Vol. 42 (2011) *ACTA PHYSICA POLONICA B* No 2

THERMODYNAMICS OF THE GRAVITY DUAL TO BOOST-INVARIANT HYDRODYNAMICS BASED ON AdS_5 BLACK HOLE

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(Received November 25, 2009; revised version received March 15, 2010; final version received December 20, 2011)

In this paper we consider the particular time-dependent background in five dimensions. We have used dual geometry in Eddington–Finkelstain type coordinates. By using the first law of black hole in AdS_5 geometry, we obtain a thermodynamics formulation. Also we obtain the dynamical surface gravity for AdS_5 black hole. We show that the first law of black hole dynamics needs some correction term.

DOI:10.5506/APhysPolB.42.179
PACS numbers: 04.60.Cf, 04.70.Dy, 11.25.Mj

1. Introduction

We note that one of the important applications of the AdS/CFT [1, 2] has been the study of the quark–gluon plasma (QGP) which is produced in heavy ion collisions. It is believed that the QGP formed at RHIC is strongly coupled [3] and also well defined in terms of ideal fluid hydrodynamics [4]. QGP is strongly interacting in a certain range of temperatures, in particular at those accessible at RHIC. The properties of the strongly coupled expanding plasma are in one-to-one correspondence with the metric of a five-dimensional black hole [5]. Since the QGP at RHIC is expanding, *i.e.* is time-dependent, so the time-dependent system will be important in five dimensions. We note here QGP is best described by an ideal fluid dynamics after thermalization of nuclear matter. The relativistic hydrodynamics describes the macroscopic nature of the plasma. The AdS/CFT correspondence is an important mathematical tool for studying the time-dependent plasma, since it may describe both the microscopic and the macroscopic nature of Yang–Mills (YM) theories in a single framework.

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An interesting kinematical regime of the expanding plasma is the so-called central rapidity region, as was suggested by Bjorken [6] and considered for the first time at strong coupling by Janik and Peschanski [7]. The Bjorken flow is a simplest and standard model for describing the evolution of QGP. Here we also stress that Bjorken flow has both the regime of far-from-equilibrium and near-equilibrium dynamics. We focus on Bjorken hydrodynamics which is near-equilibrium (large proper time) regime. By using the AdS/CFT correspondence the transport coefficients at finite temperature have been calculated using the static black hole dual geometry [8]. The thermalization has been identified with a black hole formation [9]. Subsequent near-equilibrium evolution of the black hole is dual to holographic hydrodynamics. Bjorken hydrodynamics provides a simple, yet useful example of solutions of hydrodynamic equations, which provides tractable dual gravity. The dual gravity to Bjorken hydrodynamics is constructed in a perturbative expansion at late proper time [7, 10]. Also the dual gravity in zeroth, first, second and to the third order is proposed in Refs. [7,11,12,13,14]. In Ref. [7] it has been suggested that the regularity of the holographic geometry determines the hydrodynamic parameters of the corresponding gauge-theory fluid. In fact, the equation of state [7], the shear viscosity [13] and the relaxation time [14] have been concluded uniquely from the regularity. Fortunately, the results agree with those computed by other methods.

Time-dependent black holes is assumed for background geometry in the previous works [7, 10, 11, 12, 13, 14]. However, it is non-trivial to see the presence of an event horizon in a time dependent geometry. As far as we know, no rigorous proof of the presence of the event horizon on the dual time-dependent geometry has yet been reported [16]. However, the presence of the event horizon on the dual time-dependent geometry has been studied already [15, 16]. Since the event horizon is defined globally, its analysis is hard in a time-dependent geometry. In Ref. [16] instead of event horizon, there is a more convenient object proposed which is the apparent horizon defined locally. The location of the apparent horizon has been determined explicitly in a new dual geometry, to the second order of late proper time expansion. It is known that the presence of apparent horizon is a signature of singularity. We should note that the leading order geometry is recently given in terms of boosted and dilated black brane, an insight taken from fluid/gravity duality. In that case, it is trivial to get the first law of thermodynamics, since it takes literary the form of the one for a static black brane [17]. The new ingredient of this paper is discussion of the surface gravity and relation with the first law of dynamical AdS_5 black hole.

The relationship between black hole evolution and thermodynamics is one of the most widely studied topics in theoretical physics. Bekenstein suggested that the black hole entropy is proportional to the area of its event horizon divided by the Planck area [18]. Later, Hawking showed that a black hole with surface gravity k emits radiation with temperature $T_{\mathrm{H}} = k/2\pi$ [19]. By identifying T_{H} with the temperature of the black hole, Hawking could show that black hole entropy is determined to be 1/4 of the horizon area. The expression for black hole entropy is called the Bekenstein–Hawking formula. If black hole has no entropy, mass falling into the black hole, may break the second law of thermodynamics. For black hole mechanics fourth law is suggested, by B. Carter, S. Hawking and J. Bardeen. The zeroth law of the black hole is, that for a stationary black hole the horizon has constant surface gravity [20]. Surface gravity k, is gravitational acceleration experienced at the surface of an object. It can be thought as the acceleration due to gravity experienced by test body which is very close to the object's surface and which, in order not to disturb the system, has negligible mass. But for a black hole one can not define a surface gravity as the acceleration experienced by a test particle at the object's surface is infinite. In fact, the surface gravity of a general black hole is not well defined. However, one can define the surface gravity for a black hole whose event horizon is a Killing horizon.

The first law of black hole is similar to the first law of thermodynamics,

$$dM = \frac{k}{8\pi} dA + \Omega dJ + \Phi dQ \,. \tag{1}$$

In the second law of the black hole the horizon area is a non-decreasing function of time and finally the third law implies is not possible to obtain $k = 0$. In other words, a black hole with disappearing surface gravity can not form. In these four laws, especially the first law, there is relation between infinitesimal variations of parameters of stationary black holes, such as, horizon area, mass, angular momentum, *etc.* As we know these quantities are not time-dependent. So, in order to study the dynamical version of such quantities we need a dynamical version of the first law of black holes [21]. Ref. [22] discussed the first law of black hole dynamics in four dimensional space-time assuming the spherical symmetry. In that case the dynamical surface gravity is also obtained. Also Ref. [23] discussed the first law of black hole dynamics in four-dimensional space-time but without assuming any symmetry or any asymptotic conditions. But, we consider for the first time the first law of black hole dynamics in AdS_5 space-time. In that case we obtain the dynamical surface gravity and first law of black hole dynamics. We show that the first law of black hole dynamics will agree with [22,23] up to some correction term.

2. Bjorken hydrodynamics

Let us review the properties of the Bjorken flow (BF) which is the main phase in post-thermalization evolution of the QGP formed in heavy-ion collisions. It is described by the finite temperature theory of a variant of $\mathcal{N}=4$ SYM. In the Bjorken's model, the system endure one-dimensional expansion (Bjorken expansion) along the collision axis of the heavy ions, and the fluid of the quarks and gluons has boost symmetry in the so-called central rapidity region [6]. One can consider this model of the plasma as quasi-ideal hydrodynamics. Let (τ, y, x^2, x^3) be the local rest frame (LRF) of our Bjorken flow on which the fluid is at rest. Here, τ is the proper-time, y is the rapidity, x^2 and x^3 are the perpendicular directions to the collisions axis. We assume that the fluid extends homogeneously in the perpendicular directions, and we have translational and the rotational symmetries on the (x^2, x^3)-plane. We note here the spherical symmetry is really a symmetry of a plane in the context of Bjorken flow. Also we assume that the boost invariance is the translational symmetry in the y direction (for simplicity $y \longrightarrow -y$). However, in the realistic QGP, the boost invariance is realized at the central rapidity region where y is small. But we assume that the boost invariance holds the entire region of y for simplicity. Now by introducing proper time τ and rapidity y coordinates in the longitudinal position plane

$$x^0 = \tau \cosh y\,, \qquad x^1 = \tau \sinh y\,, \tag{2}$$

we have a following Minkowski metric

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_\perp^2\,, \tag{3}$$

where $x_\perp$ denotes the transverse directions. For a conformally invariant fluid, energy momentum conservation and stress tensor is traceless

$$\nabla_\mu T^{\mu\nu} = 0\,, \qquad T^\mu_\mu = 0\,. \tag{4}$$

Thus all components of the energy momentum tensor can be constrain in terms of a single function $\epsilon(\tau)$ of energy density [7]. Then for an ideal conformal fluid, the energy density and temperature will be

$$\epsilon(\tau) = \frac{\epsilon_0}{\tau^{\frac{4}{3}}}\,, \qquad T \sim \tau^{\frac{-1}{3}}\,. \tag{5}$$

Also the entropy per unit rapidity remains constant. The AdS/CFT correspondence for dual gravity helps us to derive the transport properties of the plasma.

3. The gravity dual to Bjorken flow

The first dual gravity to the hydrodynamic description of the YM fluid is proposed as a five-dimensional metric. Einstein's equation together with the boundary condition given by the energy-momentum tensor at the boundary lead us to the metric in five dimensions [8, 10]. So, we consider the general asymptotically AdS metrics in the Fefferman–Graham coordinates [5, 7, 12],

$$ds^2 = r_0^2 \frac{g_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2}\,, \tag{6}$$

where $x^\mu = (\tau, y, x^2, x^3)$ and $r_0 \equiv (4\pi g_\mathrm{s} N_c \alpha'^2)^{1/4}$ is the length scale given by the string coupling g_s and the number of the colors N_c. The four-dimensional metric $g_{\mu\nu}$ is expanded with respect to z, [7]. For extracting a non-trivial result in [8] we take the limit such that $\tau \longrightarrow \infty$, where convention $v = \frac{z}{\tau^{1/3}}$ is fixed. This analysis leads to the above mentioned perturbation expansion in $\tau^{-2/3}$. Then Einstein's equations are solved to the requested order in a power series in $\tau^{-2/3}$ at late times. Finally, analytic expression of the late time metric is obtained:

$$ds^2 = \frac{1}{z^2}\left[-\frac{1-\left(\frac{\rho_0}{3}\frac{z^4}{z^{4/3}}\right)^2}{1+\frac{\rho_0}{3}\frac{z^4}{z^{4/3}}} d\tau^2 + \left(1+\frac{\rho_0}{3}\frac{z^4}{z^{4/3}}\right)\left(\tau^2 dy^2 + dx_\perp^2\right)\right] + \frac{dz^2}{z^2}\,. \tag{7}$$

Notice that the metric (7) is a black hole in AdS space with time-dependent horizon. The time dependence of the entropy and the Hawking temperature from the metric (7) reproduces the Bjorken's results [6]. Also it is observed that the regularity of the geometry at the horizon uniquely selects the power of time evolution of energy density $\rho \sim \tau^{-4/3}$ which makes sense of the hydrodynamics. For details we refer the reader to [7, 12].

Since the space-time of the dual geometry is required to be regular, it was shown that the shear-viscosity of the plasma saturates the universal lower bound $\eta/s = 1/4\pi$. In [14, 16] this geometry is expanded to higher orders, and the space-time at the third order appears to be singular.

However, the Fefferman–Graham (FG) coordinates are very useful for the description of the holographic renormalization [24, 25], and have been used to describe the holographic dual of Bjorken flow in Refs. [7, 11, 12, 13, 14], but in [16] a very important problem in FG coordinates is pointed out which prevents us from examining (dynamical) apparent horizons in the dual geometries. In fact we cannot see any trapped region and cannot examine the location of the horizon in a well-defined way, in these coordinates. Then, it is necessary to construct the dual geometry in a better coordinate system based on a well-defined approximation. In [15, 16] the dual geometry in the Eddington–Finkelstein type coordinates is proposed, where the trapped region and the entrapped region are packed into a single coordinate patch. Also it was argued that the dual geometry of the Bjorken flow on the EF coordinates is actually regular.

4. Gravity dual of Bjorken flow on Eddington–Finkelstein type coordinates

We begin with a short review of this construction by conventions of Ref. [16]. The dual geometry should be a solution to the $10d$ type IIB super-gravity equation. However, for our systems, this equation is reduced to a five-dimensional ($5d$) Einstein's equation with a negative cosmological constant $\Lambda = -6$ [26]

$$R^{\mu}_{\nu} - \frac{1}{2} g^{\mu\nu} R - 6 g^{\mu\nu} = 0\,. \tag{8}$$

Also we use τ rather than τ_+ in [16]. Since we want the bulk geometry to be asymptotic to (3) on the boundary and naturally adapt the coordinate chart to ingoing null geodesics, the boundary condition is given by the LRF (3). So, the Dirichlet boundary condition is applied to the metric rather than the Neumann boundary condition. Then by prescription [2] the non-normalizable mode of the bulk metric is identified with the $4d$ metric. One can consider the following Eddington–Finkelstein type coordinates system

$$ds^2 = -r^2 a d\tau^2 + 2 d\tau dr + r^2 \tau^2 e^{2(b-c)} \left(1 + \frac{1}{u\tau^{2/3}}\right)^2 dy^2 + r^2 e^c dx_{\perp}^2\,, \tag{9}$$

where $u \equiv r\tau^{1/3}$ and the functions a, b, c depend on u and r. In the limit $\tau \longrightarrow \infty$, u is fix. In order to obtain components of a, b and c we expand the parameters as

$$\begin{aligned} a(\tau, u) &= a_0(u) + a_1(u)\tau^{-2/3} + a_2(u)\tau^{-4/3} + O\left(\tau^{-2k/3}\right)\,, \\ b(\tau, u) &= b_0(u) + b_1(u)\tau^{-2/3} + b_2(u)\tau^{-4/3} + O\left(\tau^{-2k/3}\right)\,, \\ c(\tau, u) &= c_0(u) + c_1(u)\tau^{-2/3} + c_2(u)\tau^{-4/3} + O\left(\tau^{-2k/3}\right)\,. \end{aligned} \tag{10}$$

Then we solve the $5d$ Einstein's equation order by order in the large τ expansion to determine $a(n)$, $b(n)$, $c(n)$. To solve the Einstein equations, one requires the boundary conditions

$$a|_{u=\infty} = 1\,, \qquad b|_{u=\infty} = 1\,, \qquad c|_{u=\infty} = 1\,, \tag{11}$$

which guarantee the boundary condition of our metric consistent with the Bjorken flow (3). Finally, the metric $g^{(k)}$ is obtained by this procedure at order $O(\tau^{-2k/3})$, so that $g^{(k)}$ is specified completely by the functions $a_i(u)$, $b_i(u)$ and $c_i(u)$ for $i \ll k$.

Then, by solving Einstein's equations order by order in the late time expansion, one can show that the solutions depend only on a set of arbitrary constants. These constants can be fully determined order by order by requiring that the geometry is asymptotically AdS. Also in [16] it was found that the transport coefficients, such as shear viscosity are determined by the regularity condition uniquely, which agrees with the results achieved by other methods. Also it was shown by choosing the transport coefficients appropriately that the regularity of the dual geometry is realized at all orders. So, the logarithmic singularity discussed in Ref. [14] appear in our dual geometry. The Kretschmann scalar is regular except at the origin $r = 0$ [16, 26]. Moreover, the presence of the apparent horizon, shows that the geometry is actually a dynamical black hole. The singularity at the origin is not a naked singularity and the dual geometry is totally healthy. By selecting the proper transport coefficients the metric is found to be regular.

Now we express the result of [16] at zeroth order of the late-time approximation by the following formulae:

$$\begin{aligned} a_0(u) &= \frac{(1-\xi_0/4)^4 - \omega^4 u^{-4}}{(1-\xi_0/4)^2}\,, \\ b_0(u) &= 3\log(1-\xi_0/4)\,, \\ c_0(u) &= 2\log(1-\xi_0/4)\,. \end{aligned} \tag{12}$$

For the first order we obtain

$$\begin{aligned} a_1(u) &= -\frac{2}{3}\frac{(1+\xi_1)u^4 + \xi_1\omega^4 - 3\eta_0 u\omega^4}{u^5}\,, \\ b_1(u) &= -\frac{1+\xi_1}{u}\,, \\ c_1(u) &= \frac{1}{3\omega}\left[\arctan\left(\frac{u}{\omega}\right) - \frac{\pi}{2} + \frac{1}{2}\left(\frac{u-\omega}{u+\omega}\right)\right] \\ &\quad - \frac{\eta_0}{2}\log\left(1-\omega^4 u^{-4}\right) - \frac{2\xi_1}{3u}\,, \end{aligned} \tag{13}$$

where ξ_0 and ξ_1 are integration constants not fixed by the boundary data and are remaining gauge degrees of freedom. One useful gauge choice is that singularity of the Kretschmann scalar is located at the origin, which is given by:

$$\xi_0 = 0\,, \qquad \xi_1 = -1\,. \tag{14}$$

Finally, by using (9) and (12) in the metric in zero order we found the following line element

$$ds^2 = -r^2\left(1-\frac{\omega^4}{u^4}\right)d\tau^2 + 2d\tau dr + r^2\tau^2\left(1+\frac{1}{u\tau^{2/3}}\right)dy^2 + r^2 dx_\perp^2\,, \quad (15)$$

where ω is a constant. For $\omega \longrightarrow 0$ it reduces to pure AdS space. It is obvious that (15) is a black hole metric with the location of the horizon being given by the zero locus of $g^{(0)}_{\tau\tau}$, *i.e.*

$$r(\tau) = \frac{\omega}{\tau^{1/3}}\,. \quad (16)$$

5. Apparent horizon for the BF space-time

We show the presence of an apparent horizon instead of the event horizon, because an examination of the existence of the event horizon in a time-dependent setup is not easy. Usually apparent horizon definition is based on the double-null formalism [27,28,29]. We foliate the five-dimensional space-time by null-hypersurfaces $\Sigma^\pm$ each of which is parameterized by a scalar $\xi^\pm$, respectively. Let us consider normal 1-forms to $\Sigma^\pm$ which we define as $n^\pm = -d\xi^\pm$. The 1-forms have the null character: $g^{-1}(n^\pm, n^\pm) = g_{\mu\nu} = n^\pm_\mu n^\pm_\nu$.

The normal 1-forms $n^\pm_\mu dx^\mu$ on our geometry on the (τ, y, x_2, x_3, r) coordinates for metric (15) are given by

$$n^-_\mu = F^-(1,0,0,0,0)\,, \qquad n^+_\mu = F^+\left(r^2 a, 0,0,0,-2\right)\,. \quad (17)$$

The relative normalization of the null normals defines a function f as $g^{-1}(n^+, n^-) = -e^{-f}$, and normalization factors $F^\pm$ are determined by the integrability conditions $d(d\xi^\pm) = 0$.

The intersections of $\Sigma^+(\xi^+)$ and $\Sigma^-(\xi^-)$ are the $3d$ surfaces, where τ and r are constants spanned by y, x_2, x_3. We define a two-parameter family of three-dimensional spacelike surfaces $S(\Sigma_+, \Sigma_-)$. Hence, by introducing an intrinsic coordinate system (y, x_2, x_3) of the three-surfaces, the foliation is described by the imbedding $x = x(\xi^+, \xi^-; y, x_2, x_3)$. The intrinsic metric on the three-surfaces is found to be

$$h = g + e^{-f}\left(n^+ n^- \otimes n^- n^+\right)\,, \quad (18)$$

and in terms of components is

$$h_{MN} = g_{MN} + e^{-f}\left(n^+_M n^-_M \otimes n^-_M n^+_N\right)\,. \quad (19)$$

Next, we define null normal vectors $l_\pm$ on the $\Sigma^\pm$ by $l_\pm \equiv e^{-f} g^{-1}(n^\pm)$ which are explicitly given by

$$l_-^\mu = \frac{1}{2F^-}\left(-2,0,0,0,-r^2 a\right) , \qquad l_+^\mu = \frac{1}{2F^+}(0,0,0,0,1) , \tag{20}$$

where we have defined

$$e^f = g^{-1}(n^+, n^-) = -g^{\mu\nu} n^+_\mu n^-_\nu = 2F^+ F^- . \tag{21}$$

The geometrical quantities such as null normal expansions $\theta_\pm$ are defined by

$$\theta_\pm = L_\pm \log \mu , \tag{22}$$

where $L_\pm$ denotes the Lie derivative along $l^a_\pm$ and μ is the volume element of the undersurface

$$\mu = r^3 \tau e^{\tilde{b}} , \tag{23}$$

where $\tilde{b} \equiv b + \log(1 + 1/\tau r)$.

The expansion $\Theta = \theta_+\theta_-$ is an invariant, and hence the location of the apparent horizon for $r(\tau)$ can be found by solving the equation

$$\Theta = 0 . \tag{24}$$

We can determine the location of the apparent horizon in the power series, because the geometry is expanded in τ for large proper time

$$r_A(\tau) = r_{A0}\tau^{-1/3} + r_{A1}\tau^{-1} + r_{A2}\tau^{-5/3} + O\left(\tau^{-7/3}\right) . \tag{25}$$

By using the above equation, one can find $r(\tau)$ order by order in $\tau^{-2/3}$. So, the location of the apparent horizon for the second order is [16]:

$$r_{A0}(\tau)^{(2)} = \omega , \qquad r_{A1}(\tau)^{(2)} = -\tfrac{1}{2} , \qquad r_{A2}(\tau)^{(2)} = \frac{8 + 3\pi - 4\ln 2}{72\omega} . \tag{26}$$

Notice that the apparent horizon in the zeroth order coincides with the event horizon.

6. Surface gravity and the first law of the dynamical AdS_5 black hole

In this section, we want to derive the first law of this dynamical black hole. The first law we shall obtain holds for any trapping horizon such as apparent horizon.

We would like to derive the first law in two cases: First, by considering really AdS$_5$ geometry without any additional symmetry. Second, since AdS geometry corresponds to a manifold which is most symmetrical, we assume it is a $5d$ spherically symmetric space-time.

We are going to derive the first law on the base of the double-null formalism or $(2+2)$ decomposition of general relativity discussed in the previous section. Let us notice that $4+1$ dimensional space time $2+3$ decomposition is a trivial generalization. Moreover, the apparent horizon in the gravity dual to boost-invariant flow is unique due to high symmetry of the space-time. Thus we can use $2+2$ decomposition. This is a very special feature which will not generalize to more involved examples. Also, we adopt decomposition based on Lie derivatives w.r.t. null vectors developed by one of the authors [27].

In the first case, before deriving the first law we have to define energy and surface gravity in a quasi-local way in the general. In spherical symmetry there is a widely accepted energy such as Misner–Sharp (MS) energy [29]. The MS energy is used to derive the first law of black hole dynamics in spherical symmetry in Ref. [30]. Now we use the Hawking energy [31], which reduces to the MS energy in spherical symmetry. It is defined by

$$E(\xi^+,\xi^-) = \frac{r}{16\pi} \int\limits_{S(\Sigma_+,\Sigma_-)} d^3\theta\sqrt{h}\left[{}^{(3)}R + e^f\theta_+\theta_-\right] , \tag{27}$$

where h is the determinant of the three-dimensional metric h_{ab}.

Now, we recall the volume element of the intersection of the null hyper surfaces (three-surface). From Eq. (23) we have

$$A = \sqrt{\det{}^{(3)}h} = r^3\tau e^{\tilde{b}} , \qquad \tilde{b} = b + \log\left(1 + \frac{1}{r\tau}\right) \tag{28}$$

in zero and first order of late time expansion. Using the relation (26), $b = 0$, we have

$$A = r^3\tau + r^2 . \tag{29}$$

The area of $A_{\rm H}$ of the apparent horizon is given by

$$A_{\rm H} = \omega^3 - \frac{\omega^2}{2}\tau^{-2/3} + \frac{\omega}{24}\left(4 + \pi + 12\omega^2\lambda + 4\log 2\right)\tau^{-4/3} + O\left(\tau^{-23}\right) , \tag{30}$$

where $\omega = r_0\tau^{1/3}$, and r_0 is a static AdS black hole event horizon. The parameter λ was extracted in [16] as transport coefficient of hydrodynamics.

The expansion parameter $\theta_\pm (= L_\pm \log\mu)$, is written in form

$$\theta_\pm \equiv \frac{1}{\sqrt{\det{}^{(3)}h}}\left(l_\pm\sqrt{\det{}^{(3)}h}\,\right) . \tag{31}$$

A natural generalization of definition of dynamical surface gravity to a not necessarily spherically symmetric case which is proposed by [33], is

$$k(\Sigma_+,\Sigma_-) = \frac{-1}{16\pi r} \int\limits_{S(\Sigma_+,\Sigma_-)} d^3\theta\sqrt{h}\; e^f (L_+\theta_+ L_-\theta_- + \theta_+\theta_-)\,. \tag{32}$$

We now try to derive the first law of black hole dynamics by using the Hawking energy (27) and the surface gravity defined in (32). First, we define differentiation w.r.t. of the parameters $\xi^\pm$ of the space-time foliation, denoted by 'd', as

$$d = d\xi^+\partial_+ + d\xi^-\partial_+\,. \tag{33}$$

Then from (27) and (29) we obtain

$$dE = \left(\frac{E}{r}\right) dr + rd\left(\frac{E}{r}\right)\,, \tag{34}$$

and

$$dA = \left(3r^2\tau + 2r\right) dr + r^3 d\tau\,. \tag{35}$$

By using the relations (34) and (35) we finally obtain

$$dE - \frac{k}{8\pi}dA = \omega_1 A dr + \omega_2 A d\tau + rd\left(\frac{E}{r}\right)\,, \tag{36}$$

where

$$\omega_1 = \frac{1}{A}\left(\frac{E}{r} - \frac{k\left(3r^2\tau + 2r\right)}{8\pi}\right)\,, \qquad \omega_2 = -\frac{1}{A}\frac{r^3 k}{8\pi}\,. \tag{37}$$

Notice that Eq. (36) is independent of the definitions of E and k.

Now, we define a marginal surface which is a surface where one of the expansions $\theta_\pm$ vanishes. Since the Gauss–Bonnet theorem says that

$$\int\limits_{S(\Sigma_+,\Sigma_-)} d^3\theta\sqrt{h}^{(3)}R = 8\pi(1-\gamma)\,, \tag{38}$$

where γ is the genus or number of handles of $S(\Sigma_+,\Sigma_-)$, the energy divided by area radius is given by $E/r = (1-\gamma)/2$ on a marginal surface and is a constant. Therefore, from Eq. (36) we get

$$E' = \frac{k}{8\pi}A' + \omega_1 A r' + \omega_2 A\tau'\,, \tag{39}$$

where the prime denotes the derivative along the trapping horizon. This is the first law of black hole dynamics. Note that this also holds along any hypersurface foliated by three-surfaces on which E/r is constant. The terms $\omega_1 A r'$ and $\omega_2 A\tau'$ may be work terms done along the horizon.

As we have seen before in Eq. (5) the temperature of ideal conformal fluid was $T \sim \tau^{-1/3}$ which is dependent on time. So, for AdS$_5$ black hole the corresponding temperature is given by

$$T_{\mathrm{H}} = \frac{\hbar k}{2\pi}\,, \tag{40}$$

where k is the surface gravity defined in (32). Now by using the relations (5) and (40) we have

$$k \sim \frac{2\pi}{\hbar}\tau^{-1/3}\,. \tag{41}$$

Thus the surface gravity for general dynamical AdS$_5$ black hole is time independent.

Now we want to discuss consequences of thermodynamics of the black hole assuming spherically symmetric space. Let us notice that in Ref. [32] the new definition of the surface gravity is introduced which is parameterization of the outgoing null geodesics by using a natural spherically symmetric foliation of ingoing null hypersurfaces. Any observer moving along a time-like orbit can precisely measure the apparent change of frequency of a standard radio or light signal falling in from the far away region. The authors found an explicit relation between this frequency change and the acceleration of the observer. For any geodesic observer crossing the horizon, the proper time derivative of the red shift of the infalling wave is exactly equal to the surface gravity.

For a foliation by ingoing null hypersurfaces, which is parameterized by a function υ, for any radial geodesic with tangent vector k^α satisfying $k^\alpha \nu_{;\alpha} = 1$, the surface gravity k describes the inaffinity of the geodesic as

$$k = -k^\beta k^\beta \nu_{;\alpha\beta}\,. \tag{42}$$

Physical meaning of k is the red shift of an outgoing light signal moving along the horizon, which is measured with respect to the Killing time. Using the Christoffel symbols, k takes the new form [33]

$$k = k^\alpha k^\beta \Gamma^1_{\alpha\beta}\,. \tag{43}$$

With the above definition of the surface gravity k for spherically symmetric space-time, we obtain the surface gravity k for our metric (9). Since $\Gamma^1_{22} = \Gamma^1_{12} = 0$, using (12) and (13) we obtain the surface gravity for zero and first order of late time as

$$k_0 = r + \frac{\omega^4}{r^3}\tau^{-4/3}\,, \tag{44}$$

and

$$k_1 = -\tfrac{4}{3}\,\omega^3 r^{-4}\tau^{-4/3} - \tfrac{5}{3}\,\omega^4 r^{-5}\tau^{-5/3}\,. \tag{45}$$

This result for surface gravity of spherical symmetric space-time is time dependent. Using k_0 and k_1 we obtain analogy of first law of dynamical AdS black hole for zero and first order of late time as

$$E'_0 = \frac{k_0}{8\pi} A' \,, \qquad E'_1 = \frac{k_1}{8\pi} A' \,, \tag{46}$$

where A is the area of a $5d$ spherical space.

Another way to extract the first law in a general, dynamical, spherically symmetric metric, which can be written in advanced Eddington–Finkelstein coordinates, is proposed in Ref. [34]. Usually the Misner–Sharp energy is used to derive the first law of black hole dynamics in spherical case. In [35] it was shown how the use of the Misner–Sharp mass can lead to a preferred normalization for the surface gravity in spherically symmetric space-time. The surface defined by

$$r = 2m(\nu, r) \,, \tag{47}$$

where at $r = r_{\rm H}$ defines a marginally trapped surface, in many situations is also a dynamical horizon or trapping horizon. General differentiation of this equation with respect to any parameter ξ, labeling spherically symmetric foliations of the horizon, gives

$$\frac{dr}{d\xi} = 2\frac{\partial m}{\partial \nu}\frac{d\nu}{d\xi} + 2\frac{\partial m}{\partial r}\frac{dr}{d\xi} \,. \tag{48}$$

If we take $\xi = \nu$ then we obtain,

$$\frac{\partial m}{\partial \nu} = \frac{1 - 2m'}{2}\frac{dr}{d\nu} \,, \tag{49}$$

and by using the volume element of surface in spherical case, we find the formula for the area of a $5d$ spherical space,

$$A^{(5d)} \simeq 2\pi^{5/2} r^4 \,. \tag{50}$$

Thus we obtain the following relation:

$$\frac{\partial m}{\partial \tau} = \frac{(1 - 2m')}{16\pi^{5/2} r^3}\frac{\partial A}{\partial \tau} \,, \tag{51}$$

where $m' = \partial m / \partial r$. By comparing with the first law of black hole thermodynamics $dm = (k/8\pi)\, dA$ we can take surface gravity

$$k = \frac{1 - 2m'}{2\pi 3/2 r_{\rm AH}^3} \,, \tag{52}$$

where r_{AH} is the location of the apparent horizon. Equation (48) has a natural normalization of the surface gravity in terms of the Misner–Sharp mass function, which we can interpret as the mass of the black hole contained within the radius r_{AH}. From Eq. (12) we consider the value of the first order apparent horizon, $r_{\mathrm{AH}} = -1/2$, which gives

$$k = -\frac{4(1-2m')}{\pi^{3/2}}\,. \tag{53}$$

Now we rewrite Eq. (49) in terms of k,

$$E' = \frac{k}{8\pi} A'\,. \tag{54}$$

As we know this equation is the first law of black hole, which for a dynamical five dimensional spherically symmetric black hole has the same form as the static black hole. But the surface gravity has a different definition.

7. Conclusion

In this paper we start with the ideal fluid Bjorken flow that is boost invariant. With the AdS/CFT correspondence and two types of coordinates the hydrodynamics describes a dynamical AdS_5 black hole. We used the Eddington–Finkelstein coordinates and results of [16]. As we know the dual gravity is really a dynamical black hole which for all orders is regular. We show tha presence of the apparent horizon on this dual geometry. Then we considered this $5d$ black hole in general metric (9). Using the Hawking energy (27) and surface gravity (32) in general case without assuming any addition symmetry for AdS space-time, we derived the first law of black hole dynamics (39). In that case two correction terms in this equation should work along the horizon. Also with temperature of black hole, we show that the surface gravity has a time-dependent form. Next we assumed that the AdS space-time is a $5d$ spherically symmetric space-time. Then by definition (33) for dynamical surface gravity we obtained the zero and first order of surface gravity for our dynamical black hole, Eq. (44) and (45). We have also shown that these results are time-dependent. Finally, we considered Misner–Sharp energy for spherical case and obtained the surface gravity (52) and analogy of the first law for this black hole (54).

We have greatly benefited from suggestions and comments with referee. We offer our special thanks to him.

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