\left(t_{\boldsymbol{i}},\boldsymbol{X}^k\right)\right)\,. \tag{8}
$$

This analogy then carries over to the backreaction term, that may be written in the same form as in the Newtonian case

$$ {}^{\mathrm{RZA}}\mathcal{Q}_{\mathcal{D}} = \frac{\dot{\xi}^2 \left(\gamma_1 + \xi\gamma_2 + \xi^2\gamma_3\right)}{\left(1 + \xi\langle \mathrm{I_i}\rangle_{\mathcal{C_D}} + \xi^2\langle \mathrm{II_i}\rangle_{\mathcal{C_D}} + \xi^3\langle \mathrm{III_i}\rangle_{\mathcal{C_D}}\right)^2} \,, \tag{9} $$

but with coefficients γ_i and $\mathrm{I_i}$ that are now functions of the initial coframe perturbation $\dot{\mathscr{P}}^a_{\;i}$ and no longer only of the scalar gravitational potential as in the Newtonian case. For flat initial conditions the numerical difference between the two sets of coefficients, however, is expected to be small. Therefore, the evolution of $\mathcal{Q}_{\mathcal{D}}$ is basically Newtonian. Nevertheless, in GR $\mathcal{Q}_{\mathcal{D}}$ triggers non-trivial curvature and leads in this way to different results than in the Newtonian framework.

4. Explicit evaluation of the fluctuations

4.1. Large scale results

To show the importance of the variations in the cosmic parameters between different domains in the Universe, we calculate their fluctuations on a given region "today" using Eqs. (5). We choose domains close to those used by observers, *i.e.* either cone- or slice-like. The calculation used a decomposition of these windows into spherical harmonics, to separate the model independent angular part from the model dependent radial coefficients.

The results for geometries corresponding to the currently biggest low z galaxy surveys, are shown in Fig. 1. It shows that even for the large volumes realised by the SDSS and 2dF survey, the uncertainties in the cosmic parameters are still important. For the main sample of the SDSS, being volume limited up to a redshift of about 0.1, this means, that the uncertainty in the curvature is still around 5%. Even for a (hypothetical) full sky survey this value does not drop below 1%. Converted into Mpc this means that spheres of $270\,\mathrm{h}^{-1}$ Mpc in radius still suffer from curvature fluctuations of 1%. Therefore, the typical scale of 1% fluctuations in the curvature parameter is of the order of $600\,\mathrm{h}^{-1}$ Mpc. This is actually not that small as the last scattering surface at $z \approx 1100$ is only $9600\,\mathrm{h}^{-1}$ Mpc away. One of these regions therefore fills more than 5% of the way to that surface.

The formulae of Eqs. (5) are also useful to determine the fluctuations in other cosmologically interesting parameters. One is the dark energy (DE) equation of state w. In an inhomogeneous Universe, the measured value of w is influenced by curvature and backreaction. Even if they vanish globally, $\Omega_{\mathcal{R}}^{\mathcal{D}}$ and $\Omega_{\mathcal{Q}}^{\mathcal{D}}$ may be different from zero on a finite domain $\mathcal{D}$. The equation of state one measures for the component that is not made of matter, is then a combination of equations of state of the components $\Omega_{\mathcal{R}}^{\mathcal{D}}$, $\Omega_{\mathcal{Q}}^{\mathcal{D}}$ and $\Omega_{\Lambda}^{\mathcal{D}}$.

Fig. 2 shows that in the regions of interest for the measurement of the equation of state, *i.e.* for redshifts $z > 0.3$ the effect is completely negligible. Only if one was to use a method to determine it "today", the error due to the modified dynamics of the local volume would be important, but drops below 1% above $z \approx 0.1$. However, in a different way the error in the curvature parameter may induce one on w. As shown above, even large volumes may possess an untypical amount of curvature. If one used these local volumes to determine the curvature, one may find a value different from the true background value. If one then used this local value as the basis for the determination of the equation of state of DE at higher redshifts, the error might be huge even for per mille errors on the measured curvature. This has been shown in [10]. Therefore, one could mistakenly measure dynamical DE even if in reality it was a cosmological constant. So it is crucial to know how big the curvature fluctuation still is on the observational domains used.

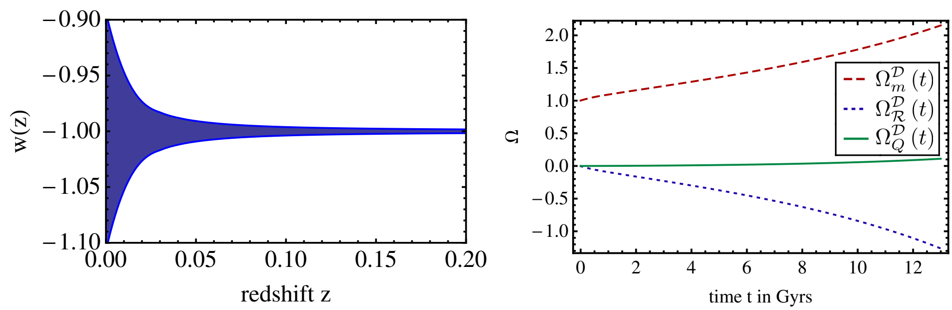

Fig. 2. Left: Uncertainty in the value of the dark energy equation of state w due to inhomogeneities, in terms of the radius of the averaging sphere $\mathcal{D}$ given in units of redshift. Right: Evolution of the domain dependent cosmic parameters of Eq. 4 with cosmic time. The background is the EdS model with $\Omega_m = 1$ ($h = 0.5$, $\sigma_8 = 1$). The plot shows values for a spherical domain of 50 Mpc radius with one-σ fluctuations of the initial invariants of the perturbation one form.

4.2. Small scale results

Already from Fig. 1 we see, that, on small scales, deviations from the background are quite large. To quantify them using the relativistic Zel'dovich approximation of Section 3.2, we choose an EdS and a ΛCDM background. In Fig. 2 we look at the temporal evolution of a typical overdense 50 Mpc sphere, one-σ away from the EdS background. On these scales the density contrast is already so big that the $\Omega_m^{\mathcal{D}}$ parameter is 100% off the global value. This is compensated by a large positive curvature. The backreaction is of the order of 15%, significantly influencing the evolution. In the ΛCDM case the deviations from the background are less important, as structure stops growing relatively early. This hinders the emergence of a sizeable backreaction component. Also the curvature does not grow as big.

4.3. Global results

As we have seen in the section above and from Fig. 1, the backreaction term is small on scales larger than the homogeneity scale. From its definition in Eq. (3) this means that either the expansion fluctuation and the shear fluctuation are both zero, or they cancel each other. To disentangle these two possibilities one may calculate their expected values for a given domain $\mathcal{D}$, using the formalism of Section 3.2. The surprising result is (see [3]), that this expectation value is independent of the size of the domain. For the expansion fluctuation one finds a value that would correspond to $\Omega_{\mathcal{Q}}^{\mathcal{D}} \approx 0.73$ today, if the shear was zero and $\mathcal{Q}_{\mathcal{D}}$ only consisted of expansion fluctuations. This shows that there is indeed a cancellation like in the Newtonian case.

5. Conclusion

We have discussed how to address the problem of quantifying the influence of inhomogeneities on the cosmic evolution in the averaging framework of [7,8]. This is advantageous because it allows to define local Ω-parameters obeying the extended Friedmann equation (2) for arbitrary inhomogeneities. The effective terms figuring in these equations (*e.g.* backreaction), give a modification of the small scale evolution, as was shown in Sect. 4.2. However, the direct influence of inhomogeneities encoded in $\Omega_{\mathcal{Q}}^{\mathcal{D}}$, dies out at the homogeneity scale (but persist on a low level see [11]), as can be seen from Fig. 1. Sect. 4.1 showed, that for the other cosmic parameters this behaviour is less drastic and leads to sizeable fluctuations on scales above the 100 Mpc threshold. For cosmic curvature, even scales of the order of $600\,\mathrm{h}^{-1}$ Mpc show 1% fluctuations. The results presented are useful for the correct interpretation of cosmological observations. They may be used for a determination of the modified growth rate Eq. (7) and therefore, for a measurement of the background cosmology. To extend them to the higher redshifts of future galaxy surveys one may use the new averaging techniques developed by [12], or directly the calculation of the fluctuations in the luminosity distance by [13]. However, this is only necessary if one wants to extend the averaging domain to the entire survey volume. For small enough subsamples the formulae presented here are completely sufficient.

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