

SIMPLIFYING 5-POINT TENSOR REDUCTION*

J. FLEISCHER

Fakultät für Physik, Universität Bielefeld
Universitätsstr. 25, 33615 Bielefeld, Germany

T. RIEMANN

Deutsches Elektronen-Synchrotron, DESY
Platanenallee 6, 15738 Zeuthen, Germany*(Received October 18, 2011)*

The 5-point tensors have the property that after insertion of the metric tensor $g^{\mu\nu}$ in terms of external momenta, all $g^{\mu\nu}$ -contributions in the tensor decomposition cancel. If furthermore the tensors are contracted with external momenta, the inverse 5-point Gram determinant $(\)_5$ cancels automatically. If the remaining 4-point sub-Gram determinant $\binom{s}{s}_5$ is not small then this approach appears to be particularly efficient in numerical calculations. We also indicate how to deal with small $\binom{s}{s}_5$. Explicit formulae for tensors of degree 2 and 3 are given for large and small (sub-) Gram determinants.

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1. Introduction

In [1] we have worked out an algebraic method to present one-loop tensor integrals in terms of scalar one-loop 1-point to 4-point functions. The tensor integrals are defined as

$$I_n^{\mu_1 \cdots \mu_R} = \int \frac{d^d k}{i\pi^{d/2}} \frac{\prod_{r=1}^R k^{\mu_r}}{\prod_{j=1}^n c_j}, \quad (1.1)$$

with denominators c_j , having *chords* q_j ,

$$c_j = (k - q_j)^2 - m_j^2 + i\epsilon. \quad (1.2)$$

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For the tensor decomposition we use Davydychev's approach [2], recursion relations as given in [3] and make detailed use of *modified Cayley determinants* introduced for this purpose in [4]. All these techniques we ask the reader to look up in [1]. The following linear combinations of the chords have proven as particularly useful

$$Q_s^\mu = \sum_{i=1}^5 q_i^\mu \frac{\binom{s}{i}_5}{\binom{}{5}_5}, \quad s = 0 \dots 5, \quad (1.3)$$

$$Q_s^{t,\mu} = \sum_{i=1}^5 q_i^\mu \frac{\binom{st}{it}_5}{\binom{t}{t}_5}, \quad s, t = 1 \dots 5. \quad (1.4)$$

2. Explicit examples

According to [2] we write the tensor of rank 2 as $([d+]^l = d + 2l, d = 4 - 2\varepsilon)$

$$I_5^{\mu\nu} = \sum_{i,j=1}^4 q_i^\mu q_j^\nu \nu_{ij} I_{5,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu} I_5^{[d+]}. \quad (2.1)$$

Inserting

$$\frac{1}{2} g^{\mu\nu} = \sum_{i,j=1}^5 \frac{\binom{i}{j}_5}{\binom{}{5}_5} q_i^\mu q_j^\nu \quad (2.2)$$

and using the recursion relation

$$\nu_{ij} I_{5,ij}^{[d+]^2} = -\frac{\binom{0}{j}_5}{\binom{}{5}_5} I_{5,i}^{[d+]} + \sum_{s=1, s \neq i}^5 \frac{\binom{s}{j}_5}{\binom{}{5}_5} I_{4,i}^{[d+],s} + \frac{\binom{i}{j}_5}{\binom{}{5}_5} I_5^{[d+]}, \quad (2.3)$$

we see that the $g^{\mu\nu}$ cancels with the result

$$I_5^{\mu\nu} = I_5^\mu Q_0^\nu - \sum_{s=1}^5 \left\{ Q_0^{s,\mu} I_4^s - \sum_{t=1}^5 Q_t^{s,\mu} I_3^{st} \right\} Q_s^\nu, \quad (2.4)$$

$$I_5^\mu = E Q_0^\mu - \sum_{s=1}^5 I_4^s Q_s^\mu, \quad E = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \binom{s}{0}_5 I_4^s. \quad (2.5)$$

A more direct approach is the use of the 5-point recursion in terms of 4-point functions [5]

$$I_5^{\mu_1 \dots \mu_{R-1} \mu} = I_5^{\mu_1 \dots \mu_{R-1}} Q_0^\mu - \sum_{s=1}^5 I_4^{\mu_1 \dots \mu_{R-1}, s} Q_s^\mu. \quad (2.6)$$

This formula in general takes into account some cancellations of the $g_{\mu\nu}$. Only $g_{\mu\nu}$ -contributions from 4-points have still to be dealt with — and they are simpler to handle. This we demonstrate for the tensor of degree 3. As a special case of (2.6) we have

$$I_5^{\mu\nu\lambda} = I_5^{\mu\nu} \cdot Q_0^\lambda - \sum_{s=1}^5 I_4^{\mu\nu, s} \cdot Q_s^\lambda. \quad (2.7)$$

The corresponding 4-point function reads ($q_5 = 0$)

$$\begin{aligned} I_4^{\mu\nu, s} &= \sum_{i,j=1}^4 q_i^\mu q_j^\nu \nu_{ij} I_{4,ij}^{[d+]^2, s} - \frac{1}{2} g^{\mu\nu} I_4^{[d+], s}, \\ \nu_{ij} I_{4,ij}^{[d+]^2, s} &= -\frac{\binom{0s}{js}_5}{\binom{s}{s}_5} I_{4,i}^{[d+], s} + \frac{\binom{is}{js}_5}{\binom{s}{s}_5} I_4^{[d+], s} + \sum_{t=1}^5 \frac{\binom{ts}{js}_5}{\binom{s}{s}_5} I_{3,i}^{[d+], st}, \end{aligned} \quad (2.8)$$

and with

$$\frac{\binom{is}{js}_5}{\binom{s}{s}_5} = \frac{\binom{i}{j}_5}{\binom{}{ }_5} - \frac{\binom{s}{i}_5 \binom{s}{j}_5}{\binom{}{ }_5 \binom{s}{s}_5} \quad (2.9)$$

we again observe the possibility to cancel $g^{\mu\nu}$ with the result

$$\begin{aligned} I_4^{\mu\nu, s} &= Q_0^{s, \mu} Q_0^{s, \nu} I_4^s - \frac{\binom{}{ }_5}{\binom{s}{s}_5} Q_s^\mu Q_s^\nu I_4^{[d+], s} - \sum_{t=1}^5 \left\{ Q_t^{s, \mu} Q_0^{s, \nu} I_3^{st} + Q_t^{s, \nu} I_3^{\mu, st} \right\}, \\ I_3^{\mu, st} &= -\sum_{i=1}^4 q_i^\mu I_{3,i}^{[d+], st} = Q_0^{st, \mu} I_3^{st} - \sum_{u=1}^5 Q_u^{st, \mu} I_2^{stu}. \end{aligned} \quad (2.10)$$

3. Contracting the tensor integrals

Scalar expressions, contracting with chords, are

$$q_{i_1 \mu_1} \cdots q_{i_R \mu_R} I_5^{\mu_1 \dots \mu_R} = \int \frac{d^d k}{i \pi^{d/2}} \frac{\prod_{r=1}^R (q_{i_r} \cdot k)}{\prod_{j=1}^5 c_j}, \quad (3.1)$$

$$g_{\mu_1, \mu_2} q_{i_1 \mu_3} \cdots q_{i_R \mu_R} I_5^{\mu_1 \dots \mu_R} = \int \frac{k^2 d^d k}{i \pi^{d/2}} \frac{\prod_{r=3}^R (q_{i_r} \cdot k)}{\prod_{j=1}^5 c_j}, \quad (3.2)$$

etc.; (3.1) and (3.2) define the contraction of all tensor indices with chords and the direct contraction of two tensor indices, respectively. These are obtained in realistic matrix element calculations by constructing projection operators or by constructing scalar differential cross sections ($\text{Born} \times 1\text{-loop}$) before loop integration. As a result of these contractions the $1/(\)_5$ cancels already.

To begin with, we have a look at

$$q_{a\mu}q_{b\nu}I_5^{\mu\nu} = (q_a \cdot I_5)(q_b \cdot Q_0) - \sum_{s=1}^5 \left\{ (q_a \cdot Q_0^s) I_4^s - \sum_{t=1}^5 (q_a \cdot Q_t^s) I_3^{st} \right\} (q_b \cdot Q_s) . \quad (3.3)$$

For $q_n = 0$, $a = 1, \dots, n-1$, $s = 1, \dots, n$

$$(q_a \cdot Q_0) = \sum_{j=1}^{n-1} (q_a \cdot q_j) \frac{\binom{0}{j}_n}{\binom{0}{n}} = -\frac{1}{2} (Y_{an} - Y_{nn}) , \quad (3.4)$$

$$(q_a \cdot Q_s) = \sum_{j=1}^{n-1} (q_a \cdot q_j) \frac{\binom{s}{j}_n}{\binom{s}{n}} = \frac{1}{2} (\delta_{as} - \delta_{ns}) , \quad (3.5)$$

and

$$(q_a \cdot I_5) = E(q_a \cdot Q_0) - \sum_{s=1}^5 I_4^s(q_a \cdot Q_s) . \quad (3.6)$$

Further

$$(q_a \cdot Q_0^s) = \frac{1}{\binom{s}{s}_5} \Sigma_a^{2,s} , \quad (q_a \cdot Q_t^s) = \frac{1}{\binom{s}{s}_5} \Sigma_a^{1,st} , \quad (3.7)$$

where the sums $\Sigma_a^{2,s}$ and $\Sigma_a^{1,st}$ are given in [6]. Both are linear combinations of $\binom{s}{s}_5$ and Kronecker- δ 's, $(\delta_{as} - \delta_{5s})$. Indeed the fact that there is no inverse $(\)_5$ anymore is due to relations (3.4) and (3.5). The second scalar which can be constructed from the tensor of degree 2 is $g_{\mu\nu}I_5^{\mu\nu}$. Due to (2.4) we need to evaluate the following scalar products

$$\begin{aligned} (Q_0 \cdot Q_0) &= \frac{1}{2} \left[\frac{\binom{0}{0}_5}{\binom{0}{5}} + Y_{55} \right] , \\ (Q_0 \cdot Q_s) &= \frac{1}{2} \left[\frac{\binom{s}{0}_5}{\binom{s}{5}} - \delta_{s5} \right] , \\ (Q_0^s \cdot Q_s) &= -\frac{1}{2} \delta_{s5} , \\ (Q_t^s \cdot Q_s) &= 0 . \end{aligned} \quad (3.8)$$

In this case the terms with $1/(\)_5$ cancel and, not surprisingly, the result finally is

$$g_{\mu\nu}I_5^{\mu\nu} = \frac{Y_{55}}{2}E + I_4^5. \quad (3.9)$$

To calculate $g_{\mu\nu}I_5^{\mu\nu\lambda}$ we need $g_{\mu\nu}I_4^{\mu\nu,s}$ and thus further scalar products, see (2.10)

$$\begin{aligned} (Q_0^s \cdot Q_0^s) &= \frac{1}{2\binom{s}{s}_5} \left[\binom{0s}{0s}_5 + 2\binom{s}{0}_5 \delta_{s5} \right] + \frac{1}{2}Y_{55}, \\ (Q_s \cdot Q_s) &= \frac{1}{2} \frac{\binom{s}{s}_5}{\binom{s}{s}_5}, \\ (Q_t^s \cdot Q_0^s) &= \frac{1}{2\binom{s}{s}_5} \left[\binom{ts}{0s}_5 - \binom{s}{s}_5 \delta_{t5} + \binom{s}{t}_5 \delta_{s5} \right], \\ (Q_t^s \cdot Q_0^{st}) &= \frac{1}{2\binom{s}{s}_5} \left[-\binom{s}{s}_5 \delta_{t5} + \binom{s}{t}_5 \delta_{s5} \right], \\ (Q_t^s \cdot Q_u^{st}) &= 0, \end{aligned} \quad (3.10)$$

which yields

$$g_{\mu\nu}I_4^{\mu\nu,s} = \frac{Y_{55}}{2}I_4^s + I_3^{s5} + \frac{\delta_{s5}}{\binom{s}{s}_5} \left[\binom{s}{0}_5 I_4^s - \sum_{t=1}^5 \binom{s}{t}_5 I_3^{st} \right], \quad (3.11)$$

and finally

$$g_{\mu\nu}I_5^{\mu\nu\lambda} = -\frac{Y_{55}}{2} \sum_{s=1}^5 I_4^s Q_s^{0,\lambda} + I_4^5 Q_0^{5,\lambda} - \sum_{t=1}^4 I_3^{5t} Q_t^{5,\lambda}. \quad (3.12)$$

It is remarkable that (3.11) is trivial again for $s \neq 5$. For $s = 5$, however, the standard cancellation of propagators does not work and for this case (3.11) is indeed a useful result. For further contraction of (3.12) with a vector q_λ again (3.7) can be applied.

4. Avoiding inverse 4-point Gram determinants

While in the above approach of taking scalar products of the tensors with chords the inverse $(\)_5$ Gram determinant cancels already, there still remains the inverse $\binom{s}{s}_5$ sub-Gram determinant of the 4-point functions. Therefore, we have to choose a different approach for the case if the latter becomes small. This approach consists in avoiding the inverse $(\)_5$ already

in the 5-point tensors from the very beginning and keeping only 4-point integrals in higher dimensions (*i.e.* integrals with only powers 1 of the scalar propagators), which for small $\binom{s}{5}_5$ should be evaluated in a different manner than by standard recursion, see [1]. If one does not want to reintroduce the inverse $(\)_5$ to see the cancellation of the $g^{\mu\nu}$, one can explicitly see its cancellation also after taking contractions. For the tensor of degree 2 we refer to [6] for the contraction with two chords. For the self-contraction of the tensor indices no simpler result than (3.9) can be achieved anyway. For the tensor of degree 3 we present new results for the contraction with three chords and a self-contraction.

The tensor can be written as follows (see [1] (4.35)–(4.37))

$$I_5^{\mu\nu\lambda} = \sum_{i,j,k=1}^5 q_i^\mu q_j^\nu q_k^\lambda E_{ijk} + \sum_{k=1}^5 g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \quad (4.1)$$

with

$$E_{00k} = \sum_{s=1}^5 \frac{1}{\binom{0}{5}_5} \left[\frac{1}{2} \binom{0s}{0k}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{k}_5 I_4^{[d+]^2,s} \right], \quad (4.2)$$

$$E_{ijk} = - \sum_{s=1}^5 \frac{1}{\binom{0}{5}_5} \left\{ \left[\binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\}. \quad (4.3)$$

Contraction of the tensor with three chords yields

$$\begin{aligned} q_{a\mu} q_{b\nu} q_{c\lambda} I_5^{\mu\nu\lambda} &= \sum_{i,j,k=1}^4 (q_a \cdot q_i)(q_b \cdot q_j)(q_c \cdot q_k) E_{ijk} \\ &+ \sum_{k=1}^4 [(q_a \cdot q_b)(q_c \cdot q_k) + (q_a \cdot q_c)(q_b \cdot q_k) + (q_b \cdot q_c)(q_a \cdot q_k)] E_{00k}. \end{aligned} \quad (4.4)$$

Introducing $Y_a = Y_{a5} - Y_{55}$, $D_a^s = \delta_{as} - \delta_{5s}$ and the following kinematical objects,

$$\begin{aligned} P_{I4} &= \frac{1}{8} \frac{1}{\binom{0s}{5}_5} \left\{ \binom{s}{s}_5 \left[\binom{s}{0}_5 Y_a Y_b Y_c + \binom{0}{0}_5 (Y_a Y_b D_c^s + Y_a Y_c D_b^s + Y_b Y_c D_a^s) \right] \right. \\ &\quad \left. + \binom{0}{0}_5 \left[\binom{s}{0}_5 (Y_a D_b^s D_c^s + Y_b D_a^s D_c^s + Y_c D_a^s D_b^s) + \binom{0}{0}_5 D_a^s D_b^s D_c^s \right] \right\}, \end{aligned} \quad (4.5)$$

$$P_{Z4} = P_{I4} - \frac{1}{12} \binom{0}{5}_5 \{ Y_a Y_b D_c^s + Y_a Y_c D_b^s + Y_b Y_c D_a^s \} \quad \text{and} \quad (4.6)$$

$$P_{I3} = \frac{1}{24} \frac{\binom{0}{5}_5}{\binom{0s}{5}_5} \left\{ [Y_a D_b^s + Y_b D_a^s] \left[\binom{0s}{0s}_5 D_c^t - \binom{0s}{0t}_5 D_c^s \right] + (a \leftrightarrow c) + (b \leftrightarrow c) \right\}, \quad (4.7)$$

we can write

$$q_{a\mu}q_{b\nu}q_{c\lambda}I_5^{\mu\nu\lambda} = \frac{d-2}{8} \frac{(\cdot)_5}{(0s)_5} (\delta_{ab}\delta_{ac}\delta_{as} - \delta_{5s})(d-1)I_4^{[d+],2,s} \\ - \frac{1}{(\cdot)_5} \left\{ P_{I_4} I_4^{[d+],s} - P_{Z_4} Z_4^{[d+],s} + P_{I_3} I_3^{[d+],st} \right. \\ \left. + \frac{1}{3} \left[\Sigma_c^{1,s} R_{ab} + \Sigma_b^{1,s} R_{ac} + \Sigma_a^{1,s} R_{cb} \right] \right\} + F_{abc}^s, \quad (4.8)$$

symmetric in the indices a, b, c and summation over s, t assumed. Here

$$R_{ab} = - \frac{1}{(\cdot)_5} \left\{ \frac{1}{(\cdot)_5} \Sigma_b^{2,s} \Sigma_a^{2,st} (d-2) I_3^{[d+],st} \right. \\ \left. + \frac{1}{(\cdot)_{st}} \Sigma_b^{2,st} \left[\Sigma_a^{3,st} (d-2) I_3^{[d+],st} - \sum_{u=1}^5 \Sigma_a^{2,stu} I_2^{stu} \right] \right\} \quad (4.9)$$

contains only 3-point functions and no inverse $(\cdot)_5$.

$$F_{abc}^s = - \frac{1}{24} \frac{(\cdot)_5}{(0s)_5} \frac{(\cdot)_5}{(\cdot)_5} [Y_c D_a^s D_b^s + Y_b D_a^s D_c^s + Y_a D_b^s D_c^s] \quad (4.10)$$

is a rational term obtained from an ε -expansion. The fact that no scalar products from (4.4) remain demonstrates that the $g^{\mu\nu}$ term has canceled.

For the selfcontracted tensor we obtain

$$q_{a\lambda} I_{5,\mu}^{\mu\lambda} = - \frac{1}{(\cdot)_5} \sum_{s=1}^5 \left\{ \Sigma_a^{1,s} \left[1 + \frac{1}{2} \frac{1}{(\cdot)_{0s}} \left(\binom{s}{s}_5 Y_{55} + 2 \binom{s}{0}_5 \delta_{s5} \right) \right] \right. \\ \left. + (\cdot)_5 (Y_{a5} - Y_{55}) \delta_{s5} \right\} I_4^{[d+],s} \\ - \frac{1}{(\cdot)_5} \sum_{s,t=1}^5 \Sigma_a^{1,s} \frac{(\cdot)_{ts}}{(\cdot)_{0s}} \frac{Y_{55}}{(\cdot)_{0st}} \left[\frac{d-2}{2} \binom{st}{st}_5 I_3^{[d+],st} + \frac{1}{2} \sum_{u=1}^5 \binom{st0}{stu}_5 I_2^{stu} \right] \\ + \frac{\delta_{s5}}{(\cdot)_5} \sum_{t=1}^5 \Sigma_a^{1,s} \frac{1}{(\cdot)_{0s} (\cdot)_{0st}} \left\{ \left[\binom{0s}{0t}_5 \binom{st}{st}_5 - \binom{ts}{0s}_5 \binom{st}{0t}_5 \right] \frac{d-2}{2} I_3^{[d+],st} \right. \\ \left. + \sum_{u=1}^5 \left[\binom{0s}{0t}_5 \binom{st0}{stu}_5 - \binom{ts}{0s}_5 \binom{st0}{0tu}_5 \right] \frac{1}{2} I_2^{stu} \right\} \\ - \frac{\delta_{t5}}{(\cdot)_5} \sum_{s=1}^5 \Sigma_a^{1,s} \frac{1}{(\cdot)_{0s} (\cdot)_{0st}} \left\{ \left[\binom{0s}{0s}_5 \binom{st}{st}_5 + \binom{ts}{0s}_5 \binom{ts}{0s}_5 \right] \frac{d-2}{2} I_3^{[d+],st} \right.$$

$$\begin{aligned}
& + \sum_{u=1}^5 \left[\binom{0s}{0s}_5 \binom{st0}{stu}_5 + \binom{ts}{0s}_5 \binom{ts0}{0su}_5 \right] \frac{1}{2} I_2^{stu} \Big\} \\
& + \frac{1}{\binom{0}{0}_5} \sum_{s,t=1}^5 \Sigma_a^{1,s} \frac{1}{\binom{0s}{0s}_5} \left\{ \left[\binom{s}{t}_5 \delta_{s5} - \binom{s}{s}_5 \delta_{t5} \right] \frac{d-2}{2} I_3^{[d+],st} - \binom{ts}{0s}_5 \frac{1}{2} I_2^{st5} \right\} \\
& + \frac{1}{\binom{0}{0}_5} \sum_{s,t=1}^5 \left[\binom{t}{0}_5 (\delta_{as} - \delta_{5s}) - \binom{s}{0}_5 (\delta_{at} - \delta_{5t}) \right] I_3^{[d+],st}. \tag{4.11}
\end{aligned}$$

Only the first two lines of (4.11) contain complete double sums while the remaining terms contribute only for specific values of s, t, u .

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