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THEORY OF STOCHASTIC CANONICAL EQUATIONS OF RANDOM MATRIX PHYSICS, SOS LAW, ELLIPTICAL GALACTIC LAW, SAND CLOCK LAW AND HEART LAW, LIFE, SOMBRERO AND HALLOWEEN LAWS*

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Devoted to the memory of Marian Ritter von Smolan Smoluchowski which after several years spent at other universities (Paris, Glasgow, and Berlin) moved to Lvov in 1899, where he took a position at the University of Lvov, before he moved to Kraków in 1913, to take over the chair in Experimental Physics Department. In 1906, independently of Albert Einstein, he described Brownian motion. Smoluchowski presented an equation which became an important basis of the theory of stochastic processes.

Our studies are essentially based on the martingale differences method developed in my previous papers for resolvents of random matrices. This method possesses the self-averaging property of the entries of resolvents of random matrices and, hence, we can deduce the stochastic canonical equation. The lecture contains the most important results from numerous papers and books dealing with the theory of Unitary random matrices and functions of random matrices. We give the REFORM method of proving of all results, avoiding the method of moments. We do not try to describe here all known properties of the eigenvalues and eigenvectors for all classes of random matrices. However, our aim is rather to present the theory of stochastic canonical equations, and to give rigorous proofs of the procedures used to deduce these equations on the base of the author's *General Statistical Analysis*. We consider special classes of analytic functions of random matrices. The description problem for normalized spectral functions of some analytic functions of random matrices is discussed in detail. Specifically, we present here the new theory: LIFE, which is the abbreviation for Limit Independence of Functions of Ensembles.

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Random matrix theory is a rapidly developing field and it has a great influence to fundamental and applied sciences: statistics, nuclear physics, and linear programming. Recent results in random matrix theory promoted the interest of researchers in the field of statistical physics to the methods and ideas developed for nuclear systems. One of the most intriguing applications of random matrix theory is the application to quantum mechanics.

We assume that energy levels of an atom are described by the eigenvalues of a random Hermitian operator, called the random Hamiltonian. It is very important that the eigenvalues of certain random matrices of large dimension converge to some nonrandom values, when the dimension of the matrix tends to infinity (see the three laws of random matrix theory in [1]). At the same time their eigenvectors remain to be random. In this manner, following E. Wigner and F. Dyson we can reach an agreement with the experimental observation of energy levels and wave functions of a nuclear of an atom on the base of stochastic canonical equations.

Most of the areas under consideration are strongly correlated with the spectral theory of nonsymmetric random matrices. The attention of scientists in the physics of random matrices is mainly focused on the matrices with zero expectations of their entries. The actual situation in the application of random matrices to physics is quite different. As a rule, the entries of matrices have nonzero means. We continue the development of a new V-analysis for nonsymmetric random matrices from *Girko's* ensemble when the pairs of the entries of random matrices are independent. Therefore, the main aim of the present lecture is to attract physicists to the new analysis of random matrices appearing in numerous contemporary problems.

If the dimensionality of observations is large, then most statisticians would agree that the efficiency of the classical parametric approaches is doubtful. In the GSA *General Statistical Analysis* we try to find new statistical estimators under two general assumptions. First, we do not require the existence of a density of observations. For example, we do not require that the observations have normal distributions. Second, we develop this analysis for the case where the number of parameters m_n can increase together with the number of observations n. Using these two assumptions we can obtain on the base of developed theory of canonical equations many new results and I am sure that the general statistical analysis will be a turning point in the multidimensional statistical analysis and Random Matrix Physics.

1. Canonical equation K_1. The main assertion

Theorem 1. *Assume that the entries* $\xi_{ij}^{(n)}; i \geq j, i, j = 1, \dots, n$, *of a symmetric random matrix* $\Xi_{n\times n} = (\xi_{ij}^{(n)})_{i,j=1}^{n}$ *are independent for each* $n = 1, 2, \dots$ *and defined on a common probability space*,

$$\boldsymbol{E}\xi_{ij}^{(n)} = a_{ij}^{(n)}, \qquad \operatorname{Var}\xi_{ij}^{(n)} = \sigma_{ij}^{(n)}, \qquad i \geq j, \ i,j = 1,\dots,n,$$

$$\sup_n \max_{i=1,\dots,n} \sum_{j=1}^{n} \sigma_{ij}^{(n)} < \infty, \tag{1.1}$$

$$\sup_n \max_{i=1,\dots,n} \sum_{j=1}^{n} \left| a_{ij}^{(n)} \right|^2 < \infty, \tag{1.2}$$

and Lindeberg's condition is satisfied, i.e., for any $\tau > 0$,

$$\lim_{n\to\infty} \max_{i=1,\dots,n} \sum_{j=1}^{n} \boldsymbol{E}\left[\xi_{ij}^{(n)} - a_{ij}^{(n)}\right]^2 \chi\left\{\left|\xi_{ij}^{(n)} - a_{ij}^{(n)}\right| > \tau\right\} = 0, \tag{1.3}$$

where χ *is the indicator of a random event,*

$$\mu_n\left\{x, \Xi_{n\times n}\right\} = n^{-1}\sum_{k=1}^{n} \chi(\omega: \ \lambda_k \ < \ x), \tag{1.4}$$

and $\lambda_1 \geq \cdots \geq \lambda_n$ *are the eigenvalues of the symmetric random matrix* $\Xi_{n\times n} = (\xi_{ij}^{(n)})_{i,j=1}^{n}$.

Then, for almost all x

$$\lim_{n\to\infty} \left|\mu_n\left\{x, \Xi_{n\times n}\right\} - F_n(x)\right| = 0, \tag{1.5}$$

with probability one. If, in addition,

$$\inf_{s,l=1,\dots,n} n\sigma_{sl}^{(n)} \geq c > 0, \tag{1.6}$$

then, with probability one,

$$\lim_{n\to\infty} \sup_x \left|\mu_n\left\{x, \Xi_{n\times n}\right\} - F_n(x)\right| = 0, \tag{1.7}$$

where $F_n(x)$ *are distribution functions whose Stieltjes transforms are equal to*

$$\int_{-\infty}^{\infty} (x-z)^{-1} d\,F_n(x) = n^{-1}\sum_{i=1}^{n} c_i(z), \qquad z = t + is, \qquad s \neq 0, \tag{1.8}$$

and the functions $c_i(z), i = 1,\dots,n$, *satisfy the canonical system of equations* K_1:

$$c_i(z) = \left\{\left[A_{n\times n} - zI_{n\times n} - \left(\delta_{pl}\sum_{s=1}^{n} c_s(z)\sigma_{sl}^{(n)}\right)_{p,l=1}^{n}\right]^{-1}\right\}_{ii}, \tag{1.9}$$

where $i = 1, \dots, n$ δ_{pl} *is the Kronecker symbol,* $A_{n\times n} = \left(a_{ij}^{(n)}\right)_{i,j=1}^{n}$, *and* $I_{n\times n}$ *is the identity matrix of the* n*-th order. There exists a unique solution* $c_i(z), i = 1, \dots, n$, *of the system of equations* K_1 *in a class of analytic functions*

$$L = \{z : \mathrm{Im} z \, \mathrm{Im} c_i(z) > 0 \,, \quad \mathrm{Im} z \neq 0 \,, \quad i = 1, \dots, n\} \tag{1.10}$$

and the functions $c_i(z)$, $i = 1, \dots, n$, *are the Stieltjes transforms of certain distribution functions.*

Note that, for some special cases, equation K_1 has been found. In the case where the matrix $A_{n\times n}$ is diagonal, the variances of the entries of a random matrix $\Xi_{n\times n}$ are equal, and Lindeberg's condition is satisfied for the components of each row vector of the matrix $\Xi_{n\times n}$, a special case of this equation was obtained by Pastur [2]. In the case where $A_{n\times n}$ is a zero matrix and the variances of the entries of a random matrix $\Xi_{n\times n}$ are bounded, it was established by F. Berezin (see [1]). The case where the matrix $A_{n\times n}$ is diagonal and the variances of the entries of a random matrix $\xi_{ij}^{(n)}$ may be different and satisfy Lindeberg's condition was studied by Girko [1].

2. Canonical equation K_{27} for normalized spectral functions of random symmetric block matrices

Consider random symmetric matrices $\Xi_{n\times n} = (\xi_{ij}^{(n)})_{i,j=1}^{n}$ with asymptotically independent entries. It is proved that, for almost all x and any $\varepsilon > 0$, under certain restrictions,

$$\lim_{n\to\infty} \boldsymbol{P}\,\{|\mu_n(x) - F_n(x)| > \varepsilon\} = 0 \,, \tag{2.1}$$

where

$$\mu_n(x) = n^{-1} \sum_{k=1}^{n} \chi(\lambda_k < x) \,, \tag{2.2}$$

$\chi(\lambda_k < x)$ is the indicator function, λ_k are eigenvalues of the matrix $\Xi_{n\times n} = (\xi_{ij}^{(n)})_{i,j=1}^{n}$, $F_n(x)$ is the distribution function whose Stieltjes transform is equal to

$$\int_{-\infty}^{\infty} (x - z)^{-1} \, dF_n(x) = n^{-1} \sum_{k=1}^{p} \mathrm{Tr}\, C_{kk}(z) \,, \qquad z = t + is \,, \qquad s \neq 0 \,, \tag{2.3}$$

and the block matrices $C_{kk}(z),\ k=1,\dots,p$, of dimensionality $q\times q$ satisfy the system of canonical equations K_{27}

$$C_{kk}(z)=\left\{\left[A_{pq}-zI_{pq}-\left(\delta_{lj}\sum_{s=1}^{p}\boldsymbol{E}\,H_{js}^{(n)}C_{ss}(z)H_{js}^{(n)*}\right)_{l,j=1}^{p}\right]^{-1}\right\}_{kk}, \quad (2.4)$$

where $k=1,\dots,p$, $A_{pq\times pq}$ is a nonrandom matrix, $I_{pq\times pq}$ is the identity matrix, $H_{js}^{(n)}$ are random matrices of dimensionality $q\times q$, p and q are some integers and notation $\{A\}_{kk}$ means the kth diagonal block of size $q\times q$ of the matrix A.

3. Manhattan project and SOS-laws

The Manhattan Project was the effort, led by the United States with participation from the United Kingdom and Canada, which resulted in the development of the first atomic bomb during World War II and was carried out in extreme secrecy. In 1939 and 1940, Eugene Paul Wigner (1902–1995) played a major role in agitating for a Manhattan Project, to build an atomic bomb. From 1942–1945, he worked on the Manhattan Project at the University of Chicago. Recall that historically Wigner limit density for the n.s.f. of symmetric random matrices was motivated by a certain model of heavy nuclei. The plot of this density is a certain semicircle (actually it is semiellipse). But this semielliptic density disappointed him and other physicists. The real densities of the energy levels of atom nucleus have another form. But we are now in a position to find such limit density for the random block matrices. For the simple random block matrices, we have Block Matrix Density which, for some matrices $A_{q\times q}$ and $B_{q\times q}$, is equal to *Sum Of Semielliptic* laws (SOS-laws) with different centers and radii. Therefore, it is possible to approximate any probability density using such SOS-law and it is possible to achieve an agreement with the observed densities of energy levels of atoms and the spectral density of our random block matrix.

To obtain the simplest result, we assume that the matrices $A_{q\times q}$ and $B_{q\times q}$ commute.

Theorem 2. *If, in addition to the conditions of Section 2, we have*

$$A_{q\times q} = H_{q\times q}\Lambda^{(1)}_{q\times q}H^{T}_{q\times q}, \qquad B_{q\times q}=H_{q\times q}\Lambda^{(1)}_{q\times q}H^{T}_{q\times q}, \quad (3.1)$$

where

$$\begin{aligned}&\Lambda^{(1)}_{q\times q}=(\delta_{ij}\lambda_i(A_{q\times q}))\,, && \Lambda^{(2)}_{q\times q}=(\delta_{ij}\lambda_i(B_{q\times q}))\,,\\ &\lambda_1(A_{q\times q})\le\cdots\le\lambda_q(A_{q\times q})\,, && \lambda_1(B_{q\times q})\le\cdots\le\lambda_q(B_{q\times q})\end{aligned} \quad (3.2)$$

are eigenvalues of matrices $A_{q\times q}$ and $B_{q\times q}$, and $H_{q\times q}$ is an orthogonal matrix, then, for all x with probability one

$$\lim_{p,q\to\infty} \left|\mu_{pq}\left(x, \varXi_{pq\times pq}\right) - F_q\left(x\right)\right| = 0\,, \tag{3.3}$$

where $F_q(x)$ is the distribution function whose density is equal to

$$\begin{aligned}\frac{d}{dx}\,F_q(x) \;=\; & \frac{1}{q}\sum_{k=1}^{q}\frac{1}{2\pi\lambda_k^2(B_{q\times q})}\chi\left\{\left[x-\lambda_k\left(A_{q\times q}\right)\right]^2 < 4\lambda_k^2(B_{q\times q})\right\} \\ & \times\sqrt{4\lambda_k^2(B_{q\times q}) - \left[x-\lambda_k\left(A_{q\times q}\right)\right]^2}\,,\end{aligned} \tag{3.4}$$

which is equal to the sum of semicircular laws (SOS-laws).

4. The canonical equation K_{96} for Girko's ensemble of random ace-matrix $\varXi_n$. Elliptical galactic law

The structure of this section is the following: at first we repeat the first 20 years old strong Elliptical law for random matrices $\varXi_n = \{\xi_{ij}^{(n)}\}$. Then we give the strong Elliptical law for random matrices $\varXi_n$ of the general form, *i.e.* when their diagonal entries $\xi_{ii}^{(n)}$ have nonzero expectations, and when we require the existence of the probability densities of the entries of random matrices and Lyapunov condition. In this case the Elliptical Galactic law means that the support of the accompanying spectral density of eigenvalues looks like the picture of several galaxies made by telescope. If the distances between the centers of these galaxies are large enough we have several almost elliptical galaxies. These statements are based on the VICTORIA-transform of random matrix which is the abbreviation of the following words: Very Important Computational Transformation Of Random Independent Arrays.

We follow the main strategy of the theory of limit theorems of the probability theory, *i.e.* we try to solve the problem of description of all limits of normalized spectral functions

$$\begin{aligned}&\nu_n\left(x, y, A_n\varXi_nB_n{+}C_n\right)\\ &= \frac{1}{n}\sum_{k=1}^{n}\chi\Big\{\mathrm{Re}\lambda_k(A_n\varXi_nB_n{+}C_n) < x\,,\ \mathrm{Im}\lambda_k(A_n\varXi_nB_n{+}C_n) < y\Big\}\,,\end{aligned} \tag{4.1}$$

where $\lambda_k(A_n\varXi_nB_n{+}C_n)$ are eigenvalues of the matrix $A_n\varXi_nB_n{+}C_n$, A_n, B_n, and C_n are nonrandom matrices, under general (as only possible) conditions on the entries $\xi_{ij}^{(n)}$ of random matrices $\varXi_n$, χ is the indicator function. We emphasize that the spectral theory of Hermitian random matrices is rather profound. For example, in 1975 Girko proved the general stochastic canonical equation for ACE (Asymptotically Constant Entries)-symmetric matrices [1, 3]: Assume that for any n, the random entries $\xi_{ij}^{(n)}$, $i \geq j$, $i,j = 1,\ldots,n$, of a symmetric matrix $\Xi_{n\times n} = \left[\xi_{ij}^{(n)} - \alpha_{ij}^{(n)}\right]_{i,j=1}^{n}$ are independent and they are ACE, *i.e.*, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \sup_{p,l=1,\ldots,n} \boldsymbol{P}\left\{\left|\xi_{pl}^{(n)}\right| > \varepsilon\right\} = 0, \alpha_{ij}^{(n)} = \int\limits_{|x|<\tau} x\, d\boldsymbol{P}\left\{\xi_{ij}^{(n)} < x\right\} \tag{4.2}$$

and $\tau > 0$ is an arbitrary constant, and that, for every $0 \leq u \leq 1$ and $0 \leq v \leq 1$,

$$K_n\left(u,v,z\right) \Rightarrow K\left(u,v,z\right)\,, \qquad -\infty < z < \infty\,, \tag{4.3}$$

where the symbol $\Rightarrow$ denotes the weak convergence of distribution when $n \to \infty$,

$$K_n\left(u,v,z\right) = n \int\limits_{-\infty}^{z} y^2\left(1+y^2\right)^{-1} d\boldsymbol{P}\left\{\xi_{ij}^{(n)} - \alpha_{ij}^{(n)} < y\right\}, \tag{4.4}$$

$in^{-1} \leq u < (i+1)\,n^{-1}$, $jn^{-1} \leq v < (j+1)\,n^{-1}$, and $K\left(u,v,z\right)$ is a nondecreasing function with bounded variation in z and continuous in u and v in the domain $0 \leq u,v \leq 1$. Then, with probability one, for almost all x,

$$\lim_{n\to\infty}\left|n^{-1}\sum_{k=1}^{n}\chi\{\lambda_k\left(\Xi_{n\times n}\right) < x\} - F\left(x\right)\right| = 0\,, \tag{4.5}$$

where $\lambda_k\left(\Xi_{n\times n}\right)$ are eigenvalues, $F(x)$ is a distribution function whose Stieltjes transform satisfies the relation

$$\int\limits_{-\infty}^{\infty} \frac{dF\left(x\right)}{1+\mathrm{i}tx} = \lim_{\alpha\downarrow 0}\int\limits_{0}^{1}\left[\int\limits_{0}^{1} x\, d_x G_\alpha\left(x,y,t\right)\right] dy\,, \tag{4.6}$$

$G_\alpha\left(x,y,t\right)$, as a function of x, is a distribution function satisfying the regularized stochastic canonical equation K_3 [2,3] at the points x of continuity,

$$G_\alpha\left(x,z,t\right) = \boldsymbol{P}\left\{\left[1+t^2\xi_\alpha\left\{G_\alpha\left(*,*,t\right),z\right\}\right]^{-1} < x\right\}, \qquad 0 \leq x \leq 1\,, \tag{4.7}$$

$\xi_\alpha \{G_\alpha(*,*,t), z\}$ is a random real functional whose Laplace transform of one-dimensional distribution is equal to

$$\boldsymbol{E}\, e^{\{-s\xi_\alpha[G_\alpha(*,*,t),z]\}}$$
$$= \exp\left\{ \int_0^1 \int_0^1 \left[\int_0^\infty \left[\exp\left\{-\frac{syx^2}{(1+\alpha|x|)^2}\right\} - 1\right] \frac{1+x^2}{x^2}\, d_x K(v,z,x)\right] d_y G_\alpha(y,v,t)\, dv \right\}, \tag{4.8}$$

where $\alpha > 0, \quad s \geq 0, \quad 0 \leq z \leq 1\,.$

The integrand $\left[\exp\left\{-syx^2(1+\alpha|x|)^{-2}\right\} - 1\right]\left(1+x^{-2}\right)$ is defined at $x = 0$ by continuity as $-sy$. There exists a unique solution of the canonical equation K_3 in the class L of functions $G_\alpha(x,y,t)$ that are distribution functions of x $(0 \leq x \leq 1)$ for any fixed $0 \leq y \leq 1$, $-\infty < t < \infty$, such that, for any integer $k > 0$ and z, the function $\int_0^1 x^k d_x G_\alpha(x,z,t)$ is analytic in t (excluding, possibly, the origin). The solution of the canonical equation K_3 can be found by the method of successive approximations.

For the first time in 1980 and in 1990 this equation was rewritten in the following form (here we use the simplest equation, when $\alpha = 0$)

$$m(s,t,z) - 1 = \int_0^\infty \exp\left\{ \int_0^1 \int_0^1 \left[\int_0^\infty \left[m\left(t^2yx^2, t, v\right) - 1\right] \times \frac{1+x^2}{x^2} d_x K(v,z,x)\right] dv \right\} \frac{\partial}{\partial y} J_0(2\sqrt{sy}) e^{-y} dy\,, \tag{4.9}$$

where

$$m(s,t,z) = \int_0^1 e^{-sx} d_x G_0(x,z,t)\,, \qquad s \geq 0\,, \tag{4.10}$$

$J_0(x)$ is the Bessel function which is equal to

$$J_0(x) = \sum_{k=0}^\infty (-1)^k x^{2k} \frac{1}{2^{2l} k! k!}\,. \tag{4.11}$$

In [4], a technical improvement and a new proof of the uniqueness of solution of canonical equation K_3 are presented, where $m(s,t,z)$ has a unique representation in the family of integrable functions. The analytic details of the statement and of the proof are elaborate.

We give here the strong Elliptical Galactic law for random matrices Ξ_n of the general form, *i.e.* their diagonal entries $\xi_{ij}^{(n)}$ have nonzero expectations and the pairs of the entries $(\xi_{ij}^{(n)}, \xi_{ji}^{(n)})$ have nonzero covariances. In this case the Elliptical Galactic law means that the support of the accompanying spectral density of eigenvalues of matrix Ξ_n looks like the picture of several galaxies made by telescope.

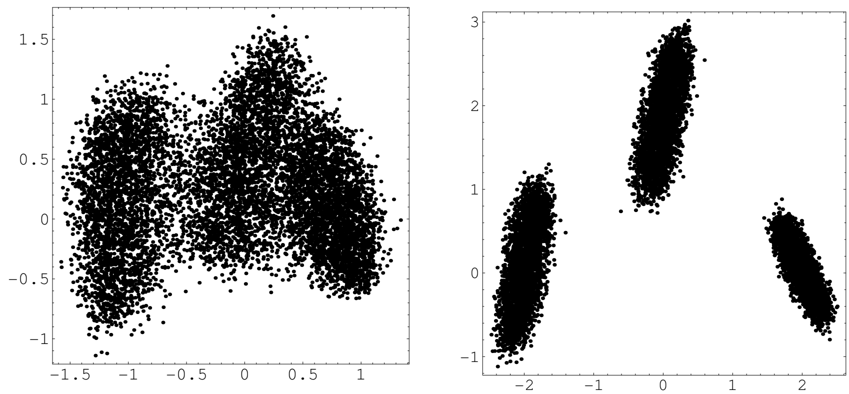


Fig. 1. See explanation in the text.

Fig. 1 (left) shows the *collision* of elliptic supports of the limit spectral density of n.s.f. of random matrix $A_n + \Lambda_n \Xi_n$, where A_n is a diagonal complex matrix with diagonal entries $(0.7, 0), (-1, 0), (0, 0.7i)$ for corresponding three equal parts of the main diagonal, and random matrix Ξ_n has equal covariances $\rho(\sqrt{\rho} = 0.2 + i0.8)$ of independent pairs of entries $(\xi_{ij}^{(n)}, \xi_{ji}^{(n)})$ with zero mean and is multiplied by diagonal matrix Λ_n with diagonal entries $(1, 0), (0.5, 0.5i), (-1, 0)$ for corresponding three equal parts of the main diagonal. We have chosen in picture 1 three different diagonal entries of the matrix A_n at a short distance. In Fig. 1 (right), we consider the diagonal matrix A_n with diagonal entries $(2, 0), (-2, 0), (0, 2i)$ at a large distant for corresponding three equal parts of the main diagonal. In the letter case we have several domains-supports like ellipses. For the exposition of the Elliptical law we have chosen the random matrix Ξ_n of dimension 30 and 300 its Monte Carlo simulation. If the distances between the centers of these *galaxies* are large enough we have several almost *elliptical galaxies.*

Fig. 2 shows the elliptic support of the limit spectral density of n.s.f. of random matrix $A_n + \Xi_n$, where A_n is a diagonal matrix with 5 different diagonal entries $(1, 0)$; $(-1, 0)$; $(-0.5, -i)$; $(0, 0.5i)$; $(0, i)$ and random matrix Ξ_n has equal covariances $\rho(\sqrt{\rho} = 0.5 + i0.5)$ of the entries (ξ_{ij}, ξ_{ji}). We have chosen five different diagonal entries of the matrix A_n at a short distance in Fig. 2 (left) and at a large $(2, 0); (-2, 0); (-1, -2i); (0, i); (0, 2i)$ in Fig. 2 (right). In the letter case we have several domains-supports like ellipses. For the exposition of the Elliptical law we have chosen the random matrix Ξ_n of dimension 50 and 300 its Monte Carlo simulation.

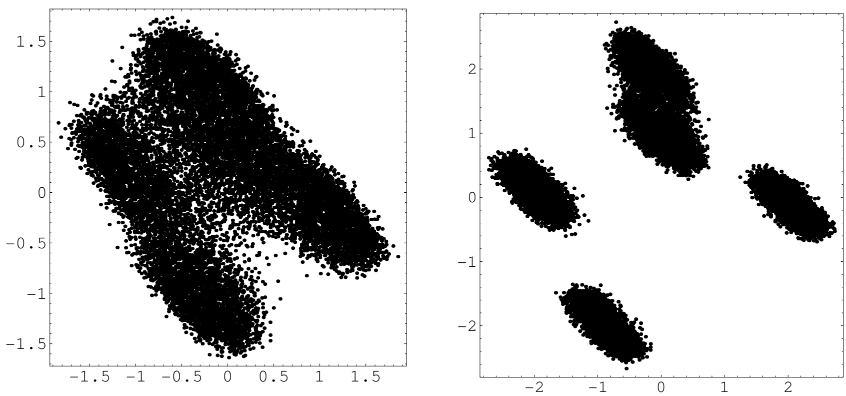


Fig. 2. See explanation in the text.

If the distances between the centers of these galaxies are large enough we have several almost elliptical galaxies.

Maybe the reader remembers the Monte Carlo simulations of eigenvalues of matrices $\Xi_n + A_n$, where Ξ_n belongs to the domain of attraction of Circular law and A_n is the diagonal matrix whose diagonal entries forms letter $\boldsymbol{R}$ on a complex plain [3]. For the case when the matrix Ξ_n belongs to the domain of attraction of Elliptical law the simulation of eigenvalues of the matrix $\Xi_n + A_n$ looks like the following picture — Fig. 3.

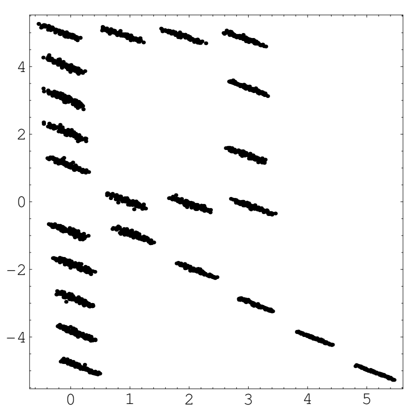


Fig. 3. See explanation in the text.

There are essentially three methods of the proof of Elliptical Laws that have been proposed: the REFORM method and Berry–Esseen inequality [1], the method of perpendiculars [1,3], the method of the central limit theorem and limit theorems for eigenvalues of random matrices [1, 3]. The main advantage of REFORM approach is that it enables the results of the previous version of Elliptical law to be extended to the case under consideration. The REFORM-method (or G-martingale approach) enables us to suggest a new method for construction of stochastic canonical equations.

We give the following *Elliptical Galactic Law* which generalizes the *Strong Circular Law* and *Weak Circular Law* (see the sketch of the proof of this law in the paper *V-transform*, Dopovidi Akademii Nauk Ukrainskoi RSR. Seriia A, Fizykotekhnichni ta matematychni nauky, 1982, N3, pp. 5–6.): For every n, let the pairs of random entries $(\xi_{ij}^{(n)}, \xi_{ji}^{(n)}); i = 1, \dots, n, j = 1, \dots, n$, of the complex matrix $\Xi_{n\times n} = (\xi_{ij}^{(n)})_{i=1,\dots,n}^{j=1,\dots n}$ be independent and given on a common probability space, $\boldsymbol{E}\xi_{ij}^{(n)} = 0, \boldsymbol{E}\left|\xi_{ij}^{(n)}\right|^2 = \sigma_{ij}^{(n)} n^{-1}, 0 < r_1 < \sigma_{ij}^{(n)} < r_2 < \infty, \boldsymbol{E}\xi_{ij}^{(n)}\xi_{ji}^{(n)} = \rho_{ij}^{(n)} n^{-1}, i \neq j,\ i, j = 1, \dots, n$, and

$$\sup_n \max_{\substack{i=1,\dots,n,\\ j=1,\dots,n}} \left\{ \sum_{j=1}^{n} \left|(A_n^{-1} C_n B_n^{-1})_{ij}\right|^2 + \sum_{i=1}^{n} \left|(A_n^{-1} B_n^{-1})_{ij}\right|^2 + \sum_{j=1}^{n} \left|(A_n^{-1} C_n B_n^{-1})_{ji}\right|^2 + \sum_{i=1}^{n} \left|(A_n^{-1} B_n^{-1})_{ji}\right|^2 \right\} < \infty\,, \tag{4.12}$$

where $A_n = \left\{a_{ij}^{(n)}\right\}_{i,j=1,\dots,n}$, $B_n = \left\{b_{ij}^{(n)}\right\}_{i,j=1,\dots,n}$ and $C_n = \left\{c_{ij}^{(n)}\right\}_{i,j=1,\dots,n}$ are nonrandom matrices, $\det A_n \neq 0, \det B_n \neq 0$, and the real and imaginary parts of entries $\sqrt{n}\xi_{ij}^{(n)}$, $\sqrt{n}\xi_{ji}^{(n)}, i > j$ have the densities

$$\begin{aligned} &p_{ij}^{(n)}(x_1, x_2, y_1, y_2) \\ &= \frac{\partial^4}{\partial x_1\, \partial x_2\, \partial y_1\, \partial y_2} \boldsymbol{P}\left\{ \mathrm{Re}\sqrt{n}\xi_{ij}^{(n)} < x_1,\ \mathrm{Re}\sqrt{n}\xi_{ji}^{(n)} < x_2\,, \right. \\ &\left. \mathrm{Im}\sqrt{n}\xi_{ij}^{(n)} < y_1,\ \mathrm{Im}\sqrt{n}\xi_{ji}^{(n)} < y_2 \right\} \end{aligned} \tag{4.13}$$

satisfying the corrected *Elliptic condition*: for some $\beta > 1$

$$\sup_n \max_{\substack{l=1,\dots,n\\ k\neq l}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max_{k=1,..,n} \left[\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} p_{kl}^{(n)}(x, y, u, v) dy \right]^{\beta} dx \right]^{1/\beta} du\, dv < \infty\,, \tag{4.14}$$

or

$$\sup_n \max_{\substack{l=1,\dots,n \\ k\neq l}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max_{k=1,..,n} \left[\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} p_{kl}^{(n)}(x,y,u,v)dx\right]^{\beta} dy\right]^{1/\beta} du\,dv < \infty\,, \tag{4.15}$$

and there exist the densities $p_{ii}^{(n)}(x)$ of the entries $\sqrt{n}\mathrm{Re}\xi_{ii}^{(n)}$, or the densities $q_{ii}^{(n)}(x)$ of the entries $\sqrt{n}\mathrm{Im}\,\xi_{ii}^{(n)}$, satisfying the condition: for some $\beta_1 > 1$

$$\sup_n \max_{k=1,\dots,n} \int_{-\infty}^{\infty} \left[p_{kk}^{(n)}(x)\right]^{\beta_1} dx < \infty\,, \tag{4.16}$$

or

$$\sup_n \max_{k=1,\dots,n} \int_{-\infty}^{\infty} \left[q_{kk}^{(n)}(x)\right]^{\beta_1} dx < \infty\,, \tag{4.17}$$

the Lyapunov condition is fulfilled: for some $\delta > 0$,

$$\max_{p,l=1,\dots,n} \boldsymbol{E}\left|\xi_{pl}^{(n)}\sqrt{n}\right|^{2+\delta} \le c < \infty\,. \tag{4.18}$$

Then, with probability one, for almost all x and y

$$\lim_{\alpha\downarrow 0}\lim_{n\to\infty} \left|\nu_n(x,y,A_n\Xi_n B_n + C_n) - F_{n,\alpha}(x,y)\right| = 0\,, \tag{4.19}$$

where

$$\nu_n(x,y,A_n\Xi_n B_n + C_n) = n^{-1}\sum_{k=1}^{n}\chi\left\{\mathrm{Re}\,\lambda_k < x, \mathrm{Im}\,\lambda_k < y\right\}, \tag{4.20}$$

λ_k are eigenvalues of the matrix $A_n\Xi_n B_n + C_n$, the Global probability density $p_{n,\alpha}(t,s) = (\partial^2/\partial t\,\partial s)F_{n,\alpha}(t,s)$ is equal to

$$p_{n,\alpha}(t,s) = \begin{cases} -\frac{1}{4\pi}\int_{\alpha}^{\infty}\left[\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}\right] b_n(y,t,s)\,dy & \text{for } (t,s)\notin G_n\,, \\ 0 & \text{for } (t,s)\in G_n, \end{cases} \tag{4.21}$$

where $\alpha > 0$,

$$\begin{aligned} b_n\Big(y,t,s\Big) &= \frac{i}{2\sqrt{y}}n^{-1}\,\mathrm{Tr}\left[I_{2n}i\sqrt{y} - Q_{2n}(y,t,s) + C_{2n}(y,t,s)\right]^{-1}\,, \\ Q_{2n}(y,t,s) &= \left(\delta_{ij}Q_{2\times 2}^{(ii)}(y,t,s)\right)_{i,j=1,\dots,n}\,, \qquad C_{2n}(t,s) = \left\{C_{2\times 2}^{(ij)}(t,s)\right\}\,, \end{aligned}$$

where $C_{2n}(t,s)=\left(c_{2\times 2}^{(ij)}(t,s)\right)_{i,j=1,\dots,n}$ is a block matrix, $C_{2\times 2}^{(ij)}=\left\{\begin{matrix} 0 & s_{ij}^{(n)} \\ \bar{s}_{ji}^{(n)} & 0 \end{matrix}\right\}$, $s_{ij}^{(n)}(t,s)$ are entries of the matrix

$$S_n(t,s)=A_n^{-1}(C_n-I_n\tau)B_n^{-1}=\left\{s_{ij}^{(n)}(t,s)\right\}, \tag{4.22}$$

and $Q_{2n}(y,t,s) = \left(\delta_{ij}Q_{2\times 2}^{(ii)}(y,t,s)\right)_{i,j=1,\dots,n}$ is the block diagonal matrices, whose diagonal block $Q_{2\times 2}^{(ii)}(t,s)$ satisfy the system of canonical equations K_{97}

$$\begin{aligned} Q_{2\times 2}^{(jj)}(y,t,s) &= \Bigg\{iI_{2n}\sqrt{y}+C_{2n}(t,s) \\ &- \left[\delta_{ij}\sum_{i=1}^{n}\boldsymbol{E}\left\{\begin{matrix} 0 & \xi_{ij} \\ \xi_{ji}^{*} & 0 \end{matrix}\right\}Q_{2\times 2}^{(ii)}(y,t,s)\left\{\begin{matrix} 0 & \xi_{ij} \\ \xi_{ji}^{*} & 0 \end{matrix}\right\}^{*}\right]_{i,j=1,\dots,n}\Bigg\}_{jj}^{-1} \end{aligned} \tag{4.23}$$

$j=1,\dots,n$, and $\bar{G}$ is a support of the Global probability density, where

$$G=\left\{(t,s): \lim_{\alpha\downarrow 0}\lim\sup_{n\to\infty}\alpha\frac{\partial}{\partial\alpha}b_n(\alpha,t,s)=0\right\}. \tag{4.24}$$

There exists a unique solution of canonical equation K_{97} in the class of positive definite block matrices $Q_{2\times 2}^{(ii)}(y,t,s)>0, y>0, i=1,\dots,n$ of the order 2×2, analytic in $y>0,\ t,\ s$.

5. The border of the support of limit spectral density $p(x,y)$ for pure G-ensemble when only two constant of diagonal matrix are pure imaginary numbers. Sand clock density

The next example is the simplest case of matrices from G-ensemble, when only two diagonal complex entries of diagonal matrix A_n are different and random matrix Ξ_n is Hermitian matrix. We can find the border support of accompanying spectral density, but even in this simple case the solution is not simple and the mathematical equation for the curve of the border of accompanying support of limit spectral density occupies almost the half of a page.

Theorem 3. *If additionally to the conditions of the previous theorem* $\rho^{(n)}=1,\ a_k=ir, k=1,\dots,[n/2]; a_l=-ir, l=[n/2]+1,\dots,n$ *then the border of the support of accompanying probability spectral density is given by the following equation*

$$
\begin{aligned}
&\Big(d^4k^4+d^3k^2\Big(\,((-2)bkl+2al^2-4akq)\,\Big)+d^2\Big(\Big(c^2k^3l+b^2k^2l^2-2abkl^3\\
&+a^2l^4+3bck^3p-5ack^2lp-3abk^2p^2+4a^2klp^2+2b^2k^3q+2abk^2lq\\
&-4a^2kl^2q+6a^2k^2q^2\Big)\Big)+d\Big(\Big(\left(-c^3\right)k^3p-b^2ck^2lp+2abckl^2p-a^2cl^3p\\
&+b^3k^2p^2+3ac^2k^2p^2-2ab^2klp^2+a^2bl^2p^2-3a^2ckp^3+a^3p^4-4bc^2k^3q\\
&-2b^3k^2lq2ac^2k^2lq+4ab^2kl^2q-2a^2bl^3q+2abck^2pq+2a^2cklpq\\
&+2a^2bkp^2q-4a^3lp^2q-4ab^2k^2q^2+2a^2bklq^2+2a^3l^2q^2-4a^3kq^3\Big)\Big)\\
&-q\Big(\Big(\left(-c^4\right)k^3-b^2c^2k^2l+2abc^2kl^2-a^2c^2l^3+b^3ck^2p+3ac^3k^2p\\
&-2ab^2cklp+a^2bcl^2p-3a^2c^2kp^2+a^3cp^3-b^4k^2q-4abc^2k^2q\\
&+2ab^3klq+3a^2c^2klq-a^2b^2l^2q+5a^2bckpq-3a^3clpq-a^3bp^2q\\
&-2a^2b^2kq^2+2a^3blq^2-a^4q^3\Big)\Big)\Big)=0\,, \qquad (5.1)
\end{aligned}
$$

where

$$
\begin{aligned}
a &= 1\,, \quad k=1\,, \quad p=0\,, \quad b=\frac{t^2}{2}-2r^2\,, \quad c=-\frac{2r^2}{s}\,,\\
d &= \frac{t^4}{16}+\frac{t^2r^2}{2}+r^4\,, \quad l=\frac{t^2}{2}-1-2r^2\,, \quad q=\frac{t^4}{16}+\frac{t^2r^2}{2}+r^4-\frac{t^2}{4}-r^2\,.
\end{aligned}
$$

6. Several examples of the border support and Monte Carlo simulations performed by `Mathematica 5` for pure G-ensemble. Sand clock density

We give here several examples. For the reader conveniences we provide them by corresponding program of `Mathematica 5`. Enjoy considering different cases of random matrices.

Fig. 4 shows the elliptic support of the limit spectral density of n.s.f. of random matrix $A_n+\Xi_n$. We have chosen the constant $r=0.5,\rho=1$. In this case we have one domain-like ellipse. For the exposition of the Elliptical law we have chosen the random Hermitian matrix Ξ_n of dimension 20 and 500 its Monte Carlo simulation.

Fig. 5 shows the elliptic support of the limit spectral density of n.s.f. of random matrix $A_n+\Xi_n$ considered in Theorem 3. We have chosen the constant $r=1,\rho=1$. In this case we have two domain like ellipses with one common point which look like *sand clock*. For the exposition of the Elliptical law we have chosen the random Hermitian matrix Ξ_n of dimension 20 and 300 its Monte Carlo simulation.

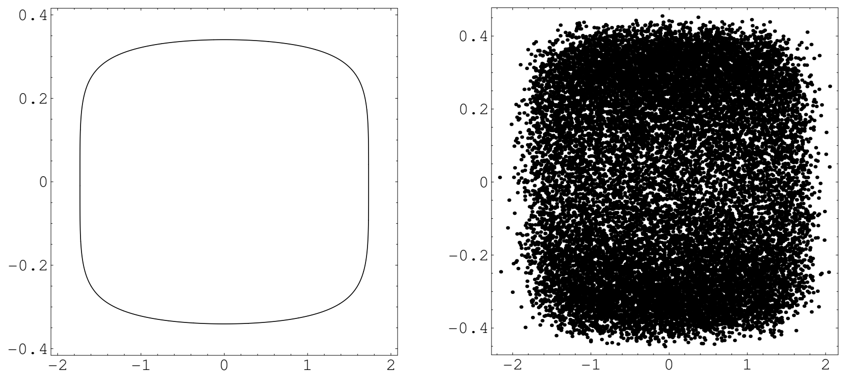


Fig. 4. See explanation in the text.

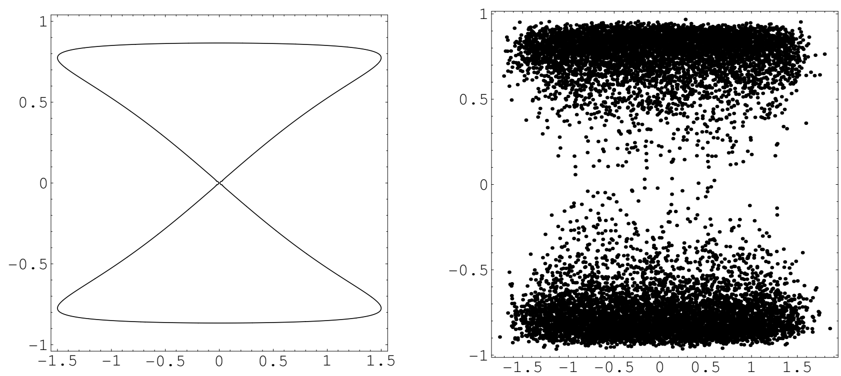


Fig. 5. See explanation in the text.

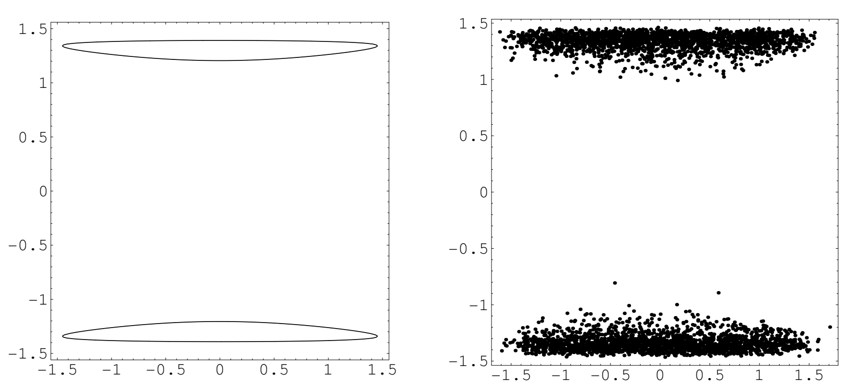


Fig. 6. See explanation in the text.

Fig. 6 shows the elliptic support of the limit *sand clock* spectral density of n.s.f. of random matrix $A_n + \Xi_n$. We have chosen the constant $r = 1.5$, $\rho = 1$. In this case we have two separated domain like ellipses. For the exposition of the Elliptical law we have chosen the random matrix Ξ_n of dimension 20 and 100 its Monte Carlo simulation.

7. VICTORIA-transform for the matriciant of the growing dimension

We consider the random matrizant

$$Z_{n\times n}^{(m)} = \prod_{i=1}^{m} \left[I_n + I_n \frac{f(\frac{i}{m})}{m} + \frac{g(\frac{i}{m})}{\sqrt{m}} \Xi_n^{(i)} \right] \tag{7.1}$$

of random ACE matrices $\Xi_n^{(i)}$ whose entries may have different variances.

We give a new method of deriving general canonical equation for the VICTORIA-transform of normalized spectral functions (n.s.f.)

$$\nu_n(u,v) = n^{-1} \sum_{k=1}^{n} \chi \left\{ \Im\lambda_k(Z_{n\times n}^{(m)}) < u, \Re\lambda_k(Z_{n\times n}^{(m)}) < u \right\} \tag{7.2}$$

of the product of random matrices (matriciant) Eq. (7.1) of the independent matrices $\Xi_n^{(i)}$, which was recently obtained for some particular cases on the base of free probability theory. Here, $\lambda_k(Z_{n\times n}^{(m)}), k = 1, \ldots, n$ mean the eigenvalues of matrix, $f(x)$ and $g(x)$ are certain functions, $I_{n\times n}$ is identity matrix and the product of random matrices is taken from the left to the right. We will use for quadratic matrices two notations: $A_{n\times n}$ and A_n. We apply the REFORM-method and the simplest and the shortest Girko's theory of the proof of the Circular law ([1, 3]) for the deduction of the system of canonical equations K_{91} for normalized spectral functions $\nu_n(u,v)$ of this matriciant $Z_{n\times n}^{(m)}$. The probability distributions of random matrices $\Xi_{n\times n}^{(i)}, i = 1, 2, \ldots$ belong to the domain of attraction of the Circular law.

8. The G-method

In this section, we show the power of our G-method in comparison to *replica trick, the supersymmetry approach and free probability theory* , on the example of the product of two random matrices. Other examples when we can consider more matrices will easily follow from this example.

Step 1. We can establish the self averaging property of n.s.f. of permutation matrices $A_{2n\times 2n}A^*_{2n\times 2n}$ due to the presence the logarithmic function in the V-transform.

Step 2. We can make the V-regularization choosing very small parameter of regularization like $\alpha = n^{-q_2}$. Where q_2 is a number and for our theory the value of this number is not important. For our purposes it is enough that this number is fixed and does not depend on n.

Step 3. Then we extend $\det[\alpha_n I_{2n\times 2n} + A_{2n\times 2n}A^*_{2n\times 2n}]$ once again:

$$\det[\alpha_n I_{2n\times 2n} + A_{2n\times 2n}A^*_{2n\times 2n}] = (-1)^{2n}\det\begin{bmatrix} i\sqrt{\alpha}_n I_{2n\times 2n} & A^*_{2n\times 2n} \\ A_{2n\times 2n} & i\sqrt{\alpha}_n I_{2n\times 2n}\end{bmatrix} \tag{8.1}$$

and now we consider the n.s.f. of Hermitian matrices

$$G_{4n\times 4n} = \begin{bmatrix} 0 & A^*_{2n\times 2n} \\ A_{2n\times 2n} & 0\end{bmatrix}. \tag{8.2}$$

Step 4. We find a canonical equation for the Stieltjes transform of the non-random accompanying n.s.f. $\mu_n(t,s,x)$.

Step 5. Then we can use the rough estimator of convergence of the n.s.f. of the corresponding random permutation matrix with the speed of convergency like n^{-q_3}. All calculations are almost the same which were used in the G-theory.

9. G-method, the Berry–Esseen inequality

We are using the Berry–Esseen inequality, the random variable γ and we do not pursue the precise order of convergency of n.s.f. $\nu_n(G_{4n\times 4n},t,s,x,)$ to the accompanying n.s.f. $\mu_n(t,s,x)$:

$$\sup_x |\nu_n(G_{4n\times 4n},t,s,x,) - \mu_n(t,s,x)| \le cn^{-q_3}\,, \qquad c > 0\,. \tag{9.1}$$

Then we can perform the limit procedure in V-transform.

10. G-method. Canonical equation K_{91} for the product of two independent matrices with independent entries

Theorem 4. *If the real matrices $\Xi^{(j)}_n = \{\xi^{(n,j)}_{pl}\}, j = 1,2; p,l = 1,\dots,n$ be independent for every $n = 1,2,\dots$ and their entries satisfy the conditions of Circular law. Then for every t and s*

$$\lim_{\alpha\downarrow 0} p\lim_{n\to\infty}\left\{\nu_n\left[t,s,\prod_{j=1}^{2}(I_n+\varepsilon_j\Xi^{(j)}_n)\right]\right.$$

$$+\frac{1}{4\pi}\int_{\alpha}^{\infty}\left(\frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial t^2}\right)\frac{1}{-2i\sqrt{y}}b_n\Big(y,t,s\Big)dy\Bigg\}=0\,, \tag{10.1}$$

where $b_n(y,t,s)=\frac{1}{n}\sum_{p=1}^{4n} r_{pp}^{(n)}(y,t,s)$ *and* $r_{pp}^{(n)}(y,t,s), p=1,\dots,4n$ *satisfy the following system of canonical equations* K_{91}*:*

$$r_{kk}^{(n)}=\left(\left[iy^{(1/2)}I_{4n\times 4n}+B_{4n\times 4n}-G_{4n\times 4n}\Big(y,t,s\Big)\right]^{-1}\right)_{kk},\quad k=1,\dots,4n\,, \tag{10.2}$$

where

$$\begin{aligned}
B_{4n\times 4n}\Big(t,s\Big) &= \left(b_{ij}^{(n)}\Big(t,s\Big)\right)_{i=1,\dots,4n}^{j=1,\dots,4n}\\
&= \left\{\begin{array}{cccc} 0 & 0 & I_n\tau^{(1/2)} & L_1\\ 0 & 0 & L_2 & I_n\tau^{(1/2)}\\ I_n\bar{\tau}^{(1/2)} & L_2^* & 0 & 0\\ L_2^* & I_n\bar{\tau}^{(1/2)} & 0 & 0\end{array}\right\},\\
L_i &= I_n+\varepsilon_i \boldsymbol{E}\varXi_n^{(i)} \qquad (10.3)\\
G_{4n\times 4n}\Big(y,t,s\Big) &= \left(g_{ij}^{(n)}\Big(y,t,s\Big)\right)_{i=1,\dots,4n}^{j=1,\dots,4n}\\
&= \left\{\begin{array}{cccc} \varepsilon_1^2 G_n^{(1)} & 0 & 0 & 0\\ 0 & \varepsilon_2^2 G_n^{(2)} & 0 & 0\\ 0 & 0 & \varepsilon_1^2 G_n^{(3)} & 0\\ 0 & 0 & 0 & \varepsilon_2^2 G_n^{(4)}\end{array}\right\}, \qquad (10.4)
\end{aligned}$$

$$\begin{aligned}
G_n^{(1)}\Big(y,t,s\Big) &= \left[\delta_{il}\frac{1}{n}\sum_{p=3n}^{4n} r_{pp}^{(n)}\Big(y,t,s\Big)\sigma_{pl}^{(n),1}\right]_{i,l=1,\dots,n},\\
G_n^{(2)}\Big(y,t,s\Big) &= \left[\delta_{il}\frac{1}{n}\sum_{p=2n}^{3n} r_{pp}^{(n)}\Big(y,t,s\Big)\sigma_{pl}^{(n),2}\right]_{i,l=1,\dots,n},\\
G_n^{(3)}\Big(y,t,s\Big) &= \left[\delta_{il}\frac{1}{n}\sum_{p=n}^{2n} r_{pp}^{(n)}\Big(y,t,s\Big)\sigma_{pl}^{(n),1}\right]_{i,l=1,\dots,n},\\
G_n^{(4)}\Big(y,t,s\Big) &= \left[\delta_{il}\frac{1}{n}\sum_{p=1}^{n} r_{pp}^{(n)}\Big(y,t,s\Big)\sigma_{pl}^{(n),2}\right]_{i,l=1,\dots,n}. \qquad (10.5)
\end{aligned}$$

There exists a unique solution of this equation in the class of analytical functions in t *and* s*.*

11. The border $G_{93}(t,s)$ of the support of the accompanying spectral density $p_{93}(t,s)$ for random matrices whose entries have equal variances and nonzero expectations for diagonal entries

It is difficult to find the accompanying probability spectral density of the product of two matrices, but surprisingly more easily to find the border of the support G of the accompanying spectral density for random matrices.

Then from Theorem 4 we obtain the border of the support G of the accompanying spectral density for random matrices with equal variances. Now we can consider many interesting cases of distribution of eigenvalues of the product of two matrices. For example, if $\boldsymbol{E}\Xi_n(1) = \boldsymbol{E}\Xi_n(2) = A_n$, $\boldsymbol{E}|\xi_{pl}^{(j)} - a_{pl}^{(j)}|^2 = n^{-1}, \varepsilon(j) = 1, j = 1,2$, then we have equation for the border of the support of accompanying spectral density

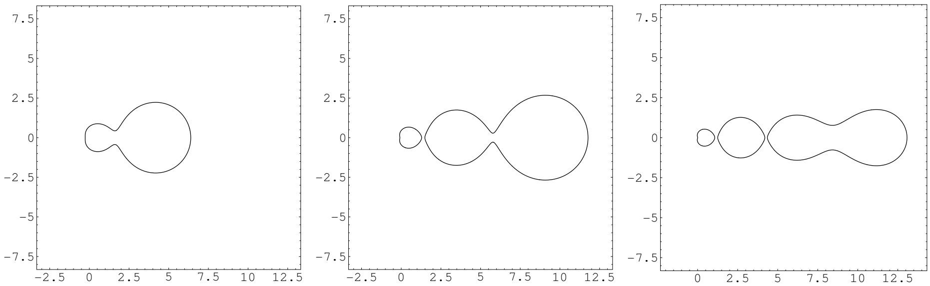
$$1 = \frac{1}{n}\operatorname{Tr}\Big\{|\tau| + (I_n + A_n)(I_n + A_n)^* - [\sqrt{\tau}(I_n + A_n)^* + \sqrt{\bar{\tau}}(I_n + A_n)]$$
$$\times [|\tau| + (I_n + A_n)(I_n + A_n)^*]^{-1}\Big[\sqrt{\tau}(I_n + A_n)^* + \sqrt{\bar{\tau}}(I_n + A_n)\Big]\Big\}^{-1}. \quad (11.1)$$

The second example, if $\boldsymbol{E}\Xi_n^{(1)} = \boldsymbol{E}\Xi_n^{(2)} = A_n$ and the matrix $I_n + A_n$ is a symmetric real matrix with eigenvalues $\lambda_k, k = 1, \ldots, n$. then the border $G_{91}(t,s)$ of the support of the limit spectral density of the product of two matrices $(I_n + \Xi_n^{(1)})(I_n + \Xi_n^{(2)})$ is equal to

$$\frac{1}{n}\sum_{k=1}^{n} \frac{1}{\sqrt{t^2+s^2} + \lambda_k^2 - |\sqrt{\bar{\tau}} + \sqrt{\tau}|^2 \frac{\lambda_k^2}{\sqrt{t^2+s^2+\lambda_k^2}}} = 1\,. \quad (11.2)$$

We have chosen in the pictures below two points $\lambda_k = 0.55$, $k = 1, \ldots, [n/2]$, $\lambda_j = 2$, $j = [n/2]+1, \ldots, n$ in two equal parts for all eigenvalues, and similarly we have chosen in the second picture eigenvalues 0.55; 1.8;3 in three equal parts and 0.55;1.6;2,4; 3,4 in four equal parts. Then we can see the structure of the border support for the limit spectral density of the product of two matrices $\left(I_n + \Xi_n^{(1)}\right)\left(I_n + \Xi_n^{(2)}\right)$.

Of course, we can consider any matrix A_n in our equation for the border support, for example, we can consider diagonal complex matrix, but behavior of border will be similar, *i.e.* if the distance between diagonal entries of diagonal matrix A_n are large enough, then the border support looks like several closed almost circles.



12. Heart law

In principal we can consider now any sum of product of independent matrices. Let us start to investigate the product of three matrices $\prod_{j=1}^{3} [A_n + \Xi_n^{(j)}]$. Under the same assumptions with respect to random matrices as in the previous theorem we get canonical equation

$$\frac{b(t,s)}{1+b(t,s)} = \frac{1}{n}\operatorname{Tr}\Big[I_n y(1+b(t,s))^2 + \left\{\begin{matrix} I_{3n}\tau^{1/3} & 0_n & A_n \\ A_n & I_n\tau^{1/3} & 0_n \\ 0_n & A_n & I_n\tau^{1/3}\end{matrix}\right\}\left\{\begin{matrix} I_n\bar\tau^{1/3} & A_n^* & 0_n \\ 0_n & I_n\bar\tau^{1/3} & A_n^* \\ A_n^* & 0_n & I_n\bar\tau^{1/3}\end{matrix}\right\}\Big]^{-1} \quad (12.1)$$

The above formula is equal to:

$$\frac{1}{n}\operatorname{Tr}\left[I_{3n}[|\tau|^{\frac{2}{3}} + y(1+b)^2] + \left\{\begin{matrix} A_nA_n^* & \tau^{1/3}A_n^* & \bar\tau^{1/3}A_n \\ \bar\tau^{1/3}A_n & A_nA_n^* & \tau^{1/3}A_n^* \\ \tau^{1/3}A_n^* & \bar\tau^{1/3}A_n & A_nA_n^*\end{matrix}\right\}\right]^{-1}. \quad (12.2)$$

We simplify this equation assuming that matrix A_n is symmetric matrix and denoting its eigenvalues by λ_k.

Then this equation is equivalent to the following

$$\frac{b(t,s)}{1+b(t,s)} = \frac{1}{n}\sum_{k=1}^{n}\Bigg[|\tau|^{2/3} + |\lambda_k|^2 + y(1+b)^2 - \left(\bar\tau^{1/3}\lambda_k, \tau^{1/3}\bar\lambda_k\right) \times \frac{\left\{\begin{matrix} |\tau|^{2/3}+|\lambda_k|^2+y(1+b)^2 & -\tau^{1/3}\bar\lambda_k \\ -\bar\tau^{1/3}\lambda_k & |\tau|^{2/3}+|\lambda_k|^2+y(1+b)^2\end{matrix}\right\}\left(\bar\tau^{1/3}\lambda_k, \tau^{1/3}\bar\lambda_k\right)^{\mathrm{T}}}{[|\tau|^{2/3}+|\lambda_k|^2 + y(1+b)^2]^2 - |\tau|^{2/3}|\lambda_k|^2}\Bigg]^{-1}$$

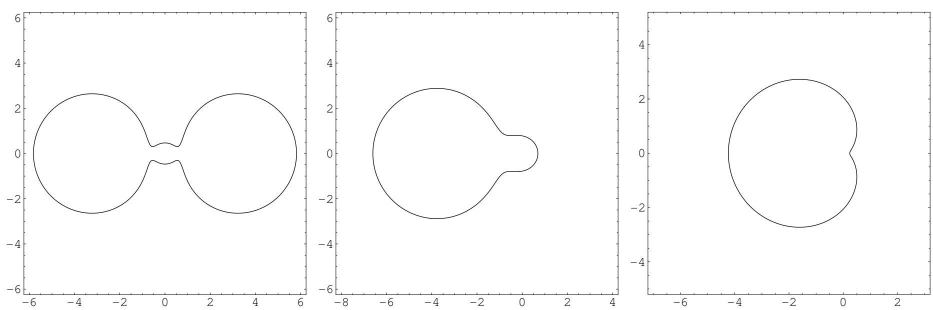
$$= \frac{1}{n}\sum_{k=1}^{n}\Bigg[|\tau|^{2/3}+|\lambda_k|^2+y(1+b)^2 - \frac{2|\tau|^{4/3}|\lambda_k|^2+2|\lambda_k|^2(|\lambda_k|^2+y(1+b)^2)-\bar{\lambda}_k^3\tau-\lambda_k^3\bar{\tau}}{|\tau|^{4/3}+[|\lambda_k|^2+y(1+b)^2]^2+|\tau|^{2/3}|\lambda_k|^2+2|\tau|^{2/3}y(1+b)^2}\Bigg]^{-1}$$

$$= \frac{1}{n}\sum_{k=1}^{n}\frac{|\tau|^{4/3}+|\lambda_k|^4+|\lambda_k|^2|\tau|^{2/3}}{|\tau|^2+|\lambda_k|^6+\bar{\lambda}_k^3\tau+\lambda_k^3\bar{\tau}}\,. \tag{12.3}$$

Using this equation we can as in Theorem 3 to find the accompanying spectral density but since in this case we should solve the equation of the third order the final formula becomes more complicated with comparison to the case of the product of two matrices. Much easily we can find the border of the support of this density. As in the case of the product of two matrices we have

$$1 = \frac{1}{n}\sum_{k=1}^{n}\frac{|\tau|^{4/3}+|\lambda_k|^4+|\tau|^{2/3}|\lambda_k|^2}{|\tau|^2+|\lambda_k|^6+\tau\bar{\lambda}_k^3+\bar{\tau}\lambda_k^3}\,. \tag{12.4}$$

Another border when the diagonal entries have nonzero different expectations is:

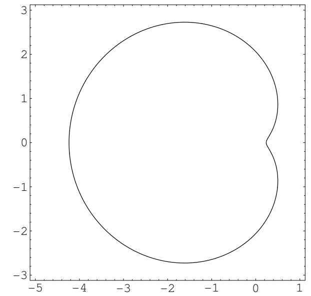


In the first picture we have chosen $a = 0,\ b = -1.42,\ c = 1.42$; in the second $a = 1.5,\ b = 0,\ c = 0$; and in the third $a = b = c = 1$.

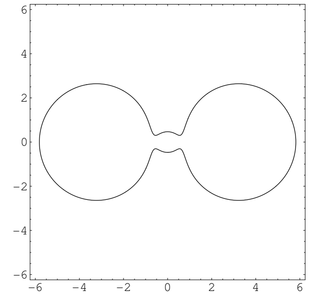
If all eigenvalues λ_k are equal to one then the equation of the border is simple

$$|\tau|^2+\tau+\bar{\tau} = |\tau|^{4/3}+|\tau|^{2/3}\,. \tag{12.5}$$

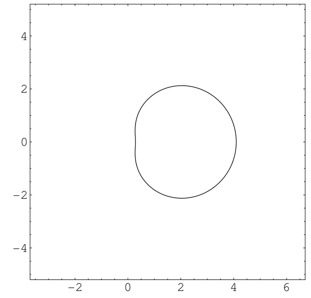
Using this equation we can find the picture of this border



The reader can compare this picture with the border which looks like heart and corresponds to the case of the product of two matrices which was found in the papers [5,6] ($a = 1.2$):



Another border when the diagonal entries have nonzero equal expectations $a = 0.3$:



After obtaining the results for n.s.f. of a single symmetric, non symmetric and unitary random matrices [1,3] we now move in this paper toward the main goal, namely to the most general solution of the problems of the limit theorems of the theory of random matrices: to find limit distributions of n.s.f. of random matrices $f[(\Xi_n^{(j)})^k, (\Xi_n^{(j)^*})^p, j, k, p = 1, 2, \ldots]$, where $f(x_1, x_2, \ldots)$

is an analytical function and $\Xi_n^{(j)}, j = 1, 2, \dots$ are independent *ACE (Asymptotically Constant Entries)*-random matrices (in particular, unitary random matrices). Particularly, using the canonical equation K_{91} we derive so called LIFE law: under a certain conditions $\prod_{j=1}^{m} \Xi_n^{(j)} \sim \widetilde{\text{LIFE}} \sim \{\Xi_n^{(1)}\}^m$.

Roughly speaking LIFE means that n.s.f. of the sum of nonrandom matrix A_n and the power of a non Hermitian matrix H_n^k with independent ACE-entries (Asymptotically Constant Entries) is approximately equal to n.s.f. of the sum of nonrandom matrix A_n and the product of k independent random matrices $H_n^{(1)} H_n^{(2)} \cdots H_n^{(k)}$ having the same structure as the initial random matrix H_n but their entries may have any distributions from a certain class $\boldsymbol{G}_2$ or $\boldsymbol{G}_3$ of matrices which were used in Girko's Circular law. The similar assertion we can prove for other certain function $f[(\Xi_n^{(j)})^k, (\Xi_n^{(j)^*})^p, j, k, p = 1, 2, \dots$ [7]. This assertion is a simple Corollary from Equation K_{91}.

By tradition of choosing the names of laws in probability theory (Arcsine law, law of iterated logarithm, *etc.*) we call this unusual behavior of the n.s.f. of the power of random matrix Ξ_n^k as the *Halloween law* keeping in mind that the appearance instead in n.s.f. of k copies of the same random matrix Ξ_n its k independent copies $\Xi_n^{(j)}, j = 1, \dots, k$ looks like *phantom or illusion.* More important that the histogram and the density of this law look like a hat that people wear during Halloween days (see Figs. 7 and 8 below).

13. The $\sim \widetilde{\textbf{LIFE}} \sim$-phenomenon

For the first time the powers of matrices Ξ_n from class $\boldsymbol{G}_1$ were investigated by Wegmann [8]. In our case when the matrix Ξ_n is non Hermitian and belongs to the class $\boldsymbol{G}_2$ or $\boldsymbol{G}_3$ the Wegmann's method is not valid. Nevertheless, we can find some relatively simple relation for the spectra of functions of random matrices using the LIFE phenomenon(the main statement): in the LIFE sense the spectra of random matrix $\Xi_n^k, \Xi_n \in \boldsymbol{G}_1 \div \boldsymbol{G}_3$, where $k > 1$ and the matrix Ξ_n is not Hermitian, approximately is equal to the spectra of the product of k independent random matrices $\prod_{j=1}^{k} \Xi_n^{(j)}$, where $\Xi_n^{(j)} \approx \Xi_n$, the symbol $\approx$ staying between two matrices Ξ_n and H_n means coincidence of distributions of these matrices. This assertion is a simple Corollary from Equation K_{91} [7].

Fig. 7 (left) shows the 300 Monte Carlo simulation of the support of the accompanying spectral density of the sum of diagonal matrix A_{60} with six different diagonal entries $a = 1$, $b = -1$, $c = -1 - i$, $d = i$, $e = -i$, $f = 1.5 + 1.5i$ chosen in equal parts and the product of five independent random matrices with independent entries $\Xi_{60}^{(p)} = (\xi_{ij}^{(p)})$, $p = 1, 2, 3, 4, 5$;

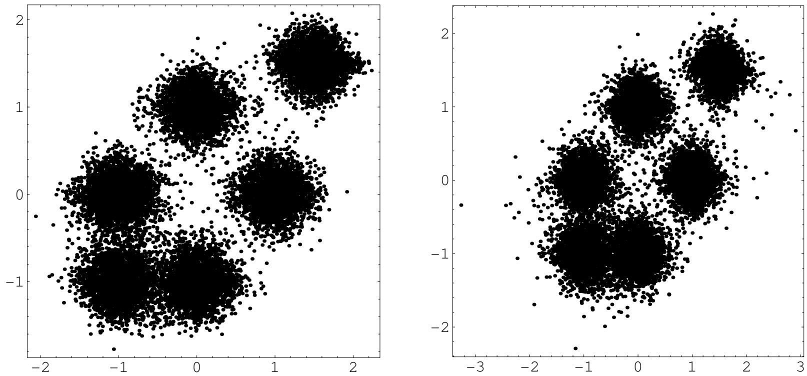


Fig. 7. See explanation in the text.

$\boldsymbol{E}\xi_{ij} = 0$, $\boldsymbol{E}[\xi_{ij}]^2 = 1/60$, $i, j = 1, \ldots, 60$. Fig. 7 (right) shows the 300 Monte Carlo simulation of the support of spectral densities of the sum of the same matrix A_{60} and the power of matrix: $[\Xi_{60}^{(1)}]^5$ and these pictures give the conformation of the LIFE law: approximately the support of spectral densities of two matrices $A_{60}+[\Xi_{60}^{(1)}]^5$ and $A_{60}+\prod_{p=1}^{5} \Xi_{60}^{(p)}$ are approximately the same.

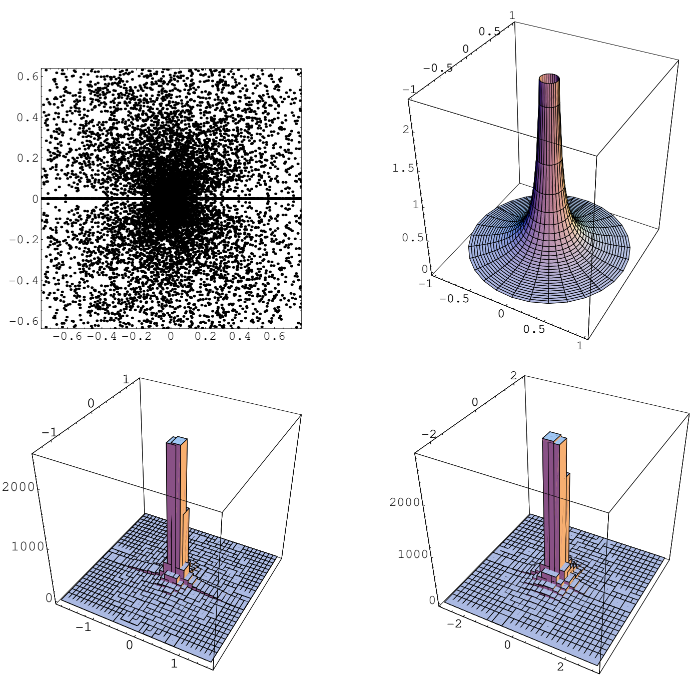


Fig. 8. See explanation in the text.

Fig. 8 (upper left) shows the 300 Monte Carlo simulation of the support of the accompanying limit spectral density $p(x,y)=\frac{1}{5\pi}(x^2+y^2)^{\frac{1}{5}-1}\chi\{x^2+y^2\leq 1\}$ of the product of five independent random matrices with independent entries $\Xi_{60}^{(p)} = (\xi_{ij}^{(p)}), p = 1,\ldots,5; \boldsymbol{E}\xi_{ij} = 0, \boldsymbol{E}[\xi_{ij}]^2 = 1/60, i,j = 1,\ldots,60$. Fig. 8 (upper right) shows the Halloween density for $k = 5$. This picture gives the conformation of the LIFE law: approximately the supports of spectral densities of the product of five matrices $\prod_{j=1}^{5} H_{60}^{(j)}$, and $\{H_{60}^{(1)}\}^5$ are the same. Fig. 8 (lower left and right) shows the histograms (Halloween law) and support of the accompanying limit spectral density.

These pictures give the conformation of the LIFE law. We see that Fig. 8 (upper right) looks indeed like a hat that some people wear during Halloween days.

14. Sombrero law for matrices $A_n\Xi_n^m$

LIFE phenomenon is working also for a matrices $A_n\Xi_n^k$, where A_n is a diagonal non random and Ξ_n is a random matrices. We do not present here corresponding calculations, because all proofs are almost the same as for matrices $A_n + \Xi_n^k$.

15. The $\sim \widetilde{\text{LIFE}} \sim$-operator

Here we give the main properties of $\sim \widetilde{\text{LIFE}} \sim$-operator:

1. First of all this operator is acting on analytical functions $f(\Xi_n(j), \Xi_n^{(j)*}, j = 1, 2\ldots)$ of non Hermitian matrices $\Xi_n(j)$ and its conjugate matrices $\Xi_n^{(j)*}$. If matrix $f(\Xi_n(j), \Xi_n^{(j)*}, j = 1, 2\ldots)$ is Hermitian then our $\sim \widetilde{\text{LIFE}} \sim$-criterion is understood as the corresponding Stieltjes transform, if this matrix is non Hermitian then we consider the corresponding VICTORIA-relation. To simplify our calculations we assume that Hermitian analytical matrix-function $f(\Xi_n(j), \Xi_n^{(j)*}, j = 1, 2\ldots)$ is a polynomial function. More precisely it is equal to

$$f\left(\Xi_n(j), \Xi_n^{(j)*}, j = 1, 2\ldots\right) = B_n + \sum_{k=0} (\Xi_n(1))^k g\left(\Xi_n(j), \Xi_n^{(j)*}, j = 2\ldots\right)\left(\Xi_n^{(1)*}\right)^k, \quad (15.1)$$

   where $g(\Xi_n(j), \Xi_n^{(j)*}, j = 2\ldots)$ is a polynomial.

2. If

$$f\left(\Xi_n(j), \Xi_n^{(j)*}, j = 1, 2\ldots\right) = \Xi_n^k, \Xi_n \in \boldsymbol{G}_3 \quad (15.2)$$

then

$$\Xi_n^k \sim \widetilde{\mathrm{LIFE}} \sim \prod_{j=1}^{k} \Xi_n^{(j)}\,, \tag{15.3}$$

where $\Xi_n^{(j)}, j=1,\dots,n$ are independent and $\Xi_n^{(j)} \approx \Xi_n$.

3. If

$$f\left(\Xi_n(j), \Xi_n^{(j)*}, j=1,2\dots\right) = \Xi_n^k \pm \left(\Xi_n^k\right)^*, \Xi_n \in \boldsymbol{G}_4 \tag{15.4}$$

then

$$\Xi_n^k \pm \left(\Xi_n^k\right)^* \sim \widetilde{\mathrm{LIFE}} \sim \prod_{j=1}^{k} \Xi_n^{(j)} \pm \left(\prod_{j=1}^{k} \Xi_n^{(j)}\right)^*. \tag{15.5}$$

16. G-method in comparison to replica trick, the supersymmetry approach and free probability theory

We give here more clarifications for the procedure described in Section 9, starting from Step 5.

Step 5. Then we can use the rough estimator of convergence of the n.s.f. of the corresponding random permutation matrix with the speed of convergency like n^{-q_3}. All calculations are almost the same which were used in the G-theory. We are using the Berry–Esseen inequality, the random variable γ and we do not pursue the precise order of convergency of n.s.f. $\nu_n(G_{4n\times 4n}, t, s, x,)$ to the non-random accompanying n.s.f. $\mu_n(t,s,x)$:

$$\sup_x |\nu_n(G_{4n\times 4n}, t, s, x,) - \mu_n(t,s,x)| \le cn^{-q_3}\,, \quad c>0\,. \tag{16.1}$$

Step 6. Then we can perform the limit procedure in V-transform replacing n.s.f. of permutation random matrices by non random accompanying n.s.f.

$$\int_0^\infty \ln\left(i\sqrt{\alpha_n}+x\right) d\nu_n(G_{4n\times 4n}, t, s, x,) = \int_0^\infty \ln\left(i\sqrt{\alpha_n}+x\right) d\mu_n(t,s,x,) + \varepsilon_n\,, \tag{16.2}$$

where $\lim_{n\to\infty} \varepsilon_n = 0$ and receive the accompanying expression which is expressing through canonical equation K_{91}.

Consider VICTORIA-transform of matrix Ξ_n^2:

$$b_n(\alpha,t,s) = \frac{1}{n}\operatorname{Tr}\left[I_n\alpha + \left(\Xi_n^2 - \tau I_n\right)\left(\Xi_n^2 - \tau I_n\right)^*\right]^{-1}, \tag{16.3}$$

where $\alpha > 0$, $\tau = t + is$. We prepare matrix Ξ_n in the following form, considering the first row vector $\vec{\xi}_1$ of the matrix Ξ_n: (For other row vectors $\vec{\xi}_j, j = 2, \dots, n$ we have similar formulas).

Despite that, even in this case, the structure of this matrix is quite complicated, we prove the following assertion [7].

Theorem 5. *If $\Xi_n \in \mathbf{G_2}$, then the following canonical equation K_{71} is valid*

$$\Xi_n \Xi_n^* + \Xi_n^* \Xi_n \sim \widetilde{LIFE} \sim \Xi_n \Xi_n^* + H_n^* H_n \,, \tag{16.4}$$

where random matrices Ξ_n and H_n are independent and $\Xi_n \approx H_n$. Here the symbol $\approx$ staying between two matrices Ξ_n and H_n means coincidence of distributions of these matrices.

Proof. The idea of the proof of this assertion consists in the following. Firstly, we prepare the matrices in such a way that their entries should be symmetric. Since this step is very important we consider this method in the special section. But now we should establish the self averaging property of n.s.f.

17. Self averaging of normalized traces of resolvents of random matrices in law of independency of ensembles of random matrices (LIFE)

For the proof of self averaging property of normalized traces of resolvents of random matrices $\Xi_n \, \Xi_n^* + \Xi_n^* \, \Xi_n$ we can use the main statement which was proven in [3] Chapter 1, Volume I. As the reader may remember, we have used the invariance principal for matrices Ξ_n and if the entries of this matrix are independent with zero expectations and the same variances, then we have replaced matrix Ξ_n by Gaussian matrix and its distribution is invariant with respect to the orthogonal transformation. We have proved that matrices $\Xi_n \, \Xi_n^*, \Xi_n^* \, \Xi_n$ are asymptotically stochastically independent. But in the general case of the distribution of matrix Ξ_n this method does not work, therefore we follow here the LIFE procedure.

Theorem 6. *If $\Xi_n \in \boldsymbol{G}_2$, then for every $t > 0$*

$$\lim_{n\to\infty} \boldsymbol{E} \left| \frac{1}{n} \operatorname{Tr} [I_n t + \Xi_n \Xi_n^* + \Xi_n^* \Xi_n]^{-1} - \boldsymbol{E} \frac{1}{n} \operatorname{Tr} [I_n t + \Xi_n \Xi_n^* + \Xi_n^* \Xi_n]^{-1} \right|^2 = 0 \,. \tag{17.1}$$

18. The Wegmann's canonical equation $\boldsymbol{K_{73}}$

Due to the LIFE-statement we can find the canonical equation for the Stieltjes transforms of n.s.f. of matrices $\Xi_n^k (\Xi_n^*)^k$.

Theorem 7. *If matrices Ξ_n belong to $\boldsymbol{G}_1$-class of random matrices, i.e. if the variances of the entries $\xi_{pl}^{(n)}, p, l = 1, \dots, n$ are equal to n^{-1} and expectation of all entries $\xi_{pl}^{(n)}$, p, $l = 1, \dots, n$ of matrices Ξ_n are equal to zero, for an integer $k > 1$*

$$\sup_n \max_{i,j=1,\dots,n} \boldsymbol{E}|\xi_{ij}\sqrt{n}|^{2k+\delta} < \infty \,, \tag{18.1}$$

then with probability one for this integer k for almost all $x \geq 0$

$$\lim_{n\to\infty} \mu_n \left\{ x, \Xi_n^k (\Xi_n^*)^k \right\} = F^{(k)}(x) \,, \tag{18.2}$$

where $F^{(k)}(x)$ is the probability distribution function whose Stieltjes transform

$$m(z) = \int_0^\infty (x-z)^{-1} dF^{(k)}(x), \mathrm{Im} z > 0 \,, \tag{18.3}$$

satisfies canonical equations K_{73} [8]

$$(-1)^{k+1} z^k m^{k+1}(z) + z m(z) + 1 = 0 \,. \tag{18.4}$$

There exist the unique solution to canonical equations K_{73} in the class L of analytic functions L=$\{m(z) : \mathrm{Im} z > 0, \mathrm{Im} m(z) > 0\}$.

Proof. Using the proof of Theorem 5 we have

$$\Xi_n^k (\Xi_n^*)^k \sim \widetilde{\mathrm{LIFE}} \sim \prod_{j=1}^k \Xi_n^{(j)} \left[\prod_{j=1}^k \Xi_n^{(j)} \right]^* \,, \tag{18.5}$$

where matrices $\Xi_n^{(j)}, j = 1, \dots, k$ are independent and their entries also independent and are distributed by normal law $N(0, n^{-1})$. Denote

$$f_k(z) = \frac{1}{n} \boldsymbol{E} \,\mathrm{Tr} \left\{ -I_n z + \prod_{j=1}^k \Xi_n^{(j)} \left[\prod_{j=1}^k \Xi_n^{(j)} \right]^* \right\}^{-1} \,, \tag{18.6}$$

$\Lambda_n^{(s)} = \{\delta_{ij} \lambda_i(s)\}_{j,i=1,\dots,n}$ are eigenvalues of the matrix $\prod_{j=1}^s \Xi_n^{(j)} [\prod_{j=1}^s \Xi_n^{(j)}]^*$.

In order to explain this result we need to prepare our resolvent for the further calculations. Firstly, using the invariance principle we can change approximately matrix Ξ_n by standard Gaussian random matrix H_n. Then, we know that $H_n = U_n \Lambda_n V_n$, where the matrices U_n and V_n are independent and are distributed by probabilistic Haar measure, and Λ_n is a diagonal random matrix of corresponding eigenvalues of matrix $(H_n H_n^*)^{1/2}$. Then, due to our theory

$$f_k(z) = \boldsymbol{E}\frac{1}{n}\operatorname{Tr}\left\{-zI_n + \Xi_n^{(1)}\Lambda_n^{(k-1)}\Xi_n^{(1)*}\right\}^{-1} + o(1)\,. \tag{18.7}$$

Then, using the canonical equation K_7 [3] we obtain

$$\begin{aligned} f_k(z) &= \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{E}\left[-z - zf_k(z)\lambda_i(k-1)\right]^{-1} + o(1) \\ &= -\frac{1}{zf_k(z)}\sum_{i=1}^{n}\frac{1}{n}\boldsymbol{E}\left[\frac{1}{f_k(z)} + \lambda_i(k-1)\right]^{-1} + o(1)\,. \end{aligned} \tag{18.8}$$

or

$$-\,zf_k^2(z) = f_{k-1}\left(\frac{1}{f_k(z)}\right) + o(1)\,, \tag{18.9}$$

where

$$f_{k-1}\left(\frac{1}{f_k(z)}\right) = \sum_{i=1}^{n}\frac{1}{n}\boldsymbol{E}\left[\frac{1}{f_k(z)} + \lambda_i(k-1)\right]^{-1}\,. \tag{18.10}$$

Continuing such process of calculations we get

$$f_{k-1}^2\left(\frac{1}{f_k(z)}\right) = f_k(z)f_{k-2}\left(-\frac{1}{f_{k-1}}\right) + o(1)\,. \tag{18.11}$$

We simplify calculations writing

$$f_{s-1}\left(\frac{1}{f_s(z)}\right) = f_{s-1}(z), s = 1, \ldots, k\,. \tag{18.12}$$

Then we have the system of canonical equations

$$\begin{cases} f_{k-1}^2(z) = f_k(z)f_{k-2}(z) + o(1)\,, \\ f_{k-2}^2(z) = f_{k-1}(z)f_{k-3}(z) + o(1)\,, \\ \qquad\vdots \\ f_5^2(z) = f_6(z)f_4(z) + o(1)\,, \\ f_4^2(z) = f_5(z)f_3(z) + o(1)\,, \\ f_3^2(z) = f_4(z)f_2(z) + o(1)\,, \\ f_2^2(z) = f_3(z)f_1(z) + o(1)\,, \end{cases} \tag{18.13}$$

and the last equations is a simple consequence of the theory of Gram random matrices:

$$f_1(z) = f_2(z)\frac{1}{1+f_1(z)} + o(1)\,. \tag{18.14}$$

Now finding from the last equation

$$f_2(z) = f_1(z)(1+f_1(z)) + o(1) \tag{18.15}$$

and substituting $f_2(z)$ in the previous two last equations we obtain

$$f_3(z) = f_1(z)[1+f_1(z)]^2 + o(1)\,, \tag{18.16}$$

$$f_3^2(z) = f_4(z)f_1[1+f_1(z)] + o(1)\,. \tag{18.17}$$

Hence, we have the system of equations

$$\begin{cases} -zf_k^2(z) = f_{k-1}(z) + o(1)\,, \\ f_{k-1}^2(z) = f_k(z)f_{k-2}(z) + o(1)\,, \\ f_{k-2}^2(z) = f_{k-1}(z)f_{k-3}(z) + o(1)\,, \\ \quad\vdots \\ f_3^2(z) = f_4 f_1(z)(1+f_1(z)) + o(1)\,, \\ f_1(z)(1+f_1(z))^2 = f_3(z) + o(1)\,. \end{cases} \tag{18.18}$$

Solving it we get

$$-zf_k^2(z) = f_1(z)(1+f_1(z))^{k-2} + o(1)\,, \tag{18.19}$$

$$f_k(z) = f_1(z)(1+f_1(z))^{k-1} + o(1)\,. \tag{18.20}$$

From this system we obtain canonical equation K_{73}

$$(-1)^{k+1}z^k f_k^{k+1}(z) + zf_k(z) + 1 = 0\,. \tag{18.21}$$

Theorem 7 is proved.

Remark The corresponding Wegmann's equation (see [8]) follows from this equation if we will put $f_k(z) = -x^{-1}f(x-1)$.

19. The Quatriq law

Corollary 1. [8] *If in addition to the condition of Theorem 7, $k = 3$ then with probability one*

$$\lim_{n\to\infty} \mu_n\left(x, \Xi_n^3(\Xi_n^*)^3\right) = F^{(3)}(x)\,, \tag{19.1}$$

where $F^{(3)}(x)$ *is one quarter circle distribution function whose density* $p(x)$ *is equal to*

$$\frac{\partial F(x)}{\partial x} = \begin{cases} \frac{\sqrt{3}}{2\pi}\left[\left(q(x)/2+\sqrt{Q(x)}\right)^{1/3} - \left(q(x)/2-\sqrt{Q(x)}\right)^{1/3}\right], & Q(x)>0\,, \\ 0\,, & Q(x)<0\,, \end{cases} \tag{19.2}$$

where $Q\,(x) = [\theta\,(x)/3]^3 + [q\,(x)/2]^2$, $\theta = -a^2/3 + b$, $q = 2(a/3)^3 - ab/3 + d$, $a = 0,\ b = -1/x,\ c = -1/x^2$.

Proof. In this case the canonical equation K_{73} has a form

$$z^3 m^4(z) + zm(z) + 1 = 0\,. \tag{19.3}$$

As in previous chapters we prove that there exists the limits

$$q(x) = \lim_{\varepsilon \downarrow 0} \operatorname{Re} m(x + i\varepsilon), p(x) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \operatorname{Im} m(x + i\varepsilon)\,. \tag{19.4}$$

Therefore, using equation (18.21) we have for these functions

$$x^3[q(x) + i\pi p(x)]^4(z) + x[q(x) + i\pi p(x)] + 1 = 0\,. \tag{19.5}$$

From this equation considering separately the real and imaginary parts and taking into account that we are interested in positive probability density we have two equations $4x^2q^2(x) - 4x[\pi p(x)]^2 + 1 = 0$, $x^3q^4(x) - 4x^3[\pi p(x)]^2 + x^3[\pi p(x)]^4 - 2x^3q^2(x)[\pi p(x)]^2 + xq(x) + 1 = 0$.

Now, using formulas for the roots of polynomial of fourth degree as in [3] we complete the proof of Corollary 1.

20. The Cubic law

Corollary 2. [8] *If in addition to the condition of Theorem 7* $k = 2$ *then with probability one*

$$\lim_{n\to\infty} \mu_n\left(x, \Xi_n^2(\Xi_n^*)^2\right) = F^{(2)}(x)\,, \tag{20.1}$$

where $F^{(2)}(x)$ *is one quarter circle distribution function whose density* $p(x)$ *is equal to where* $F(x)$ *is the probability distribution function whose density* $p(x)$ *is equal to*

$$\frac{\partial F(x)}{\partial x} = \begin{cases} \frac{\sqrt{3}}{2\pi}\left[\left(q(x)/2+\sqrt{Q(x)}\right)^{1/3} - \left(q(x)/2-\sqrt{Q(x)}\right)^{1/3}\right], & Q(x) > 0\,, \\ 0\,, & Q(x) < 0\,, \end{cases} \tag{20.2}$$

where $Q(x) = [\theta(x)/3]^3 + [q(x)/2]^2$, $\theta = -a^2/3 + b$, $q = 2(a/3)^3 - ab/3 + d$, $a = 0, b = -1/x$, $c = -1/x^2$.

Proof. In this case the canonical equation K_{73} has a form

$$-z^2 m^3(z) + zm(z) + 1 = 0\,. \tag{20.3}$$

Now using Cardano formulas as in [3] we complete the proof of Corollary 2.

21. The one quarter circle law

Corollary 3. [8] *If in addition to the condition of Theorem 7, $k = 1$ then with probability one*

$$\lim_{n\to\infty} \mu_n(x, \Xi_n(\Xi_n^*)) = F^{(1)}(x)\,, \tag{21.1}$$

where $F^{(1)}(x)$ is one quarter circle distribution function whose density $p(x)$ is equal to

$$p(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\chi\{0 < x < 4\}\,. \tag{21.2}$$

Proof. In this case the canonical equation K_{73} has a form

$$zm^2(z) + zm(z) + 1 = 0\,. \tag{21.3}$$

Now solving this equation as in [3] we complete the proof of Corollary 3.

22. Application of extended *V*-transform

Of course, we can find also other canonical equations for our matrices using the extended V-transform. We give here only the sketch of this approach. In this case we already have the product of $2k$ matrices:

$$\begin{aligned}
&\mathrm{Tr}\left[\Xi_n^k \Xi_n^{k*} - I_n z\right]^{-1} = -\frac{\partial}{\partial z}\ln\det\left[\Xi_n^k\Xi_n^{k*} - I_{2n}z\right] + nz^{-1} \\
&= -\frac{\partial}{\partial z}\ln\left|\det\left\{\begin{matrix} I_{2n}z^{1/4} & 0_{2n} & 0_{2n} & G_{2n}^{(1)} \\ G_{2n}^{(2)} & I_{2n}z^{1/4} & 0_{2n} & 0_{2n} \\ 0_{2n} & G_{2n}^{(3)} & I_{2n}z^{1/4} & 0_{2n} \\ 0_{2n} & 0_{2n} & G_{2n}^{(4)} & I_{2n}z^{1/4}\end{matrix}\right\}\right| + nz^{-1}\,, \quad (22.1)
\end{aligned}$$

where

$$G_{2n}^{(1)} = \left\{ \begin{matrix} A_n & H_n \\ 0_n & 0_n \end{matrix} \right\}, \qquad G_{2n}^{(2)} = \left\{ \begin{matrix} H_n & 0_n \\ 0_n & B_n \end{matrix} \right\},$$
$$G_{2n}^{(3)} = \left\{ \begin{matrix} B_n & 0_n \\ 0_n & H_n^* \end{matrix} \right\}, \qquad G_{2n}^{(4)} = \left\{ \begin{matrix} H_n^* & 0_n \\ A_n & 0_n \end{matrix} \right\}.$$

Now we can apply for this expression the VICTORIA-transform and find n.s.f. of the matrix $\left\{ \begin{matrix} 0_{8n} & Q_{8n} \\ Q_{8n}^* & 0_{8n} \end{matrix} \right\}$, where

$$Q_{8n} = \left\{ \begin{matrix} 0_{2n} & 0_{2n} & 0_{2n} & G_{2n}^{(1)} \\ G_{2n}^{(2)} & 0_{2n} & 0_{2n} & 0_{2n} \\ 0_{2n} & G_{2n}^{(3)} & 0_{2n} & 0_{2n} \\ 0_{2n} & 0_{2n} & G_{2n}^{(4)} & 0_{2n} \end{matrix} \right\}.$$

The next step consists in derivation of canonical transform($16n$ equations, or, we can replace matrix Q_{8n} by block matrix (every block has dimensions $2k \times 2k$) and derive one equation for block matrices of dimension $2k$.

So, we have two proofs: one is based on the existence of probability densities of the entries of random matrices and the second is based on the G-Lindeberg condition. But the final formulas for the limit spectral densities coincide. Therefore we can simplify our first final formula.

23. The $\sim \widetilde{\text{LIFE}} \sim$-operator for block random matrices

We can repeat all previous calculations for the block random matrices. See corresponding material in Volumes I and II of [3]. As a result we have the following assertion.

Theorem 8. *If $\Xi_n \in \boldsymbol{G}_1$, then the following canonical equation K_{69} is valid*

$$f(\Xi_n\Xi_n^*) + g(\Xi_n^*\Xi_n) \sim \widetilde{LIFE} \sim f(\Xi_n\Xi_n^*) + g(H_n^*H_n)\,, \tag{23.1}$$

where $f(x)$, $g(x)$ are analytic functions, the blocks of random matrices Ξ_n and H_n are independent and $\Xi_n \approx H_n$. Here the symbol $\approx$ staying between two matrices Ξ_n and H_n means coincidence of distributions of these matrices.

24. The canonical equation K_{74}

Repeating the proof of Theorem 7, when matrices Ξ_n belong to the class of matrices $\boldsymbol{G_1}$ we have proved the following assertion.

Theorem 9. *Let as in the previous chapter we have the same matrix Ξ_n, i.e. we require that their entries $\xi_{ij}, i,j = 1,\dots,n$ have zero expectations and equal variances n^{-1}, for some $\delta > 0$ and integer $k > 0$*

$$\sup_n \max_{p,l=1,\dots,n} \boldsymbol{E} \left| \xi_{pl}^{(n)} \sqrt{n} \right|^{4k+\delta} < c < \infty , \tag{24.1}$$

let matrix $A_n = (a_{ij})$ be symmetric and

$$\sup_n \max_{i,j=1,\dots,n} \sum_{j=1}^{n} a_{ij}^2 < c < \infty . \tag{24.2}$$

Then with probability one for almost all x and this $k > 0$

$$\lim_{n\to\infty} \left\{ \mu_n \left\{ x, A_n + \Xi_n^k (\Xi_n^k)^* \right\} - L_n^{(k)}(x) \right\} = 0 , \tag{24.3}$$

where $L_n^{(k)}(x)$ is the probability distribution function whose Stieltjes transform

$$m_n(z) = \int_0^\infty (x-z)^{-1} dL_n^{(k)}(x) , \ \mathrm{Im} z > 0 \tag{24.4}$$

satisfies the canonical equations K_{74}

$$m_n(z) = \int_{-\infty}^{\infty} \frac{dF_n(y)}{-z + y + m_n^{-1}(z) - b\left[-m_n^{-1}(z)\right]} , \tag{24.5}$$

where $F_n(y)$ is the n.s.f. of the matrix A_n, and $b(z)$ satisfies the canonical equation K_{74}

$$(-1)^k z^{k-1} b^k(z) + z b(z) + 1 = 0 . \tag{24.6}$$

There exist the unique solution to canonical equations K_{74} in the class L of analytic functions

$$L = \{ m(z) , \ \mathrm{Im}\, z > 0 , \ \mathrm{Im}\, m(z) > 0 \} . \tag{24.7}$$

25. The canonical equation K_{79}

Let us find the equation for the Stieltjes transform of n.s.f. for matrices belonging to the class of matrices $\Xi_n^k (\Xi_n^*)^k + (\Xi_n^*)^k \Xi_n^k$.

Theorem 10. *If matrices Ξ_n belonging to the class of G-matrices and their entries have expectations zero, variances n^{-1} and for a certain $\delta > 0$*

$$\max_{i,j=1,\dots,n} \boldsymbol{E} \left| \xi_{ij}^{(n)} \sqrt{n} \right|^{4k+\delta} \le c , \tag{25.1}$$

then with probability one for every $k > 0$

$$\lim_{n\to\infty} \mu_n \left\{ x, \Xi_n^k (\Xi_n^*)^k + (\Xi_n^*)^k \Xi_n^k \right\} = F(x)\,, \tag{25.2}$$

where $F(x)$ *is the probability distribution function whose Stieltjes transform*

$$m(z) = \int_{-\infty}^{\infty} (x-z)^{-1} dF(x)\,, \quad \mathrm{Im} z > 0\,, \tag{25.3}$$

satisfies canonical equations K_{79} *[8]*

$$(-1)^{k-1} \left[\frac{zm(z)-1}{2} \right]^k m(z) + \frac{zm(z)-1}{2} + 1 = 0\,. \tag{25.4}$$

There exist the unique solution to canonical equations K_{80} *in the class* L *of analytic functions* L=$\{m(z),\ \mathrm{Im} z > 0,\ \mathrm{Im} m(z) > 0,\ \lim_{|z|\to\infty} m(z) = 0\}$.

Proof. In this case

$$\begin{aligned} m_n(z) &= \frac{1}{n} \boldsymbol{E}\, \mathrm{Tr} \left[-zI_n + \Xi_n^k (\Xi_n^*)^k + (\Xi_n^*)^k \Xi_n^k \right]^{-1} \\ &= \frac{1}{n} \boldsymbol{E}\, \mathrm{Tr} \left[-zI_n + \Xi_n \Lambda_n \Xi_n^* + \tilde{\Xi}_n \Lambda_n \tilde{\Xi}_n \right]^{-1} + o(1)\,, \end{aligned} \tag{25.5}$$

where Λ_n is the diagonal matrix of eigenvalues of matrix $\Xi_n^{k-1} (\Xi_n^*)^{k-1}$. Obviously

$$m_n(z) = \frac{1}{n} \boldsymbol{E}\, \mathrm{Tr} [-zI_n + B_n B_n^*]^{-1} = \frac{1}{n} \boldsymbol{E}\, \mathrm{Tr} [-zI_{2n} + B_n^* B_n]^{-1} + \frac{1}{z}\,, \tag{25.6}$$

where

$$B_n = [\, \Xi_n \Lambda_n, \quad \tilde{\Xi}_n \Lambda_n \,]\,.$$

Using canonical equation K_5 we get

$$\begin{aligned} m_n(z) &= \frac{2}{n} \sum_{j=1}^{n} \boldsymbol{E} \left[-z - zm_n(z) \lambda_j^{(k-1)} \right]^{-1} + \frac{1}{z} + o(1) \\ &= -\frac{2}{zm_n(z)} f\left(\frac{1}{m_n(z)}\right) + \frac{1}{z} + o(1)\,, \end{aligned} \tag{25.7}$$

where $f(-u)$ satisfy the canonical equation K_{75}

$$(-1)^k u^{k-1} f^k(-u) + uf(-u) + 1 = 0\,, \quad \mathrm{Im} u > 0\,. \tag{25.8}$$

Putting in this equation $u = -1/(m_n(z))$ we get

$$(-1)^k \frac{(-1)^{(k-1)}}{m_n^{k-1}(z)} f^k \left(\frac{1}{m_n(z)}\right) - \frac{1}{m_n(z)} f\left(\frac{1}{m_n(z)}\right) + 1 = 0\,. \tag{25.9}$$

Then taking into account that from (79.66) we have

$$f\left(\frac{1}{m_n(z)}\right) = -\frac{zm_n^2(z) - m_n(z)}{2} \tag{25.10}$$

we get canonical equation K_{79}

$$\frac{(-1)^{k-1}}{m_n^{k-1}(z)} \left(\frac{zm_n^2(z) - m_n(z)}{2}\right)^k + \frac{1}{m_n(z)} \left(\frac{zm_n^2(z) - m_n(z)}{2}\right) + 1 = 0\,. \tag{25.11}$$

Hence, we obtain equation K_{79}. Theorem 10 is proved.

Corollary 4. [8] *If in addition to the condition of Theorem 10, $k = 1$ then with probability one*

$$\lim_{n\to\infty} \mu_n \left\{x, \Xi_n^k (\Xi_n^*)^k + (\Xi_n^*)^k \Xi_n^k\right\} = M(x)\,, \tag{25.12}$$

where $M(x)$ is the probability distribution function whose density $p(x)$ is equal to

$$p(x) = \frac{1}{2\pi x}\sqrt{-x^2 + 6x - 1}\chi\left\{3 - \sqrt{8} < x < 3 + \sqrt{8}\right\}\,. \tag{25.13}$$

26. The canonical equation K_{82}

Let us find the equation for the Stieltjes transform of n.s.f. for matrices belonging to the class of G-matrices.

Theorem 11. *If matrices belonging to the class of $G^{(k)}$-matrices, i.e. if matrices Ξ_n belonging to the class of G-matrices and their entries have expectations zero, variances n^{-1} and for a certain $\delta > 0$*

$$\max_{i,j=1,\ldots,n} \boldsymbol{E}\left|\xi_{ij}^{(n)}\sqrt{n}\right|^{4k+\delta} \le c\,, \tag{26.1}$$

then with probability one for integer $k > 0$

$$\lim_{n\to\infty} \mu_n \left\{x, \Xi_n^k (\Xi_n^*)^k - \Xi_n^k (\Xi_n^*)^k\right\} = F(x)\,, \tag{26.2}$$

where $F(x)$ is the probability distribution function whose Stieltjes transform

$$m(z) = \int_{-\infty}^{\infty} (x - z)^{-1} dF(x)\,,\ \mathrm{Im} z > 0\,, \tag{26.3}$$

satisfies canonical equations K_{82} [8]

$$zm(z)^2 + m(z) = f(m(z)) + f(-m(z))\,, \tag{26.4}$$

where the analytic function $f(m(z))$ satisfies equations

$$-\frac{f(m(z))^k}{m(z)^{k-1}} - \frac{1}{m(z)} f(m(z)) + 1 \ = \ 0\,, \tag{26.5}$$

$$(-1)^k \frac{f(-m(z))^k}{m(z)^{k-1}} + \frac{1}{m(z)} f(-m(z)) + 1 \ = \ 0\,. \tag{26.6}$$

There exist the unique solution to canonical equations K_{82} in the class L of analytic functions $L = \{m(z),\ \mathrm{Im} z > 0,\ \mathrm{Im} m(z) > 0\}$.

Proof. In this case

$$\begin{aligned} m_n(z) \ &= \ \frac{1}{n} \boldsymbol{E}\, \mathrm{Tr} \left[-zI_n + \Xi_n^k (\Xi_n^*)^k - (\Xi_n^*)^k \Xi_n^k \right]^{-1} \\ &= \ \frac{1}{n} \boldsymbol{E}\, \mathrm{Tr} \left[-zI_n + \Xi_n \Lambda_n \Xi_n^* - \tilde{\Xi}_n \Lambda_n \tilde{\Xi}_n \right]^{-1} + o(1)\,, \end{aligned} \tag{26.7}$$

where Λ_n is the diagonal matrix of eigenvalues of matrix $\Xi_n^{k-1} (\Xi_n^*)^{k-1}$. Obviously

$$m_n(z) = \frac{1}{n} \boldsymbol{E}\, \mathrm{Tr} \left[-zI_{2n} + B_n^* B_n \right]^{-1} + \frac{1}{z} + o(1)\,, \tag{26.8}$$

where

$$B_n^* = \begin{bmatrix} \Lambda_n \Xi_n \\ \Lambda_n \tilde{\Xi}_n \end{bmatrix}\,. \tag{26.9}$$

Using canonical equation K_5 we get

$$\begin{aligned} m_n(z) \ &= \ \frac{1}{n} \sum_{j=1}^{n} \boldsymbol{E}\, \mathrm{Tr} [-z - z m_n(z) \lambda_j]^{-1} \\ &\quad + \frac{1}{n} \sum_{j=1}^{n} \boldsymbol{E}\, \mathrm{Tr} [-z + z m_n(z) \lambda_j]^{-1} + \frac{1}{z} + o(1) \\ &= \ -\frac{1}{z m_n(z)} \left[f(\frac{1}{m_n(z)}) + f(\frac{-1}{m_n(z)}) \right] + \frac{1}{z} + o(1)\,, \end{aligned} \tag{26.10}$$

where

$$f(-u) \tag{26.11}$$

satisfy the canonical equation K_{75}

$$(-1)^k u^{k-1} f^k(-u) + uf(-u) + 1 = 0\,, \ \mathrm{Im} u > 0\,. \tag{26.12}$$

Putting in these equations

$$u = -\frac{1}{m_n(z)} \tag{26.13}$$

we get canonical equation K_{82}. Theorem 11 is proven.

Corollary 5. (Cubic law) [8] *If in addition to the condition of Theorem 4 $k = 1$, matrices Ξ_n belonging to the class of $G^{(k)}$-matrices, i.e. their entries have expectations zero, variances n^{-1} and for a certain $\delta > 0$*

$$\max_{i,j=1,\dots,n} \boldsymbol{E} \left| \xi_{ij}^{(n)} \sqrt{n} \right|^{4k+\delta} \le c\,, \tag{26.14}$$

then with probability one

$$\lim_{n\to\infty} \mu_n\{x, \Xi_n\Xi_n^* + \Xi_n^*\Xi_n\} = M(x)\,, \tag{26.15}$$

where $F(x)$ is the probability distribution function whose density $p(x)$ is equal to

$$\frac{\partial F(x)}{\partial x} = \begin{cases} \frac{\sqrt{3}}{2\pi} \left[\left(q(x)/2 + \sqrt{Q(x)} \right)^{1/3} - \left(q(x)/2 - \sqrt{Q(x)} \right)^{1/3} \right], & Q(x) > 0\,, \\ 0\,, & Q(x) < 0\,, \end{cases} \tag{26.16}$$

where

$$\begin{aligned} Q(x) &= \left[\frac{\theta(x)}{3} \right]^3 + \left[\frac{q(x)}{2} \right]^2, \\ \theta &= -a^2/3 + b\,, \qquad q = 2(a/3)^3 - ab/3 + d\,, \\ a &= -\frac{1}{x}\,, \qquad b = \frac{x-1}{x}\,, \qquad c = \frac{1}{x}\,. \end{aligned}$$

Proof. In this case the system of canonical equations K_{81} has a form

$$\begin{aligned} -zm^2(z) + m(z) &= f(m(z)) + f(-m(z))\,, \\ -f(m(z)) - m^{-1}(z) f(m(z)) + 1 &= 0\,, \\ -f(-m(z)) + m^{-1}(z) f(-m(z)) + 1 &= 0\,. \end{aligned} \tag{26.17}$$

Solving it we obtain the cubic equation

$$zm^3(z) - m^2(z) + m(z)(z-1) + 1 = 0\,. \tag{26.18}$$

Now using Cardano formulas as in [3] we complete the proof of Corollary 5.

27. The canonical equation K_{85}

In the particular case when the expectations of the entries $\xi_{ij}^{(n)}$ of random matrices Ξ_n are equal to zero and variances are equal to

$$\boldsymbol{E}\left|\Xi_{ij}^{(n)}\right|^2 = n^{-1}\,, \tag{27.1}$$

we have the following assertion which appeared for the first time in [8] in 1976.

Theorem 12. [3] *If additionally to the conditions of this theorem the variances are equal n^{-1} and expectation of all entries of matrices are equal to zero, then with probability one*

$$\lim_{n\to\infty} \mu_n\left(x, \Xi_n^k + (\Xi_n^*)^k\right) = G(x)\,, \tag{27.2}$$

where $G(x)$ is the probability distribution function whose Stieltjes transform

$$m(z) = \int_{-\infty}^{\infty} (x-z)^{-1} dG(x)\,, \quad \mathrm{Im} z > 0 \tag{27.3}$$

satisfies canonical equations K_{85} [8]

$$[z + m(z)]^{k-1} m^{k+1}(z) + zm(z) + m^2(z) + 1 = 0\,. \tag{27.4}$$

There exist the unique solution to canonical equations K_{75} in the class L of analytic functions $L = \{m(z), \mathrm{Im} z > 0\,,\ \mathrm{Im} m(z) > 0\}$.

Proof. Using (27.2) we have

$$\begin{aligned} m_n(z) &= \frac{1}{n}\boldsymbol{E}\,\mathrm{Tr}\left[-zI_n + \Xi_n^k + (\Xi_n^*)^k\right]^{-1} \\ &= \frac{1}{n}\boldsymbol{E}\mathrm{Tr}[-zI_n + \Xi_n^{(1)} A_n + A_n^* \Xi_n^{(1)}]^{-1} + o(1)\,, \end{aligned} \tag{27.5}$$

where

$$A_n = \Xi_n^{k-1}\,.$$

Due to invariance principle (Volume 1) we can assume that the matrix $\Xi_n^{(1)}$ is the standard Gaussian. Therefore, using the invariance property of the standard Gaussian matrices with respect to orthogonal transformations we obtain

$$m_n(z) = \frac{1}{n}\boldsymbol{E}\,\mathrm{Tr}\left[-zI_n + \Xi_n^{(1)}\Lambda_n + \Lambda_n\Xi_n^{(1)*}\right]^{-1} + o(1)\,, \qquad (27.6)$$

where $\Lambda_n = (\delta_{ij}\lambda_i)$ is the diagonal matrix of eigenvalues of the Hermitian matrix $[A_nA_n^*]^{1/2}$. Therefore, using the canonical equation K_1 we get

$$m_n(z) = \frac{1}{n}\sum_{j=1}^{n} c_j(z) + o(1)\,, \qquad (27.7)$$

where the analytic functions

$$c_j(z), j = 1,\dots,n \qquad (27.8)$$

satisfy the system of equations:

$$\begin{aligned} c_j(z) &= \left\{-z - \frac{1}{n}\sum_{s=1}^{n}\boldsymbol{E}[\xi_{js}\lambda_j + \xi_{sj}\lambda_s]^2 c_s(z)\right\}^{-1}, j = 1,\dots,n \\ &= \boldsymbol{E}\left\{-z - \frac{1}{n}\sum_{s=1}^{n}\lambda_s^2 c_s(z) - \lambda_j^2\sum_{s=1}^{n} c_s(z)\right\}^{-1}, j = 1,\dots,n\,. \end{aligned} \qquad (27.9)$$

Since this system of equations has unique solution in a certain class of analytic functions we can choose solution

$$c_j(z) = m(z)\,, \qquad j = 1,\dots,n \qquad (27.10)$$

which satisfies such system of equations.

Now we apply for thus expression an unusual result of LIFE which easily follows from the previous calculations. Changing the power of a matrix by the product of corresponding independent matrices we get

Lemma *Under conditions of Theorem 12 for any* $k = 1, 2, \dots$

$$\frac{1}{n}\boldsymbol{E}\,\mathrm{Tr}\,\Xi_n^k(\Xi_n^*)^k = 1 + o(1)\,. \qquad (27.11)$$

Using this assertion we get from (27.21)

$$\begin{aligned} m_n(z) &= \frac{1}{n}\sum_{j=1}^{n}\left\{-z - m_n(z) - \lambda_j^2 m_n(z)\right\}^{-1}, \quad j = 1,\dots,n \\ &= -\frac{1}{m_n(z)}\sum_{j=1}^{n}\frac{1}{n}\boldsymbol{E}\left\{[z + m_n(z)]\,m_n^{-1}(z) + \lambda_j\right\}^{-1}\,. \end{aligned} \qquad (27.12)$$

Considering

$$[z + m_n(z)]m_n^{-1} \tag{27.13}$$

as a parameter we can use equation K_{78}

$$-m_n^2(z) = f\left\{[z + m_n(z)]\, m_n^{-1}\right\}, \tag{27.14}$$

where the function $f(u)$ satisfies the following algebraic equations

$$(-1)^k u^{k-1} f^k(-u) + uf(-u) + 1 = 0\,. \tag{27.15}$$

Putting

$$u = -[z + m_n(z)]m_n^{-1}(z) \tag{27.16}$$

in this equation we obtain

$$(-1)^{k-1}[z{+}m_n(z)]^{k-1}m_n^{-k+1}(z)m_n^{2k}(z){-}[z{+}m_n(z)]m_n^{-1}(z)(-m_n^2(z)){+}1{=}0\,. \tag{27.17}$$

Hence

$$\frac{[z + m_n(z)]^{k-1}}{m_n^{k-1}(z)}m_n^{2k}(z) + [z + m_n(z)]m_n(z) + 1 = 0\,. \tag{27.18}$$

Or, the Stieltjes transform

$$m(z) = \int_{-\infty}^{\infty} (x - z)^{-1} dG(x)\,,\ \mathrm{Im} z > 0\,, \tag{27.19}$$

satisfies canonical equations K_{82} [8]

$$[z + m(z)]^{k-1} m^{k+1}(z) + zm(z) + m^2(z) + 1 = 0\,. \tag{27.20}$$

Theorem 12 is proved.

Corollary 6 [8] *If additionally to the conditions of Theorem 12 the variances are equal n^{-1}, expectation of all entries of matrices are equal to zero and $k = 1$ then with probability one*

$$\lim_{n\to\infty} \mu_n(x, \Xi_n + \Xi_n^*) = G(x)\,, \tag{27.21}$$

where $G(x)$ is the probability distribution function whose density $p(x)$ is equal to

$$p(x) = \frac{1}{4\pi}\sqrt{8 - x^2}\,\chi\left\{|x| < \sqrt{8}\right\}\,. \tag{27.22}$$

28. Simple example of LIFE for matrices $\Xi_n^k - (\Xi_n^*)^k$

We start simple example of matrices for which we can apply results of LIFE using the REFORM method.

Theorem 13. *Let as in the previous chapters we have the same matrix Ξ_n. Then as in the sense of normalized common traces the main assertion of LIFE is the following:*

$$\Xi_n^k - (\Xi_n^*)^k \sim \widetilde{LIFE} \sim \prod_{j=1}^{k} \Xi_n^{(j)} - \left\{ \prod_{j=1}^{k} \Xi_n^{(j)} \right\}^* , \tag{28.1}$$

where the matrices $\Xi_n^{(j)}, j = 1, \dots, k$ are independent and $\Xi_n^{(j)} \approx \Xi_n, j = 1, \dots, n$.

29. The canonical equation K_{87} for matrices $A_n + \Xi_n^k + \Xi^{*k}$

Theorem 14. *If additionally to the conditions of Theorem 13 the variances are equal n^{-1} and expectation of all entries of matrices are equal to zero, A_n is a symmetrical matrix with bounded eigenvalues, then with probability one*

$$\lim_{n\to\infty} \left\{ \mu_n \left(x, A_n + \Xi_n^k + (\Xi_n^*)^k \right) - G_n(x) \right\} = 0 \,, \tag{29.1}$$

where $G_n(x)$ is the probability distribution function whose Stieltjes transform

$$m(z) = \int_{-\infty}^{\infty} (x-z)^{-1} dG_n(x) \,, \quad \mathrm{Im} z > 0 \,, \tag{29.2}$$

satisfies the system of canonical equations K_{58} [3]

$$\left\{ \begin{array}{l} m_n(z) = \displaystyle\int_{-\infty}^{\infty} \frac{d\mu_n(x, A_n)}{x - z + f_n(z)} \,, \\ m_n(z) = \displaystyle\int_{-\infty}^{\infty} \frac{d\nu(x)}{x - z + g_n(z)} \,, \\ 1 = f_n(z) m_n(z) - (z - g_n(z)) m_n(z) \,. \end{array} \right. \tag{29.3}$$

There exists a unique solution

$$\{m_n(z), f_n(z), g_n(z)\} \,, \quad z = t + is \tag{29.4}$$

of the system of canonical equations K_{58} *in the class of analytic functions*

$$L = \Big\{ [m_n(z), f_n(z), g_n(z)] : \ \mathrm{Im}\, m_n(z) > 0\,, \ \mathrm{Im}\, f_n(z) > 0, \ \mathrm{Im}\, g_n(z) > 0\,,$$
$$z = t + is, \ \mathrm{Im}\, z > 0, \ \lim_{s\to\infty} \sup_{|t|} \frac{1}{s} \left[|f_n(z)| + |g_n(z)| + |m_n(z)| \right] = 0 \Big\}. \quad (29.5)$$

The Stieltjes transform

$$b(z) = \int_{-\infty}^{\infty} \frac{d\nu(x)}{x - z} \quad (29.6)$$

of distribution function $\nu(x)$ *satisfies equation* K_{81}

$$[z + b(z)]^{k-1} b^{k+1}(z) + z b(z) + b^2(z) + 1 = 0\,. \quad (29.7)$$

There exist the unique solution to canonical equations K_{75} *in the class* L *of analytic functions*

$$L = \{ b(z)\,, \ \mathrm{Im}\, z > 0\,, \ \mathrm{Im}\, b(z) > 0 \}\,. \quad (29.8)$$

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