GEOMETRIES, QUANTUM GRAVITY AND QUANTUM MATTER IN 4-DIMENSIONS*

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We present arguments that in 4-dimensions quantum matter, geometry and gravity are related in a special, new way. This is based on the geometry of exotic smooth $\mathbb{R}^4_k$, k even, which on the one hand underlies the effective states of quantum matter, as in Kondo effect, and on the other, refers to exact superstring backgrounds. This kind of link of geometry and quantum matter allows for quantum treatment of gravity confined to exotic $\mathbb{R}^4_k$.

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1. Introduction

Let us consider a quantum particle which travels through some 4-dimensional (4d) background. When scale of energy increases, there must exist a regime where quantum gravity description is necessary. If additionally, the background manifold is curved the question arises about the fate of this 4d smooth curved pseudo-Riemannian geometry in the quantum gravity (QG) limit. The correct answer requires QG calculations in dimension 4.

There exist several candidates for the theory of quantum gravity among which superstring theory is probably the most promising one. However, the formulation of superstring theory in 4 dimensions is not possible due to very fundamental reasons — consistency requires 10 space-time dimensions. Moreover, supersymmetry plays a crucial role in string theory and any direct quantization of general relativity (GR) in 4 space-time dimensions is by now far from being complete.

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On the other hand, formal path integral over space-time geometries teaches us that we have to include different smooth structures of space-time [1,2].

In the following section, we present smooth 4-geometries and quantum spin matter in the Kondo effect seen through this 4d structures. In the last section, a pattern for QG calculations in 4d emerges. One finds the hints that in 4 dimensions the QG regime is more likely and effectively assigned to the correlated matter states rather than to the fundamental fields and particles from the standard model of these.

2. 4-geometries and 2d conformal field theory

The standard $\mathbb{R}^4_{\rm std}$ is the only differential structure inherited from the topological product of axes $\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. An exotic $\mathbb{R}^4$ is the same topological 4-manifold $\mathbb{R}^4$ — but with a different smooth structure (*i.e.* non-diffeomorphic with the standard one). The exotic $\mathbb{R}^4$ is the only Euclidean space $\mathbb{R}^n$ with an exotic smoothness structure. Here we deal with *small* exotic $\mathbb{R}^4$'s which emerge due to the failure [3,4] of the h-cobordism theorem in dimension 5. Even though exotic $\mathbb{R}^4$'s are smooth 4-manifolds, a big mathematical problem, which also limits their applications in physics, is to find a suitable effective coordinate presentation such that one can do calculus respecting the exoticness of these manifolds.

Recently, it was established how to relate these 4-exotics with some structures on S^3 (see *e.g.* [5,7,6]). This S^3 has to be placed as a part of the boundary of some compact contractible 4-submanifold with boundary, *i.e.* Akbulut cork. If so, one can prove that exotic smoothness of the $\mathbb{R}^4$ is tightly related with codimension-one foliations of this S^3, hence with the 3-rd real cohomology classes of S^3. In this sense we classify exotic smooth $\mathbb{R}^4$'s from the radial family by $H^3(S^3,\mathbb{R})$ [5,8].

Small exotic $\mathbb{R}^4$ is determined by the compact 4-manifold A with boundary ∂A which is homology 3-sphere, and attached several *Casson handles* (CHs). A is the Akbulut cork and CH is built from many stages towers of immersed 2-disks. These 2-disks cannot be embedded and the intersection points can be placed in general position in 4D in separated double points. Every CH has infinite many stages of intersecting disks. However, CH is topologically the same as (homeomorphic to) an open 2-handle, *i.e.* $D^2\times\mathbb{R}^2$. Now if one replaces CHs, from the above description of small exotic $\mathbb{R}^4$, by ordinary open 2-handles (with suitable linking numbers in the attaching regions) the resulting object is standard $\mathbb{R}^4$. The reason is the existence of infinite (continuum) many diffeomorphism classes of CH, even though all are topologically the same.

In the particular case of integral $H^3(S^3, \mathbb{Z})$ one yields the relation of exotic $\mathbb{R}^4_k$, $k[\,] \in H^3(S^3, \mathbb{Z})$, $k \in \mathbb{Z}$ with the Wess–Zumino (WZ) term of the k Wess–Zumino–Witten (WZW) model on SU(2). This is because the integer classes in $H^3(S^3, \mathbb{Z})$ are of special character. Topologically, this case refers to flat $\mathrm{PSL}(2, \mathbb{R})$-bundles over the space $(S^2 \setminus \{k \text{ punctures}\}) \times S^1$ and due to the Heegard decomposition one obtains the relation [5]

$$\frac{1}{(4\pi)^2} \left\langle \mathrm{GV}(\mathcal{F}), \left[S^3\right] \right\rangle = \frac{1}{(4\pi)^2} \int\limits_{S^3} \mathrm{GV}(\mathcal{F}) = \pm(2-k) \tag{1}$$

the sign depends on the orientation of the fundamental class $[S^3]$. We can interpret the Godbillon–Vey (GV) invariant of the foliation of S^3 as a WZ term. Namely, we consider a smooth map $G : S^3 \to \mathrm{SU}(2)$ and a 3-form $\Omega_3 = \mathrm{Tr}((G^{-1}dG)^3)$ so that the integral

$$\frac{1}{8\pi^2} \int\limits_{S^3=\mathrm{SU}(2)} \Omega_3 = \frac{1}{8\pi^2} \int\limits_{S^3} \mathrm{Tr}\left(\left(G^{-1}dG\right)^3\right) \in \mathbb{Z}$$

is the winding number of G. Thus indeed every Godbillon–Vey class with integer value like (1) is generated by a 3-form Ω_3. Therefore, the Godbillon–Vey class is the WZ term of the $\mathrm{SU}(2)_k$ WZW model. Since the foliation of S^3 with this GV class is generated by some exotic $\mathbb{R}^4$, we obtain the relation:

The structure of exotic $\mathbb{R}^4_k$'s, $k \in \mathbb{Z}$ from the radial family determines the WZ term of the $k-2$ WZW model on SU(2).

This WZ term enables the cancellation of quantum anomaly due to conformal invariance of the classical σ-model on SU(2). Thus we have a way of how to include this cancellation term from smooth 4-geometry: if smoothness of the ambient 4-space, in which S^3 is placed as a part of the boundary of the cork, is that of exotic $\mathbb{R}^4_k$, then the WZ term of the classical σ-model with target $S^3 = \mathrm{SU}(2)$, *i.e.* $\mathrm{SU}(2)_k$ WZW, is precisely generated by this 4-smoothness. An important property follows then:

The change of smoothness of exotic $\mathbb{R}^4_k$ to exotic $\mathbb{R}^4_l$, $k, l \in \mathbb{Z}$ both from the radial family, corresponds to the change of the level k of the WZW model on SU(2), i.e. $k\,\mathrm{WZW} \to l\,\mathrm{WZW}$.

The end of the exotic $\mathbb{R}^4_k$ *i.e.* $S^3 \times \mathbb{R}$ cannot be standard smooth and it is, in fact, fake smooth $S^3 \times_{\Theta_k} \mathbb{R}$, [9]. So we have determined, via WZ term, the "quantized" geometry of $\mathrm{SU}(2)_{k-2} \times \mathbb{R}$ as corresponding to the exotic geometry of the end of $\mathbb{R}^4_k$.

3. Kondo effect

Let us see that the $\mathrm{SU}(2)_k$ WZW model is well suited to the description of the k-channel Kondo effect. The symmetry of the Kondo state is based on infinite dimensional Kac–Moody algebra. This affine algebra $\mathrm{SU}(2)_k$ is spanned on 3-component currents $\vec{\mathcal{J}}_n$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$

$$\left[\mathcal{J}^a_n, \mathcal{J}^b_m\right]_k = i\epsilon^{abc}\mathcal{J}^c_{n+m} + \tfrac{1}{2}kn\delta^{ab}\delta_{n,-m}\,. \tag{2}$$

Next, we decompose the currents $\vec{\mathcal{J}}_n$ as $\vec{\mathcal{J}}_n = \vec{J}_n + \vec{S}$, where J_n corresponds to conduction electrons and S is the spin of the impurity. Then $\vec{J}_n$ obey the same Kac–Moody algebra, *i.e.* $[J^a_n, J^b_m]_k = i\epsilon^{abc}J^c_{n+m} + \frac{1}{2}kn\delta^{ab}\delta_{n,-m}$ and usual relations for $\vec{S}$ hold true, *i.e.* $[S^a, S^b] = i\epsilon^{abc}S^c$, $[S^a, J^b_n] = 0$. From the point of view of field theories describing the interacting currents with spins, $\vec{\mathcal{J}}_n$ correspond to the effective infrared fixed point of the theory of interacting spins $\vec{S}$ with $\vec{J}_n$. Then, the interaction Hamiltonian of the theory for $k = 1$ reads

$$H_s = c\left(\frac{1}{3}\sum_{-\infty}^{+\infty}\vec{J}_{-n}\cdot\vec{J}_n + \lambda\sum_{-\infty}^{+\infty}\vec{J}_n\cdot\vec{S}\right)\,. \tag{3}$$

The new Hamiltonian where $\vec{S}$ is now effectively absent (for $k = 1$) is given by $H = c'\sum_{-\infty}^{+\infty}\left(\vec{\mathcal{J}}_{-n}\cdot\vec{\mathcal{J}}_n - \frac{3}{4}\right)$. For arbitrary integer k the infrared effective fixed point is now attained for $\lambda = \frac{2}{2+k}$. The spin part of the Hamiltonian reads $H_{s,k} = \frac{1}{2\pi(k+2)}\vec{J}^{\,2} + \lambda\vec{J}\cdot\vec{S}\delta(x)$. The spins $\vec{S}$ reappear as the boundary conditions in the Boundary Conformal Field Theory (BCFT) represented by the WZW model on SU(2). This model defines the Verlinde fusion rules. The following *fusion rules hypothesis*, proposed in [10], explains creation and nature of the multichannel Kondo states:

The infrared fixed point in the k-channel spin-s Kondo problem is given by fusion with the spin-s primary for $s \leqq k/2$ or with the spin $k/2$ primary for $s > k/2$. The level k Kac–Moody algebra, as in the level k WZW SU(2) model, governs the behavior of the Kondo state in the presence of k channels of conducting electrons and magnetic impurity of spin s.

This is also the reason why, already at low temperatures, entangled magnetic matter of impurities and conduction electrons, indicates the exotic 4-geometry. First, every CH generates a fermion field. Every small exotic $\mathbb{R}^4$ can be represented as handlebody where Akbulut cork has several CHs attached. Let us remove a single CH from the handlebody $\mathbb{R}^4_k$. The result is $\mathbb{R}^4_k \setminus \mathrm{CH}$. The boundary of it reads $\partial(\mathbb{R}^4_k \setminus \mathrm{CH})$. The contribution to the Einstein action $\int_{\mathbb{R}^4_k \setminus \mathrm{CH}} R\sqrt{g}d^4x$ from this boundary is the suitable surface term

$$\int_{\partial(\mathbb{R}^4_k \setminus \mathrm{CH})} R\sqrt{g}d^4x + \int_{\partial(\mathbb{R}^4_k \setminus \mathrm{CH})} K_{\mathrm{CH}}\sqrt{g_\partial}d^3x\,,$$

where K_{CH} is the trace of the 2-nd fundamental form and g_∂ is the metric on the boundary [11]. But as shown in [11], this term is expressed by the spinor field ψ describing the immersion of D^2 into $\mathbb{R}^3$, which extends to the immersion of $D^2 \times (0,1)$ into $\mathbb{R}^4$

$$\int_{\partial(\mathbb{R}^4_k \setminus \mathrm{CH})} K_{\mathrm{CH}}\sqrt{g_\partial}d^3x = \int_{\partial(\mathbb{R}^4_k \setminus \mathrm{CH})} \psi\gamma^\mu D_\mu\overline{\psi}\sqrt{g_\partial}d^3x\,. \tag{4}$$

This can be extended to 4-dimensional Einstein–Hilbert action with the source depending on the CH, hence on exotic $\mathbb{R}^4_k$

$$S_4^{\mathrm{CH}}(\mathbb{R}^4_k) = \int_{\mathbb{R}^4_k \setminus \mathrm{CH}} \left(R + \psi\gamma^\mu D_\mu\overline{\psi}\right)\sqrt{g}d^4x\,. \tag{5}$$

Again, it was shown in [11] that the spinor field ψ extends over whole 4-manifold so that the 4D Dirac equations are fulfiled. In this way, we have fermion fields which are determined by CH. Moreover, these fermions plays a role of gravity sources as in (5). In fact, every infinite branch of the CH determines some 4D fermion.

Second, given exotic $\mathbb{R}^4_p$ we have r Casson handles in its handlebody. These r CHs generate effective $q(r)$-many infinite branches. Each such branch generates a fermion field. Attaching the CHs to the cork results in exotic $\mathbb{R}^4_p$. Hence p is the function of q in general, $p = p(q(r))$.

Let us assign now the simplest possible CH to every CH in the handlebody of exotic $\mathbb{R}^4$, such that replacing the original CH by this simple one does not change the exotic smoothness. This is the model handlebody we refer to in the context of the Kondo effect.

The k-channel Kondo state in the k-channel Kondo effect is the entangled state of conducting electrons in k bands and the magnetic spin s impurity. The physics of resulting state is described by BCFT by the Verlinde fusion rules in $\mathrm{SU}(2)_k$ WZW model. To have the WZ term in this WZW model one certainly needs $p = k$. This kWZ term is generated by exotic $\mathbb{R}^4_{k+2}$ (see Sec. 2). The general correspondence appears:

One assigns the 4-smooth geometry on $\mathbb{R}^4$ to the k-channel Kondo effect such that k corresponds to the number of infinite branches of CHs in the handlebody. This 4-geometry is $\mathbb{R}^4_p$, where $p = p(k)$, $p, k \in \mathbb{N}$. The *change between the physical Kondo states, from this emerging in k_1 channel Kondo effect to this with k_2 channels, $k_1 \neq k_2$, corresponds to the change between 4-geometries, from exotic $\mathbb{R}^4_{p_1}$ to $\mathbb{R}^4_{p_2}$, $p_1 \neq p_2$, $p_1, p_2 \in \mathbb{N}$, such that $p_1 = p_1(k_1)$ and $p_2 = p_2(k_2)$ as above.*

Whether actually $p = k$ or not is the question about the level of the SU(2) WZW model and the corresponding fusion rules in use. If $k = p$ the exotic geometry gives the same fusion rules as the Affleck proposed. In the case of $k \neq p$ and $k < p$ in the k-channel Kondo effect the fusion rules derived from the exotic geometry are those of the $\mathrm{SU}(2)_p$ WZW model. In high energies, if Kondo state survives, possibly 4-exotic geometry dominates and fermion fields as conducting electrons could be created by the CH of this geometry. This discrepancy in fusion rules can bear experimental signature and is the content of *relativistic fusion rule* (RFR) hypothesis.

4. 4d matter and quantum gravity

We described so far the relation of specific 4-geometries on Euclidean $\mathbb{R}^4$ with quantum spin matter of entangled states of conduction electrons and magnetic impurities. This is based on the relation, from the previous section, between these geometries and $\mathrm{SU}(2)_k$ WZW models of BCFT. Let us now make a step further and consider gravity included in exotic $\mathbb{R}^4$ in the regime of QG. As we know it requires QG in 4d which does not exist yet. Instead, we analyze the geometry of the topological end $S^3 \times \mathbb{R}$ of $\mathbb{R}^4$ which in the exotic case is replaced by $\mathrm{SU}(2)_k \times \mathbb{R}$. Exactly the same replacement is proposed in superstring theory where one wants to include strong magnetic field and its gravitational backreactions in 4d. This $\mathrm{SU}(2)_k \times \mathbb{R}$ as above is a part of exact 10d heterotic superstring background [12]. Moreover, 4d results were obtained in heterotic superstring theory based on these backgrounds. This coincidence enables us to translate these results to those derived on exotic $\mathbb{R}^4_k$, k even, in the QG regime. In this way, we have a natural candidate for dealing with QG results in 4 dimensions at least for geometries (hence gravity) confined to these 4-exotics. Thus one calculates [12] the energy spectrum of scalar charge particle which travels through 4d exotic geometry, as

$$\begin{aligned}\Delta E^k_{j,m,\overline{m}} &= \frac{1}{k+2}\left[j(j+1) - m^2\right] \\ &\quad + \frac{\left(2\sqrt{k+2}eH - \left(\lambda + \frac{1}{\lambda}\right)m - \left(\lambda - \frac{1}{\lambda}\right)\sqrt{(1+2/k)}\overline{m}\right)^2}{4(k+2)(1-2H^2)}\,. \end{aligned} \tag{6}$$

Here, k stands for the effects from different exotic $\mathbb{R}^4_k$, λ is the moduli representing the gravitational backreaction of the magnetic field H, e is the charge of a particle, $j, m, \overline{m}$ are quantum numbers due to the symmetry of the contracted exotic $\mathbb{R}^4_k$. Similarly, the spectrum for the particle with spin S reads

$$\Delta E_{j,m,S} = \frac{1}{k+2}\left[j(j+1) - (m+S)^2 + \frac{(eHR - m - S)^2}{(1-2H^2)}\right]. \qquad (7)$$

Very important feature of calculations on exotic $\mathbb{R}^4_k$ is the appearance of the mass gap in mass spectra of particles in the theory. This mass gap μ^2 depends on exotic $\mathbb{R}^4_k$ via k as: $\mu^2 \sim \frac{1}{k+2}$ [12]. Such an approach is the realization of the idea that superstring theory refers to physical 4-dimensions via its relation to exotic small $\mathbb{R}^4$'s. New constraints reducing the multitude of possible superstring vacua are thus proposed [7, 8, 13, 14, 15].

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