NONLINEAR WAVES IN A HOMOGENEOUS MULTIPHASE AND MULTICOMPONENT RELATIVISTIC MIXTURE

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Classical and relativistic multiphase and multicomponent flows represent an interesting field of research due to their various applications. In order to simulate multiphase and multicomponent flows, the effect due to interfaces between constituents has to be analyzed. In the approach following in this paper, the constituents are averaged to lead to a *homogeneous mixture*, thus only one set of equations for the total mass, momentum, and energy of the mixture, supplemented by equations for the mass or volume fraction of the constituents has to be solved. The main purpose of this paper is to develop the relativistic generalization of a recent classical approach to the study of multiphase and multicomponent homogeneous mixture. An hyperbolic system of equations is founded, made by particle number and energy-tensor conservations equations, supplemented by mass or volume fraction equations for the constituents. Thus, a non linear wave propagation compatible with this system is considered.

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1. Introduction

Due to their various applications, classical and relativistic multiphase and multicomponent flows represent an interesting field of research. In classical framework, multiphase and multicomponent mixture are common in a lot of engineering applications, as for example fuel sprays in combustion processes, liquid-jet machining of materials, and stream generation and condensation in nuclear reactors [1, 2, 3, 4, 5, 6, 7]. Also in General Relativity there are many topics where matter can be represented as a multiphase/multicomponent mixture, see bibliography in [8, 9, 10]. They

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concerned, for example, null fluid with string fluid [11], radiation fluid in addiction to a string fluid [12, 13, 14] and, for most of the history of the Universe, the dominant matter content is a mixture of matter and radiation [15, 16, 17, 18, 19].

The physical mechanisms underlying classical and relativistic multiphase and multicomponent flows as well as the interplay of these mechanisms are very complex. In those flows the phases and/or components can assume a large number of complicated configurations; small-scale interactions between the phases can have a deep impact on macroscopic flow properties [20].

In order to simulate multiphase and multicomponent flows, the effect due to interfaces between constituents has to be analyzed, as for example the large or discontinuous property variations across them. Two approaches are commonly used to simulate these flows. In the first one the interfaces between the phases and/or components are tracked explicitly [21, 22, 23, 24, 25, 26]. The prediction of the motion of large bubbles in a liquid, the motion of liquid after a dam break, the prediction of jet breakup, and the tracking of any liquid-gas interface are typical application of this approach. In the second one, the constituents are averaged to lead to a homogeneous mixture. Many dispersed flows including bubbly flow of air in water or mist flow can be considered as homogeneous mixture. Moreover, the homogeneous mixture is said to be in equilibrium if it is in both mechanical and thermal equilibrium while, on the contrary, it is said to be in non-equilibrium. The advantage of the homogenized-mixture approach with respect to the interface-tracking approach is that it solves, in classical framework, only one set of equations for the total mass, momentum, and energy of the mixture, supplemented by equations for the mass or volume fraction of the constituents [27]. In relativistic framework, the set of equations is made by particle number and energy-tensor conservations equations, supplemented by mass or volume fraction equations for the constituents. However, there are challenges associated with the use of homogenized-mixture approach, like the mathematical closure of the system, that is acoustically and thermodynamically consistent.

The main purpose of this paper is to build up a relativistic formulation of some recent results on classical dynamics of multiphase/multicomponent flows with an arbitrary number of constituents (which can be either phases, either components) starting from the idea of Lagumbay *et al.* [28, 29], based on an equilibrium-homogenized-mixture approach.

Since the phases and/or components of the mixture are sufficiently well mixed and the particle size are sufficiently small so that any significant relative motion can be ignored, the four-velocity, u^α can be supposed to be the same for each constituent of the mixture. Here, u^α is the mixture unit four-vector defined to be future-pointing

$$g_{\alpha\beta}u^{\alpha}u^{\beta} = 1\,, \tag{1}$$

where $g_{\alpha\beta}$ are the covariant components of Lorentz metric tensor with signature $+, -, -, -$. In what follows, the units are such that the velocity of light is unitary. According to the homogeneous equilibrium mixture model, the temperature T and the pressure p are the same for each phase and/or component in the mixture. Conversely, each constituent has its own particle number density, r_k, its specific internal energy, ε_k, and its energy density ρ_k [30, 31]

$$\rho_k = r_k\left(1 + \varepsilon_k\right)\,, \tag{2}$$

where the subscript k denote a specific phase or component, $1 \leq k \leq N$, and N is the total number of constituents in the flow.

In what follows, the mixture is assumed to consist of two phases, namely liquid and gas, and the gas phase is assumed to consist of two components, namely, a generic gas and a vapor. These are denoted by the subscripts 1, 2 and 3 for gas, vapor and liquid, respectively. It should be noted, however, that the homogeneous equilibrium mixture model can be extended in a straightforward fashion to an arbitrary number of phases and components. Variables without subscripts are related to the mixture as a whole. The subscript k is used to denote the variables of a specific constituent.

In the next sections, the relativistic multiphase and multicomponent flow are described and the complete system of governing differential equations are derived. Then, in Sec. 3, the propagation of weak discontinuities admitted by the model are examined and the expression for their speeds of propagation are obtained. In the last section, the relativistic generalization of the idealized fluid mixture model, obtained in a classical framework by Lagumbay *et al.* [28], is presented. The expression of the speeds of propagation of weak discontinuity in this special case is also obtained.

2. Relativistic homogeneous equilibrium mixture model

The homogeneous equilibrium mixture model is based on the notion that the four-velocity, the temperature, T, and the pressure, p, of all the phases and/or components are equal, and then an unique four-velocity, temperature and pressure are defined for the whole mixture. The quantities associated with a given constituent are averaged to give the corresponding mixture quantity. Accordingly, quantities per unit volume are averaged by their respective volume fraction X_k. For example, the mixture particle number density r is given by

$$r = \sum_{k=1,2,3} r_k X_k\,, \tag{3}$$

where r_k is the particle number density of the kth phase and/or components, and the volume fractions satisfy the constraint

$$\sum_{k=1,2,3} X_k = 1\,. \tag{4}$$

Conversely, quantities per unit mass are averaged by their respective mass fractions Y_k. For example, the specific internal energy ε of the mixture is given by

$$\varepsilon = \sum_{k=1,2,3} \varepsilon_k Y_k\,, \tag{5}$$

where ε_k is the specific internal energy of the kth mixture element and the mass fractions satisfy the constraint

$$\sum_{k=1,2,3} Y_k = 1\,. \tag{6}$$

The volume and mass fractions are related through the following relations

$$r_k X_k = r Y_k \qquad (k = 1, 2, 3)\,, \tag{7}$$

and allow to define any bulk quantity. The expressions for the energy density of the mixture, ρ, the relativistic specific enthalpy of the mixture, f, and the specific heat at constant volume of the mixture, C_{V}, are

$$\rho = \sum_{k=1,2,3} \rho_k X_k\,, \tag{8}$$

$$f = \sum_{k=1,2,3} f_k Y_k\,, \tag{9}$$

$$r f = \sum_{k=1,2,3} r_k f_k X_k\,, \tag{10}$$

$$C_{\mathrm{V}} = \sum_{k=1,2,3} C_{\mathrm{V}k} Y_k\,, \tag{11}$$

where $f_k = 1 + \varepsilon_k + p/r_k$ is the relativistic specific enthalpy and $C_{\mathrm{V}k}$ is the specific heat at constant volume of the kth constituent. Moreover, the specific internal energy of the kth phase and/or components, ε_k, is supposed to be given by $\varepsilon_k = C_{\mathrm{V}k} T$.

The equations governing the evolution of the relativistic homogeneous equilibrium mixture are the particle number density conservation

$$\nabla_\alpha (r u^\alpha) = 0\,, \tag{12}$$

the total energy-momentum conservation

$$\nabla_\alpha T^{\alpha\beta} = 0\,, \tag{13}$$

where the stress energy tensor is given by

$$T^{\alpha\beta} = rfu^\alpha u^\beta - pg^{\alpha\beta}\,, \tag{14}$$

being f the relativistic total specific enthalpy

$$f = 1 + h = 1 + \varepsilon + \frac{p}{r}\,, \tag{15}$$

with $h = \varepsilon + p/r$ the "classical" specific enthalpy of the mixture. The balance laws for particle number density for each phase and/or component is

$$\nabla_\alpha(r_k X_k u^\alpha) = 0 \qquad (k = 1, 2, 3)\,. \tag{16}$$

Moreover, the spatial projection of the equation (13) is

$$\gamma^\lambda_\beta \nabla_\alpha T^{\alpha\beta} \equiv rfu^\alpha \nabla_\alpha u^\lambda - \gamma^{\alpha\lambda}\partial_\alpha p = 0\,, \tag{17}$$

where $\gamma^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ is the projection tensor onto the three-space orthogonal to u^α (the rest space of an observer moving with four-velocity u^α), whereas the projection of (13) along u^α is

$$u_\beta \nabla_\alpha T^{\alpha\beta} \equiv u^\alpha \partial_\alpha \rho + (\rho + p)\nabla_\alpha u^\alpha = 0\,, \tag{18}$$

being $\rho = r(1 + \varepsilon)$.

Together with (3), equations (16) implies the balance laws for the bulk particle number density (12). By relations (7), equations (16) can be also rewritten as

$$\nabla_\alpha\left(rY_k u^\alpha\right) = 0 \qquad (k = 1, 2, 3)\,, \tag{19}$$

which by virtue of (12), give the following evolution laws for the mass fractions Y_k

$$u^\alpha \partial_\alpha Y_k = 0 \qquad (k = 1, 2, 3)\,. \tag{20}$$

The constitutive equations of particle number density of the gas, vapor and liquid are assumed to take the form

$$r_k = r_k(p, T) \qquad (k = 1, 2, 3)\,. \tag{21}$$

The mathematical model here derived is general and can be used for arbitrary forms of the equation of state of each phase. The following relations can be obtained by differentiating (21)

$$dr_k = \frac{1}{\lambda_k^2}dp - \left(\frac{\beta_k}{\lambda_k}\right)^2 dT \qquad (k = 1, 2, 3)\,, \tag{22}$$

where $\lambda_k = (\partial p/\partial r_k)_T^{1/2}$ and $\beta_k = (\partial p/\partial T)_{r_k}^{1/2}$ are the isothermal speed of sound and compressibility of the kth component, respectively. Using equations (3) and (22), and the constraint (4), the differentiation of the mixture particle number density yields

$$dr = \sum_{k=1,2,3} r_k dX_k + \sum_{k=1,2,3} X_k dr_k\,, \tag{23}$$

that can be also written as

$$dr = (r_1 - r_3)dX_1 + (r_2 - r_3)dX_2 + \frac{1}{\lambda_X^2}dp - \frac{1}{\lambda_{X\beta}^2}dT\,, \tag{24}$$

where $1/\lambda_X^2 = \sum_k X_k/\lambda_k^2$ and $1/\lambda_{X\beta}^2 = \sum_k X_k\beta_k^2/\lambda_k^2$.

By rewriting equations (12), (17), (18), (20) in term of the seven independent field variables u^α, p, T, X_1, X_2, the mathematical study of the relativistic multiphase and multicomponent flow can be performed using the following set of equations

$$\begin{aligned}
u^\alpha\partial_\alpha p &= -\xi\nabla_\alpha u^\alpha\,, &\text{(25a)}\\
rfu^\alpha\nabla_\alpha u^\beta &= \gamma^{\alpha\beta}\partial_\alpha p\,, &\text{(25b)}\\
u^\alpha\partial_\alpha T &= -\frac{p}{rC_{\mathrm{V}}}\nabla_\alpha u^\alpha\,, &\text{(25c)}\\
u^\alpha\partial_\alpha X_1 &= -\omega_1 X_1\nabla_\alpha u^\alpha\,, &\text{(25d)}\\
u^\alpha\partial_\alpha X_2 &= -\omega_2 X_2\nabla_\alpha u^\alpha\,, &\text{(25e)}
\end{aligned}$$

where

$$\xi = \frac{rC_{\mathrm{V}} + p\sum_{k=1,2,3}\frac{X_k}{r_k}\left(\frac{\beta_k}{\lambda_k}\right)^2}{rC_{\mathrm{V}}\sum_{k=1,2,3}\frac{X_k}{r_k\lambda_k^2}}\,,$$

$$\omega_1 = 1 + \frac{p}{r_1 rC_{\mathrm{V}}}\left(\frac{\beta_1}{\lambda_1}\right)^2 - \frac{1}{r_1\lambda_1^2\sum_{k=1,2,3}\frac{X_k}{r_k\lambda_k^2}} - \frac{p}{r_1 rC_{\mathrm{V}}\lambda_1^2}\sum_{k=1,2,3}\frac{X_k}{r_k}\left(\frac{\beta_k}{\lambda_k}\right)^2\,,$$

$$\omega_2 = 1 + \frac{p}{r_2 rC_{\mathrm{V}}}\left(\frac{\beta_2}{\lambda_2}\right)^2 - \frac{1}{r_2\lambda_2^2\sum_{k=1,2,3}\frac{X_k}{r_k\lambda_k^2}} - \frac{p}{r_2 rC_{\mathrm{V}}\lambda_2^2}\sum_{k=1,2,3}\frac{X_k}{r_k}\left(\frac{\beta_k}{\lambda_k}\right)^2\,.$$

3. Weak discontinuities

In a domain Ω of space-time V_4, let Σ be a regular hyper-surface, not generated by the flow lines, being $\varphi(x^\alpha) = 0$ its local equation. It is set $L_\alpha = \partial_\alpha \varphi$. As it will be clear below, the hypersurface Σ is a space-like one, *i.e.* $L^\alpha L_\alpha < 0$. In the following, N_α will denote the normalized vector

$$N_\alpha = \frac{L_\alpha}{\sqrt{-L^\beta L_\beta}}\,, \qquad N_\alpha N^\alpha = -1\,.$$

A particular class of solutions of system (25) is considered, namely, weak discontinuity waves Σ, on which the field variables u^α, p, T, X_1, X_2 are continuous, but jump discontinuities may occur in their normal derivatives. In this case, if Q denotes any of these fields, then there exists [30, 32] the distribution δQ, with support Σ, such that

$$\overline{\delta}\,[\nabla_\alpha Q] = N_\alpha \delta Q\,,$$

where $\overline{\delta}$ is the Dirac measure defined by φ with Σ as support, square brackets denote the discontinuity, δ being an operator of infinitesimal discontinuity; δ behaves like a derivative insofar as algebraic manipulations are concerned.

Then, from system (25), the following linear homogeneous system in the distributions $N_\alpha \delta u^\alpha$, δp, δT, δX_1 and δX_2 is obtained

$$L\delta p + \xi N_\alpha \delta u^\alpha \;=\; 0\,, \tag{26a}$$

$$rfL\delta u^\alpha - \gamma^{\alpha\beta} N_\beta \delta p \;=\; 0\,, \tag{26b}$$

$$L\delta T + \frac{p}{rC_{\mathrm{V}}} N_\alpha \delta u^\alpha \;=\; 0\,, \tag{26c}$$

$$L\delta X_1 + \omega_1 X_1 N_\alpha \delta u^\alpha \;=\; 0\,, \tag{26d}$$

$$L\delta X_2 + \omega_2 X_2 N_\alpha \delta u^\alpha \;=\; 0\,, \tag{26e}$$

where $L = u^\alpha N_\alpha$. Moreover, from the unitary character of u^α it follows that

$$u_\alpha \delta u^\alpha = 0\,. \tag{27}$$

Now, the normal speeds of propagation of the various waves with respect to an observer moving with the mixture velocity u^α can be investigated. The normal speed λ_Σ of propagation of the wave front Σ, described by a time-like word line having tangent vector field u^α, that is with respect to the time direction u^α, is given by [30, 31, 32]

$$\lambda_\Sigma^2 = \frac{L^2}{1+L^2}\,. \tag{28}$$

The local causality condition, *i.e.* the requirement that the characteristic hyper-surface Σ has to be time-like or null (or equivalently that the normal N_α be space-like or null, that is $g^{\alpha\beta}N_\alpha N_\beta \leq 0$), is equivalent to the condition $0 \leq \lambda_\Sigma^2 \leq 1$.

System (26) admits the solution $L = 0$, which represents a wave moving with the mixture. For the corresponding discontinuities the following equations holds

$$N_\alpha \delta u^\alpha = 0\,, \qquad \delta p = 0\,. \tag{29}$$

Since the coefficients characterizing the discontinuities exhibit five degrees of freedom, then system (26) admits five independent eigenvectors corresponding to $L = 0$ in the space of the field variables.

Under the assumption $L \neq 0$, equation (26b), multiplied by N_α, gives

$$r f L N_\alpha \delta u^\alpha + l^2 \delta p = 0\,, \tag{30}$$

where $l^2 = 1 + L^2$. As a consequence, (26a) and (30) represent a linear homogeneous system in the two scalar distributions $N_\alpha \delta u^\alpha$ and δp, which may have non trivial solutions only if the determinant of the coefficients vanishes. Therefore, we find the equation

$$\mathcal{H} \equiv r f L^2 - \xi l^2 = 0\,, \tag{31}$$

which corresponds to the hydrodynamical waves propagating in such a relativistic homogeneous equilibrium mixture. Their speeds of propagation are given by

$$\lambda_\Sigma^2 = \frac{\xi}{rf} = \frac{r C_{\mathrm{V}} + p \displaystyle\sum_{k=1,2,3} \frac{X_k}{r_k}\left(\frac{\beta_k}{\lambda_k}\right)^2}{r^2 f C_{\mathrm{V}} \displaystyle\sum_{k=1,2,3} \frac{X_k}{r_k}\frac{1}{\lambda_k^2}} \tag{32}$$

and the condition $0 < (\xi/rf) \leq 1$ ensures their spatial orientation.

The associated discontinuities can be written in terms of $\psi = -n_\alpha \delta u^\alpha$ as follows

$$\delta u^\alpha = \psi n^\alpha\,, \tag{33a}$$

$$\delta T = \frac{p}{r C_{\mathrm{V}}}\frac{l}{L}\psi\,, \tag{33b}$$

$$\delta p = \xi \frac{l}{L}\psi\,, \tag{33c}$$

$$\delta X = \omega_1 X_1 \frac{l}{L}\psi\,, \tag{33d}$$

$$\delta Y = \omega_2 X_2 \frac{l}{L}\psi\,, \tag{33e}$$

where n^α is the unitary space-like four-vector defined by

$$n_\alpha = \frac{1}{l}\left(N_\alpha - L u_\alpha\right) . \tag{34}$$

Observe that if the above condition characterizing the space-like orientations of the surface is verified, then the governing equations represent a (not strictly) hyperbolic system. In fact, all velocities (eigenvalues) are real, and there is a complete set of eigenvectors in the space of field variables, *i.e.* seven independent eigenvectors (5 from $L = 0$ and 2 from $\mathcal{H} = 0$), for the seven independent field variables u^α, p, T, X_1 and X_2.

4. Application: relativistic idealized fluid mixture

In order to obtain a closed-form solution of the governing equation, the mixture entropy density is assumed to be a function of pressure p, temperature T and mass fractions Y_k

$$S = S(p, T, Y_k)\,, \tag{35}$$

so that, using the Gibbs equation, the following differential relation holds

$$T dS = d\varepsilon + p d\left(\frac{1}{r}\right) + \sum_{k=1,2,3} L_k dY_k\,, \tag{36}$$

where L_k is the latent heat phase change and is assumed to be a function of pressure and temperature, $L_k = L_k(p, T)$.

Equations (7) lead to the relation

$$\frac{1}{r} = \sum_{k=1,2,3} \frac{Y_k}{r_k}\,, \tag{37}$$

and its differentiation gives

$$d\left(\frac{1}{r}\right) = \sum_{k=1,2,3}\left(\frac{dY_k}{r_k} - Y_k \frac{dr_k}{r_k^2}\right) . \tag{38}$$

Using the constitutive equations (21), (38) can be written as

$$d\left(\frac{1}{r}\right) = \sum_{k=1,2,3}\left(\frac{dY_k}{r_k} + Y_k \frac{\partial}{\partial p}\left(\frac{1}{r_k}\right) dp + Y_k \frac{\partial}{\partial T}\left(\frac{1}{r_k}\right) dT\right) . \tag{39}$$

The differentiation of the specific internal energy of the mixture, $\varepsilon = C_{\mathrm{V}} T$, yields

$$d\varepsilon = C_{\mathrm{V}} dT + T \sum_{k=1,2,3}\left(C_{\mathrm{V}k} dY_k\right) , \tag{40}$$

and the introduction of equations (39) and (40) into (36) leads to

$$
\begin{aligned}
dS \;=\; \sum_{k=1,2,3} \Bigg\{ & \left[\frac{C_{\mathrm{V}k} Y_k}{T} + \frac{p}{T} Y_k \frac{\partial}{\partial T}\left(\frac{1}{r_k}\right) \right] dT \\
& + \left[\frac{p}{T} Y_k \frac{\partial}{\partial p}\left(\frac{1}{r_k}\right) \right] dp + \left[C_{\mathrm{V}k} + \frac{p}{T r_k} + \frac{L_k}{T} \right] dY_k \Bigg\} \,. \qquad (41)
\end{aligned}
$$

Since dS is a total derivative, equation (41) implies the following constraints

$$
\sum_{k=1,2,3} \frac{Y_k}{r_k^2} \left(\frac{\partial r_k}{\partial T} + \frac{p}{T} \frac{\partial r_k}{\partial p} \right) \;=\; 0 \,, \qquad (42)
$$

$$
\frac{\partial}{\partial T}\left(\frac{L_k}{T}\right) \;=\; \frac{C_{\mathrm{V}k}}{T} + \frac{p}{T^2 r_k} \,, \qquad (43)
$$

$$
\frac{\partial}{\partial p}\left(\frac{L_k}{T}\right) \;=\; -\frac{1}{T r_k} \,. \qquad (44)
$$

In order equation (42) to be satisfied for arbitrary mass fractions, the expression inside the brackets must be equal to zero for each constituent. Consequently, the constitutive equation for each constituent must be a function of the ratio of pressure and temperature, *i.e.*

$$
r_k = r_k \left(\frac{p}{T} \right) \,. \qquad (45)
$$

Assuming that gas and vapor follow the ideal-gas laws, respectively

$$
r_1 \;=\; \frac{p}{R_1 T} \,, \qquad (46)
$$

$$
r_2 \;=\; \frac{p}{R_2 T} \,, \qquad (47)
$$

and the specific heat at constant volume of gas and vapor is given by, respectively

$$
C_{\mathrm{V}1} \;=\; \frac{R_1}{\gamma_1 - 1} \,, \qquad (48)
$$

$$
C_{\mathrm{V}2} \;=\; \frac{R_2}{\gamma_2 - 1} \,, \qquad (49)
$$

where γ_1 and γ_2 are the specific ratio of the gas and vapor respectively.

For the liquid, it is proposed the relativistic version of the idealized liquid proposed by Lagumbay in classical framework [29]

$$
r_3 = r_o + \alpha \frac{p}{T} \,, \qquad (50)
$$

where $\alpha = T_o/\lambda_{3o}^2$, T_o is the reference temperature, λ_{3o} is the reference speed of sound, and r_o is the reference particle number density of the liquid. A liquid obeying to (50) is called relativistic idealized liquid.

Combining (46), (47) and (50) with (37), the mixture particle number density can be written as

$$r = \frac{p/T}{R_1 Y_1 + R_2 Y_2 + \frac{p}{T} Y_3 \left(r_o + \alpha \frac{p}{T} \right)^{-1}} \,. \tag{51}$$

The mixture defined by the previous relation is called relativistic idealized fluid mixture because it is derived from the relativistic idealized liquid defined above and an ideal gas and vapor. This equation can be interpreted as the equation of state of the mixture.

The normal speeds of propagation (32) of the hydrodynamical waves for this particular mixture is

$$\lambda_\Sigma^2 = \frac{1}{rf} \frac{r C_{\mathrm{V}} + p\beta^2 \displaystyle\sum_{k=1,2,3} \frac{X_k}{r_k \lambda_k^2}}{r C_{\mathrm{V}} \displaystyle\sum_{k=1,2,3} \frac{X_k}{r_k \lambda_k^2}} \,, \tag{52}$$

where $\lambda_1 = (R_1 T)^{1/2}$, $\lambda_2 = (R_2 T)^{1/2}$ and $\lambda_3 = (T/\alpha)^{1/2}$ are the isothermal speeds of sound in the gas, vapor and liquid, respectively, and $\beta = \beta_1 = \beta_2 = \beta_3 = (p/T)^{1/2}$ is the compressibility. Some simplifications yield

$$\lambda_\Sigma^2 = \frac{1}{rf} \frac{r C_{\mathrm{V}} \lambda_{X/r}^2 + p\beta^2}{r C_{\mathrm{V}}} \,, \tag{53}$$

where

$$\frac{1}{\lambda_{X/r}^2} = \sum_{k=1,2,3} \frac{X_k}{r_k \lambda_k^2} \,. \tag{54}$$

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